GTU

As per revised 2018 syllabus
First Year Engineering

MATHEMATICS-1

McGraw Hill Education

Ravish R. Singh • Mukul Bhatt

Mathematics-1

Gujarat Technological University 2018

About the Authors

Ravish R Singh is presently Director at Thakur Ramnarayan College of Arts and Commerce, Mumbai. He obtained a BE degree from University of Mumbai in 1991, an MTech degree from IIT Bombay in 2001, and a PhD degree from Faculty of Technology, University of Mumbai, in 2013. He has published several books with McGraw Hill Education (India) on varied subjects like *Engineering Mathematics, Applied Mathematics, Electrical Networks, Network Analysis and Synthesis, Electrical Engineering, Basic Electrical and Electronics Engineering*, etc., for all-India curricula as well as regional curricula of some universities like Gujarat Technological University, Mumbai University, Pune University, Jawaharlal Nehru Technological University, Anna University, Uttarakhand Technical University, and Dr A P J Abdul Kalam Technical University. Dr Singh is a member of IEEE, ISTE, and IETE, and has published research papers in national and international journals. His fields of interest include Circuits, Signals and Systems, and Engineering Mathematics.

Mukul Bhatt is presently Assistant Professor at Department of Mathematics and Statistics, Thakur Ramnarayan College of Arts and Commerce, Mumbai. She obtained her MSc (Mathematics) degree from H N B Garhwal University in 1992, and a PhD degree from Faculty of Science, PAHER University, Udaipur, Rajasthan in 2017. She has published several books with McGraw Hill Education (India) Private Limited on *Engineering Mathematics* and *Applied Mathematics* for all-India curricula as well as regional curricula of some universities like Gujarat Technological University, Mumbai University, Pune University, Jawaharlal Nehru Technological University, Anna University, Uttarakhand Technical University, and Dr A P J Abdul Kalam Technical University. Dr Bhatt has twenty six years of teaching experience at various levels in engineering colleges and her fields of interest include Integral Calculus, Complex Analysis, and Operation Research. She is a member of ISTE.

Mathematics-1

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Ravish R Singh
Director
Thakur Ramnarayan College of Arts & Commerce
Mumbai, Maharashtra

Mukul Bhatt
Assistant Professor
Thakur Ramnarayan College of Arts & Commerce
Mumbai, Maharashtra

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Dedicated
to

Aman and Aditri

Ravish R Singh

Soumya and Siddharth

Mukul Bhatt

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UNIT-2

UNIT-3

UNIT-4

UNIT-5

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Preface

Mathematics is a key area of study in any engineering course. A sound knowledge of this subject will help engineering students develop analytical skills, and thus enable them to solve numerical problems encountered in real life, as well as apply mathematical principles to physical problems, particularly in the field of engineering.

Users

This book is designed for the first year GTU engineering students pursuing the course Mathematics-1, SUBJECT CODE: 3110014 in their first year Ist Semester. It covers the complete GTU syllabus for the course on Mathematics-1, which is common to all the engineering branches.

Objective

The crisp and complete explanation of topics will help students easily understand the basic concepts. The tutorial approach (i.e., teach by example) followed in the text will enable students develop a logical perspective to solving problems.

Features

Each topic has been explained from the examination point of view, wherein the theory is presented in an easy-to-understand student-friendly style. Full coverage of concepts is supported by numerous solved examples with varied complexity levels, which is aligned to the latest GTU syllabus. Fundamental and sequential explanation of topics are well aided by examples and exercises. The solutions of examples are set following a 'tutorial' approach, which will make it easy for students from any background to easily grasp the concepts. Exercises with answers immediately follow the solved examples enforcing a practice-based approach. We hope that the students will gain logical understanding from solved problems and then reiterate it through solving similar exercise problems themselves. The unique blend of theory and application caters to the requirements of both the students and the faculty. Solutions of GTU examination questions are incorporated within the text appropriately.

Highlights

- Crisp content strictly as per the latest GTU syllabus of Mathematics-1 (Regulation 2018)
- Comprehensive coverage with lucid presentation style
- Each section concludes with an exercise to test understanding of topics
- Solutions of GTU examination questions included appropriately within the chapters
- Rich exam-oriented pedagogy:
  - ➢ Solved examples within chapters: 850+
  - ➢ Unsolved exercises: 500+
  - ➢ MCQs at the end of chapters: 300+

Chapter Organization

The content spans the following 10 chapters which wholly and sequentially cover each module of the syllabus.

❏ **Chapter 1** introduces Indeterminate Forms.

❏ **Chapter 2** discusses Improper Integrals.

❏ **Chapter 3** presents Gamma and Beta Functions.

❏ **Chapter 4** covers Applications of Definite Integrals.

❏ **Chapter 5** deals with Sequences and Series.

❏ **Chapter 6** presents Taylor's and Maclaurin's Series.

❏ **Chapter 7** discusses Fourier Series.

❏ **Chapter 8** presents Partial Derivatives.

❏ **Chapter 9** covers Multiple Integrals.

❏ **Chapter 10** deals with Matrices.

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Constructive suggestions for the improvement of the book will always be welcome.

Ravish R Singh
Mukul Bhatt

Publisher's Note

Remember to write to us. We look forward to receiving your feedback, comments and ideas to enhance the quality of this book. You can reach us at *info.india@mheducation.com*. Please mention the title and authors' name as the subject. In case you spot piracy of this book, please do let us know.

Roadmap to the Syllabus

Mathematics-1

Subject Code: 3110014

Unit-I

Indeterminate Forms and L'Hôspital's Rule.

Improper Integrals, Convergence and divergence of the integrals, Beta and Gamma functions and their properties.

Applications of definite integral, Volume using cross-sections, Length of plane curves, Areas of Surfaces of Revolution.

GO TO

CHAPTER 1: Introduction to Some Special Functions

CHAPTER 2: Improper Integrals

CHAPTER 3: Gamma and Beta Functions

CHAPTER 4: Applications of Definite Integrals

Unit-2

Convergence and divergence of sequences, The Sandwich Theorem for Sequences, The Continuous Function Theorem for Sequences, Bounded Monotonic Sequences, Convergence and divergence of an infinite series, geometric series, telescoping series, nth term test for divergent series, Combining series, Harmonic Series, Integral test, The *p*-series, The Comparison test, The Limit Comparison test, Ratio test, Raabe's Test, Root test, Alternating series test, Absolute and Conditional convergence, Power series, Radius of convergence of a power series, Taylor and Maclaurin series.

GO TO

CHAPTER 5: Sequences and Series

CHAPTER 6: Taylor's and Maclaurin's Series

Unit-3

Fourier Series of $2l$ periodic functions, Dirichlet's conditions for representation by a Fourier series, Orthogonality of the trigonometric system, Fourier Series of a function of period $2l$, Fourier Series of even and odd functions, Half range expansions.

GO TO **CHAPTER 7: Fourier Series**

Unit-4

Functions of several variables, Limits and continuity, Test for non existence of a limit, Partial differentiation, Mixed derivative theorem, differentiability, Chain rule, Implicit differentiation, Gradient, Directional derivative, tangent plane and normal line, total differentiation, Local extreme values, Method of Lagrange Multipliers.

GO TO **CHAPTER 8: Partial Derivatives**

Unit-5

Multiple integral, Double integral over Rectangles and general regions, double integrals as volumes, Change of order of integration, double integration in polar coordinates, Area by double integration, Triple integrals in rectangular, cylindrical and spherical coordinates, Jacobian, multiple integral by substitution.

GO TO **CHAPTER 9: Multiple Integrals**

Unit-5

Elementary row operations in Matrix, Row echelon and Reduced row echelon forms, Rank by echelon forms, Inverse by Gauss-Jordan method, Solution of system of linear equations by Gauss elimination and Gauss-Jordan methods. Eigen values and eigen vectors, Cayley-Hamilton theorem, Diagonalization of a matrix.

GO TO **CHAPTER 10: Matrices**

UNIT-1

Chapter 1. Indeterminate Forms
Chapter 2. Improper Integrals
Chapter 3. Gamma and Beta Functions
Chapter 4. Applications of Definite Integrals

CHAPTER 1

Indeterminate Forms

Chapter Outline

1.1 INTRODUCTION

We have studied certain rules to evaluate the limits. But some limits cannot be evaluated by using these rules. These limits are known as indeterminate forms. There are seven types of indeterminate forms:

(i) $\frac{0}{0}$ (ii) $\frac{\infty}{\infty}$ (iii) $0 \times \infty$ (iv) $\infty - \infty$

(v) 1^{∞} (vi) 0^{0} (vii) ∞^{0}

These limits can be evaluated by using L'Hospital's rule.

1.2 L'HOSPITAL'S RULE

Statement If $f(x)$ and $g(x)$ are two functions of x which can be expanded by Taylor's series in the neighbourhood of $x = a$ and if $\lim_{x \to a} f(x) = f(a) = 0$, $\lim_{x \to a} g(x) = g(a) = 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Proof Let $x = a + h$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{h\to 0}\frac{f(a+h)}{g(a+h)}$$

$$=\lim_{h\to 0}\frac{f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\dots\dots}{g(a)+hg'(a)+\frac{h^2}{2!}g''(a)+\dots\dots} \quad \text{[By Taylor's theorem]}$$

$$=\lim_{h\to 0}\frac{hf'(a)+\frac{h^2}{2!}f''(a)+\dots\dots}{hg'(a)+\frac{h^2}{2!}g''(a)+\dots\dots} \quad [\because f(a)=0,\ g(a)=0]$$

$$=\lim_{h\to 0}\frac{f'(a)+\frac{h}{2!}f''(a)+\dots\dots}{g'(a)+\frac{h}{2!}g''(a)+\dots\dots}$$

$$=\frac{f'(a)}{g'(a)}$$

$$=\lim_{x\to a}\frac{f'(x)}{g'(x)},\ \text{provided } g'(a)\neq 0.$$

Note

The following standard limits can be used to solve the problems:

(i) $\lim_{x\to 0}\frac{\sin x}{x}=1$ (ii) $\lim_{x\to 0}\frac{\tan x}{x}=1$ (iii) $\lim_{x\to 0}\frac{a^x-1}{x}=\log a$

(iv) $\lim_{x\to 0}\frac{e^x-1}{x}=1$ (v) $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$ (vi) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$

(vii) $\lim_{x\to 0}\frac{\sin^{-1}x}{x}=1$ (vii) $\lim_{x\to 0}\frac{\sinh x}{x}=1$ (ix) $\lim_{x\to 0}\frac{\tan^{-1}x}{x}=1$

1.3 TYPE 1 : $\frac{0}{0}$ FORM

Problems under this type are solved by using L'Hospital's rule considering the fact that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}\quad \text{if } \lim_{x\to a} f(x)=0 \text{ and } \lim_{x\to a} g(x)=0.$$

Example 1

Prove that $\lim_{x\to 0}\frac{(1+x)^n-1}{x}=n$.

Solution

Let

$$l=\lim_{x\to 0}\frac{(1+x)^n-1}{x} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{x\to 0}\frac{n(1+x)^{n-1}}{1} \qquad \text{[Applying L'Hospital's rule]}$$

$$=n$$

Example 2

Evaluate $\lim_{x\to 0}\frac{xe^x-\log(1+x)}{x^2}$.

Solution

Let

$$l=\lim_{x\to 0}\frac{xe^x-\log(1+x)}{x^2} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{x\to 0}\frac{e^x+xe^x-\frac{1}{1+x}}{2x} \qquad \left[\frac{0}{0}\text{ form}\right]$$

[Applying L'Hospital's rule]

$$=\lim_{x\to 0}\frac{e^x+e^x+xe^x+\frac{1}{(1+x)^2}}{2} \qquad \text{[Applying L'Hospital's rule]}$$

$$=\frac{3}{2}$$

Example 3

Evaluate $\lim_{x\to 1}\frac{x\log x-(x-1)}{(x-1)\log x}$.

Solution

Let $$l=\lim_{x\to 1}\frac{x\log x-(x-1)}{(x-1)\log x} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{x\to 1}\frac{x\cdot\frac{1}{x}+\log x-1}{\log x+(x-1)\cdot\frac{1}{x}} \qquad \text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to 1}\frac{\log x}{\log x+1-\frac{1}{x}} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}} \qquad \text{[Applying L'Hospital's rule]}$$

$$=\frac{1}{2}$$

Example 4

Evaluate $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2\log(1+x)}{x\sin x}$.

Solution

Let $$l=\lim_{x\to 0}\frac{e^x-e^{-x}-2\log(1+x)}{x\sin x} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{x\to 0}\frac{e^x+e^{-x}-2\cdot\frac{1}{1+x}}{\sin x+x\cos x} \qquad \left[\frac{0}{0}\text{ form}\right] \qquad \text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to 0}\frac{e^x-e^{-x}+\frac{2}{(1+x)^2}}{\cos x+\cos x-x\sin x} \qquad \text{[Applying L'Hospital's rule]}$$

$$=\frac{2}{2}$$

$$=1$$

Example 5

Evaluate $\lim_{x\to 0} \frac{e^x + e^{-x} - x^2 - 2}{\sin^2 x - x^2}$. **[Winter 2016; Summer 2014]**

Solution

Let

$$I = \lim_{x\to 0} \frac{e^x + e^{-x} - x^2 - 2}{\sin^2 x - x^2} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{2\sin x\cos x - 2x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{2(-\sin x)\sin x + 2\cos^2 x - 2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{-2\sin^2 x + 2\cos^2 x - 2}$$

$$= \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{2\cos 2x - 2}$$

$$= \lim_{x\to 0} \frac{e^x - e^{-x}}{(-2)2\sin 2x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{e^x - e^{-x}}{-4\sin 2x}$$

$$= \lim_{x\to 0} \frac{e^x + e^{-x}}{-8\cos 2x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{2}{-8}$$

$$= -\frac{1}{4}$$

Example 6

Evaluate $\lim_{x\to 0} \frac{2^x - 1}{(1+x)^{\frac{1}{2}} - 1}$.

Solution

Let

$$I = \lim_{x\to 0} \frac{2^x - 1}{(1+x)^{\frac{1}{2}} - 1} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x \to 0} \frac{2^x \log 2}{\frac{1}{2}(1+x)^{-\frac{1}{2}}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= 2 \log 2$$

Example 7

Evaluate $\lim_{x \to 0} \frac{\log(1+kx^2)}{1-\cos x}$.

Solution

Let
$$l = \lim_{x \to 0} \frac{\log(1+kx^2)}{1-\cos x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{\frac{1}{1+kx^2} \cdot 2kx}{\sin x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= 2k \lim_{x \to 0} \frac{1}{(1+kx^2) \cdot \frac{\sin x}{x}}$$

$$= 2k \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

Example 8

Evaluate $\lim_{x \to 0} \frac{\log(1+x^3)}{\sin^3 x}$.

Solution

Let
$$l = \lim_{x \to 0} \frac{\log(1+x^3)}{\sin^3 x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{\frac{1}{1+x^3} \cdot 3x^2}{3\sin^2 x \cos x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x \to 0} \left(\frac{x}{\sin x}\right)^2 \frac{1}{(1+x^3)\cos x}$$

$$= 1 \qquad \left[\lim_{x \to 0} \frac{x}{\sin x} = 1\right]$$

Example 9

Evaluate $\lim_{x\to 1} \frac{x - x^x}{1 + \log x - x}$.

Solution

Let $$l = \lim_{x\to 1} \frac{x - x^x}{1 + \log x - x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 1} \frac{x - e^{x\log x}}{1 + \log x - x}$$

$$= \lim_{x\to 1} \frac{1 - e^{x\log x}(1 + \log x)}{\frac{1}{x} - 1} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 1} \frac{-e^{x\log x}(1 + \log x)^2 - e^{x\log x}\left(\frac{1}{x}\right)}{-\frac{1}{x^2}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= 2$$

Example 10

Evaluate $\lim_{x\to y} \frac{x^y - y^x}{x^x - y^y}$.

Solution

Let $$l = \lim_{x\to y} \frac{x^y - y^x}{x^x - y^y} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to y} \frac{yx^{y-1} - y^x \log y}{x^x(1 + \log x) - 0} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{y^y - y^y \log y}{y^y(1 + \log y)}$$

$$= \frac{1 - \log y}{1 + \log y}$$

Example 11

Evaluate $\lim_{x\to \frac{\pi}{2}} \frac{\sin(x\cos x)}{\cos(x\sin x)}$.

Solution

Let $$l = \lim_{x\to\frac{\pi}{2}} \frac{\sin(x\cos x)}{\cos(x\sin x)} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos(x\cos x)(\cos x - x\sin x)}{-\sin(x\sin x)(\sin x + x\cos x)} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{\pi}{2}$$

Example 12

Evaluate $\lim_{x\to\frac{1}{2}} \dfrac{\cos^2 \pi x}{e^{2x} - 2xe}$. **[Winter 2015]**

Solution

Let $$l = \lim_{x\to\frac{1}{2}} \frac{\cos^2 \pi x}{e^{2x} - 2xe} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to\frac{1}{2}} \frac{2\cos \pi x(-\pi \sin \pi x)}{2e^{2x} - 2e} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to\frac{1}{2}} \frac{-\pi \sin 2\pi x}{2(e^{2x} - e)} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to\frac{1}{2}} \frac{-2\pi^2 \cos 2\pi x}{2\cdot 2e^{2x}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{\pi^2}{2e}$$

Example 13

Prove that $\lim_{\theta\to\alpha} \dfrac{1-\cos(\theta-\alpha)}{(\sin\theta - \sin\alpha)^2} = \dfrac{1}{2}\sec^2 \alpha.$

Solution

Let $$l = \lim_{\theta\to\alpha} \frac{1-\cos(\theta-\alpha)}{(\sin\theta - \sin\alpha)^2} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{\theta\to\alpha} \frac{\sin(\theta-\alpha)}{2(\sin\theta-\sin\alpha)\cos\theta} \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{\theta\to\alpha} \frac{\sin(\theta-\alpha)}{(\sin 2\theta - 2\sin\alpha\cos\theta)} \quad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{\theta\to\alpha} \frac{\cos(\theta-\alpha)}{2\cos 2\theta + 2\sin\alpha\sin\theta} \quad \text{[Applying L'Hospital's rule]}$$

$$= \frac{\cos 0}{2\cos 2\alpha + 2\sin\alpha\sin\alpha}$$

$$= \frac{1}{2(1-2\sin^2\alpha)+2\sin^2\alpha}$$

$$= \frac{1}{2-2\sin^2\alpha}$$

$$= \frac{1}{2\cos^2\alpha}$$

$$= \frac{1}{2}\sec^2\alpha$$

Example 14

Evaluate $\lim_{x\to 0} \frac{5\sin x - 7\sin 2x + 3\sin 3x}{\tan x - x}$.

Solution

Let $l = \lim_{x\to 0} \frac{5\sin x - 7\sin 2x + 3\sin 3x}{\tan x - x} \quad \left[\frac{0}{0}\text{ form}\right]$

$$= \lim_{x\to 0} \frac{5\cos x - 14\cos 2x + 9\cos 3x}{\sec^2 x - 1} \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{-5\sin x + 28\sin 2x - 27\sin 3x}{2\sec^2 x\tan x} \quad \left[\frac{0}{0}\text{ form}\right] \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{-5\frac{\sin x}{x} + 56\frac{\sin 2x}{2x} - 81\frac{\sin 3x}{3x}}{2\sec^2 x \cdot \frac{\tan x}{x}}$$

$$= \frac{-5+56-81}{2} \quad \left[\because \lim_{x\to 0}\frac{\sin nx}{nx} = 1 \text{ and } \lim_{x\to 0}\frac{\tan x}{x} = 1\right]$$

$$= -15$$

Example 15

Evaluate $\lim\limits_{x\to 0}\dfrac{2x^2-2e^{x^2}+2\cos(x^{\frac{3}{2}})+\sin^3 x}{x^2}$.

Solution

Let $l=\lim\limits_{x\to 0}\dfrac{2x^2-2e^{x^2}+2\cos x^{\frac{3}{2}}+\sin^3 x}{x^2}$ $\left[\dfrac{0}{0}\text{ form}\right]$

$$=\lim_{x\to 0}\frac{4x-2e^{x^2}(2x)-2\sin x^{\frac{3}{2}}\left(\frac{3}{2}x^{\frac{1}{2}}\right)+3\sin^2 x\cos x}{2x}\quad\left[\frac{0}{0}\text{ form}\right]$$

[Applying L'Hospital's rule]

$$=\lim_{x\to 0}\frac{4-4(e^{x^2}+xe^{x^2}\cdot 2x)-3\left(\sqrt{x}\cos x^{\frac{3}{2}}\cdot\frac{3}{2}x^{\frac{1}{2}}+\frac{1}{2\sqrt{x}}\sin x^{\frac{3}{2}}\right)+6\sin x\cos^2 x-3\sin^3 x}{2}$$

[Applying L'Hospital's rule]

$$=\frac{4-4-\lim\limits_{x\to 0}\dfrac{\sin x^{\frac{3}{2}}}{2\sqrt{x}}\cdot\dfrac{x}{x}}{2}$$

$$=-\frac{1}{2}\cdot\frac{1}{2}\lim_{x\to 0}\frac{\sin x^{\frac{3}{2}}}{x^{\frac{3}{2}}}\cdot x\qquad\left[\because\lim_{x\to 0}\frac{\sin x^{\frac{3}{2}}}{x^{\frac{3}{2}}}=1\right]$$

$$=0$$

Example 16

Evaluate $\lim\limits_{x\to 0}\dfrac{A}{x^2}\left[\dfrac{\sin kx}{\sin lx}-\dfrac{k}{l}\right]$.

Solution

Let $l=\lim\limits_{x\to 0}\dfrac{A}{x^2}\left[\dfrac{\sin kx}{\sin lx}-\dfrac{k}{l}\right]$

$$=A\lim_{x\to 0}\frac{l\sin kx-k\sin lx}{lx^2\sin lx}\qquad\left[\frac{0}{0}\text{ form}\right]$$

$$=\frac{A}{l}\lim_{x\to 0}\frac{l\sin kx-k\sin lx}{x^2\cdot\dfrac{\sin lx}{lx}\cdot lx}$$

$$= \frac{A}{l^3} \lim_{x \to 0} \frac{l \sin kx - k \sin lx}{x^3} \quad \left[\frac{0}{0} \text{ form}\right] \quad \left[\because \lim_{x \to 0} \frac{\sin lx}{lx} = 1\right]$$

$$= \frac{A}{l^3} \lim_{x \to 0} \frac{lk \cos kx - kl \cos lx}{3x^2} \quad \left[\frac{0}{0} \text{ form}\right] \quad \text{[Appling L'Hospital's rule]}$$

$$= \frac{A}{l^3} \lim_{x \to 0} \frac{-lk^2 \sin kx + kl^2 \sin lx}{6x} \quad \text{[Applying L'Hospital's rule]}$$

$$= \frac{A}{6l^3} \lim_{x \to 0} \left[-lk^2 \cdot \frac{\sin kx}{kx} \cdot k + kl^2 \cdot \frac{\sin lx}{lx} \cdot l\right]$$

$$= \frac{A}{6l^3}\left(-lk^3 + kl^3\right) \quad \left[\because \lim_{x \to 0} \frac{\sin kx}{kx} = 1 = \lim_{x \to 0} \frac{\sin lx}{lx}\right]$$

$$= \frac{Ak}{6l^2}\left(l^2 - k^2\right)$$

Example 17

Evaluate $\lim_{x \to 0} \frac{x^{\frac{1}{2}} \tan x}{(e^x - 1)^{\frac{3}{2}}}$.

Solution

Let $$l = \lim_{x \to 0} \frac{\sqrt{x} \tan x}{(e^x - 1)^{\frac{3}{2}}} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{x\sqrt{x}}{(e^x - 1)^{\frac{3}{2}}} \cdot \frac{\tan x}{x}$$

$$= \lim_{x \to 0} \frac{x\sqrt{x}}{(e^x - 1)^{\frac{3}{2}}} \cdot \lim_{x \to 0} \left(\frac{\tan x}{x}\right)$$

$$= \lim_{x \to 0} \left(\frac{x}{e^x - 1}\right)^{\frac{3}{2}} \quad \left[\because \lim_{x \to 0} \frac{\tan x}{x} = 1\right]$$

Now, $$\lim_{x \to 0} \frac{x}{e^x - 1} = \lim_{x \to 0} \frac{1}{e^x} \quad \text{[Applying L'Hospital's rule]}$$

$$= 1$$

$\therefore$ $$\lim_{x \to 0} \left(\frac{x}{e^x - 1}\right)^{\frac{3}{2}} = (1)^{\frac{3}{2}} = 1$$

Hence, $l = 1$

Example 18

Evaluate $\lim_{x\to 0}\dfrac{\log_{\sec x}\cos\frac{x}{2}}{\log_{\sec\frac{x}{2}}\cos x}$.

Solution

Let
$$l=\lim_{x\to 0}\frac{\log_{\sec x}\cos\frac{x}{2}}{\log_{\sec\frac{x}{2}}\cos x}$$
$$=\lim_{x\to 0}\frac{\log\cos\frac{x}{2}}{\log\sec x}\cdot\frac{\log\sec\frac{x}{2}}{\log\cos x}$$
$$=\lim_{x\to 0}\frac{\log\cos\frac{x}{2}}{(-\log\cos x)}\cdot\frac{\left(-\log\cos\frac{x}{2}\right)}{\log\cos x}$$
$$=\lim_{x\to 0}\left(\frac{\log\cos\frac{x}{2}}{\log\cos x}\right)^2 \qquad \left[\frac{0}{0}\text{ form}\right]$$

Now,
$$\lim_{x\to 0}\frac{\log\cos\frac{x}{2}}{\log\cos x}=\lim_{x\to 0}\frac{\frac{1}{\cos\frac{x}{2}}\cdot\left(-\frac{1}{2}\sin\frac{x}{2}\right)}{\frac{1}{\cos x}(-\sin x)} \qquad \text{[Applying L'Hospital's rule]}$$
$$=\lim_{x\to 0}\frac{\tan\frac{x}{2}}{2\tan x}$$
$$=\lim_{x\to 0}\frac{1}{4}\left(\frac{\tan\frac{x}{2}}{\frac{x}{2}}\right)\cdot\left(\frac{x}{\tan x}\right)$$
$$=\frac{1}{4}$$

$$\therefore \quad \lim_{x\to 0}\left(\frac{\log\cos\frac{x}{2}}{\log\cos x}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16} \qquad \left[\because \lim_{x\to 0}\frac{\tan x}{x} = 1\right]$$

Hence, $l = \frac{1}{16}$

Example 19

Prove that $\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}} - e}{x} = -\frac{e}{2}$.

Solution

Let $l = \lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}} - e}{x}$ $\left[\frac{0}{0}\text{ form}\right]$

$$= \lim_{x\to 0}\frac{e^{\frac{1}{x}\log(1+x)} - e}{x}$$ [Applying L'Hospital's rule]

$$= \lim_{x\to 0}\frac{e^{\frac{1}{x}\log(1+x)}\left[-\frac{1}{x^2}\log(1+x) + \frac{1}{x(1+x)}\right]}{1}$$ $\left[\frac{0}{0}\text{ form}\right]$

[Applying L'Hospital's rule]

$$= \lim_{x\to 0}(1+x)^{\frac{1}{x}}\lim_{x\to 0}\frac{[-\log(1+x)\cdot(1+x) + x]}{x^2(1+x)}$$ $\left[\frac{0}{0}\text{ form}\right]$

$$= e\lim_{x\to 0}\left[\frac{-\log(1+x) - 1 + 1}{2x + 3x^2}\right]$$ $\left[\frac{0}{0}\text{ form}\right]$ [Applying L'Hospital's rule]

$$= e\lim_{x\to 0}\left[\frac{\left(-\frac{1}{1+x}\right)}{2+6x}\right]$$

$$= -\frac{e}{2}$$

Example 20

Prove that $\lim_{x\to 0} \frac{\left(\sqrt{1-x}-1\right)^{2n}}{(1-\cos x)^n} = 2^{-n}$.

Solution

Let $l = \lim_{x\to 0} \frac{\left(\sqrt{1-x}-1\right)^{2n}}{(1-\cos x)^n} \cdot \frac{\left(\sqrt{1-x}+1\right)^{2n}}{\left(\sqrt{1-x}+1\right)^{2n}}$

$$= \lim_{x\to 0} \frac{(1-x-1)^{2n}}{\left(2\sin^2\frac{x}{2}\right)^n \left(\sqrt{1-x}+1\right)^{2n}}$$

$$= \lim_{x\to 0} \frac{(-x)^{2n}}{2^n\left(\sin\frac{x}{2}\right)^{2n} \left(\sqrt{1-x}+1\right)^{2n}} \cdot \frac{2^n}{2^n}$$

$$= \lim_{x\to 0} \left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right)^{2n} \frac{2^n}{\left(\sqrt{1-x}+1\right)^{2n}} \qquad \left[\because (-x)^{2n} = \left\{(-x)^2\right\}^n = x^{2n}\right]$$

$$= \frac{1}{2^n}$$

Example 21

Evaluate $\lim_{x\to 0} \frac{x^2+2\cos x-2}{x\sin x}$.

Solution

Let $l = \lim_{x\to 0} \frac{x^2+2\cos x-2}{x\sin x}$ $\left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x\to 0} \frac{2x-2\sin x}{\sin x + x\cos x} \qquad \left[\frac{0}{0} \text{ form}\right] \text{ [Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{2-2\cos x}{\cos x+\cos x - x\sin x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= 0$$

Example 22

Evaluate $\lim\limits_{x\to 0}\frac{e^x\sin x - x - x^2}{x^2 + x\log(1-x)}$.

Solution

Let $I = \lim\limits_{x\to 0}\frac{e^x\sin x - x - x^2}{x^2 + x\log(1-x)}$ $\left[\frac{0}{0}\text{ form}\right]$

$$= \lim_{x\to 0}\frac{e^x\sin x + e^x\cos x - 1 - 2x}{2x + \log(1-x) - \frac{x}{1-x}} \quad \left[\frac{0}{0}\text{ form}\right] \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{e^x(\sin x + \cos x) + e^x(\cos x - \sin x) - 2}{2 - \frac{1}{1-x} - \frac{1}{1-x} - \frac{x}{(1-x)^2}} \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{2(e^x\cos x - 1)}{2\left(1 - \frac{1}{1-x}\right) - \frac{x}{(1-x)^2}} \quad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{2(e^x\cos x - e^x\sin x)}{2\left[-\frac{1}{(1-x)^2}\right] - \frac{1}{(1-x)^2} - \frac{2x}{(1-x)^3}} \quad \text{[Applying L'Hospital's rule]}$$

$$= -\frac{2}{3}$$

EXERCISE 1.1

1. Prove that $\lim\limits_{x\to a}\frac{x^2\log a - a^2\log x}{x^2 - a^2} = \log a - \frac{1}{2}$.
2. Prove that $\lim\limits_{x\to 0}\frac{\log(1-x^2)}{\log\cos x} = 2$.
3. Prove that $\lim\limits_{x\to \frac{\pi}{4}}\frac{1-\tan x}{1-\sqrt{2}\sin x} = 2$.
4. Prove that $\lim\limits_{x\to 0}\frac{e^x - \sqrt{1+2x}}{\log(1+x^2)} = 1$.

5. Evaluate $\lim\limits_{x \to a} \dfrac{\sqrt{a+x}\tan^{-1}\sqrt{a^2-x^2}}{\sqrt{a-x}}$.

$$\left[\text{Hint: } \lim_{x \to a}(x+a)\frac{\tan^{-1}\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}} \text{ as } x \to a,\ a-x \to 0\right]$$

[Ans.: 2a]

6. Evaluate $\lim\limits_{x \to 0} \dfrac{e^x + \log_e \dfrac{1-x}{e}}{\tan x - x}$.

$\left[\text{Ans.: } -\dfrac{1}{2}\right]$

7. Evaluate $\lim\limits_{x \to 1} \dfrac{1-x+\log x}{1-\sqrt{2x-x^2}}$.

[Ans.: –1]

8. Prove that $\lim\limits_{x \to 0} \dfrac{xe^x - \log(1+x)}{x^2} = \dfrac{3}{2}$.

9. Prove that $\lim\limits_{x \to 0} \dfrac{e^x - \sqrt{1+2x}}{\log(1+x^2)} = 1$.

10. Prove that $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{1-\tan x}{1-\sqrt{2}\sin x} = 2$.

11. Prove that $\lim\limits_{x \to 3} \dfrac{\sqrt{3x}-\sqrt{12-x}}{2x-3\sqrt{19-5x}} = \dfrac{8}{69}$.

12. Prove that $\lim\limits_{x \to 1} \dfrac{a^{\log x} - x}{\log x} = \log\dfrac{a}{e}$.

13. Prove that $\lim\limits_{x \to a} \dfrac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}} = \dfrac{1}{\sqrt{2a}}$.

14. Prove that $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x^3} = \dfrac{1}{3}$.

1.4 TYPE 2: $\frac{\infty}{\infty}$ FORM

Problems under this type are also solved by using L'Hospital's rule considering the fact that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \text{ if } \lim_{x \to a} f(x) = \infty \text{ and } \lim_{x \to a} g(x) = \infty.$$

Example 1

Prove that $\lim\limits_{x\to\infty} \dfrac{\log x}{x^n} = 0, (n > 0)$.

Solution

Let $$l = \lim_{x\to\infty} \frac{\log x}{x^n} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{nx^{n-1}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{1}{n} \lim_{x\to\infty} \frac{1}{x^n}$$

$$= 0$$

Example 2

Evaluate $\lim\limits_{x\to 0} \dfrac{\log \sin x}{\cot x}$.

Solution

Let $$l = \lim_{x\to 0} \frac{\log \sin x}{\cot x} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{\sin x} \cdot \cos x}{-\text{cosec}^2 x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} -(\cos x \sin x)$$

$$= 0$$

Example 3

Prove that $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\log\left(x-\frac{\pi}{2}\right)}{\tan x} = 0$.

Solution

Let $$l = \lim_{x\to\frac{\pi}{2}} \frac{\log\left(x-\frac{\pi}{2}\right)}{\tan x} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\left(\frac{1}{x-\frac{\pi}{2}}\right)}{\sec^2 x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cos^2 x}{x - \frac{\pi}{2}} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{2\cos x(-\sin x)}{1} \qquad \text{[Applying L'Hospital's rule]}$$

$$= 0$$

Example 4

Prove that $\lim_{x \to a} \frac{\log(x-a)}{\log(a^x - a^a)} = 1.$

Solution

Let $l = \lim_{x \to a} \frac{\log(x-a)}{\log(a^x - a^a)} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$

$$= \lim_{x \to a} \frac{\frac{1}{(x-a)}}{\frac{1}{a^x - a^a} \cdot a^x \log a} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x \to a} \frac{1}{a^x \log a} \cdot \lim_{x \to a}\left(\frac{a^x - a^a}{x - a}\right) \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \frac{1}{a^a \log a} \cdot \lim_{x \to a} \frac{a^x \log a}{1} \qquad \text{[Applying L'Hospital's rule to second term]}$$

$$= \frac{1}{a^a \log a} \cdot a^a \log a$$

$$= 1$$

Example 5

Prove that $\lim_{x \to 0} \log_x \tan x = 1.$

Solution

Let $l = \lim_{x \to 0} \log_x \tan x$

$$= \lim_{x \to 0} \frac{\log \tan x}{\log x} \quad \left[\frac{\infty}{\infty} \text{ form}\right] \qquad \text{[Change of base property]}$$

$$= \lim_{x\to 0} \frac{\frac{1}{\tan x}\cdot \sec^2 x}{\frac{1}{x}}$$ [Applying L'Hospital's rule]

$$= \lim_{x\to 0} \frac{x}{\tan x}\cdot \lim_{x\to 0} \sec^2 x$$ $\left[\because \lim_{x\to 0} \frac{\tan x}{x} = 1\right]$

$= 1$

Example 6

Evaluate $\lim_{x\to 0} \log_{\tan x} \tan 2x$.

Solution

Let $l = \lim_{x\to 0} \log_{\tan x} \tan 2x$

$$= \lim_{x\to 0} \frac{\log \tan 2x}{\log \tan x}$$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$$= \lim_{x\to 0} \frac{\frac{1}{\tan 2x}\cdot 2\sec^2 2x}{\frac{1}{\tan x}\cdot \sec^2 x}$$ [Applying L'Hospital's rule]

$$= \lim_{x\to 0} \frac{\frac{\tan x}{x}\cdot \sec^2 2x}{\frac{\tan 2x}{2x}\cdot \sec^2 x}$$

$= 1$ $\left[\because \lim_{x\to 0} \frac{\tan x}{x} = 1\right]$

Example 7

Prove that $\lim_{x\to\infty} \frac{\sinh^{-1} x}{\cosh^{-1} x} = 1$.

Solution

Let $l = \lim_{x\to\infty} \frac{\sinh^{-1} x}{\cosh^{-1} x}$

$$= \lim_{x\to\infty} \frac{\log\left(x + \sqrt{x^2+1}\right)}{\log\left(x + \sqrt{x^2-1}\right)}$$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$$= \lim_{x\to\infty} \frac{\dfrac{1}{\left(x+\sqrt{x^2+1}\right)}\left(1+\dfrac{1}{2\sqrt{x^2+1}}\cdot 2x\right)}{\dfrac{1}{\left(x+\sqrt{x^2-1}\right)}\left(1+\dfrac{1}{2\sqrt{x^2-1}}\cdot 2x\right)}$$ [Applying L'Hospital's rule]

$$= \lim_{x\to\infty} \frac{\dfrac{\sqrt{x^2+1}+x}{\left(x+\sqrt{x^2+1}\right)\sqrt{x^2+1}}}{\dfrac{\sqrt{x^2-1}+x}{\left(x+\sqrt{x^2-1}\right)\sqrt{x^2-1}}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2-1}}{\sqrt{x^2+1}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{1-\dfrac{1}{x^2}}}{\sqrt{1+\dfrac{1}{x^2}}}$$

$$= 1$$

Example 8

Prove that $\lim\limits_{x\to\infty} \dfrac{e^{\frac{1}{x}}+e^{\frac{2}{x}}+e^{\frac{3}{x}}+\ldots+e^{\frac{x}{x}}}{x} = e-1.$

Solution

Let $$l = \lim_{x\to\infty} \frac{e^{\frac{1}{x}}+e^{\frac{2}{x}}+e^{\frac{3}{x}}+\ldots+e^{\frac{x}{x}}}{x}$$

$$= \lim_{x\to\infty} \frac{e^{\frac{1}{x}}\left[1-\left(e^{\frac{1}{x}}\right)^x\right]}{1-e^{\frac{1}{x}}}\cdot\frac{1}{x}$$ [Sum of GP]

$$= \lim_{x\to\infty} \frac{e^{\frac{1}{x}}(e-1)}{e^{\frac{1}{x}}-1}\cdot\frac{1}{x}$$

Putting $\dfrac{1}{x} = y$, when $x\to\infty$, $y\to 0$

$$l = \lim_{y\to 0} \frac{(e-1)e^{y}\,y}{e^{y}-1}$$ $\left[\dfrac{0}{0}\text{ form}\right]$

$$= \lim_{y \to 0} \frac{(e-1)(ye^y + e^y)}{e^y}$$ [Applying L'Hospital's rule]

$$= e - 1$$

Example 9

Prove that $\lim_{x \to \infty} \frac{x^n}{e^{kx}} = 0$.

Solution

Let $l = \lim_{x \to \infty} \frac{x^n}{e^{kx}}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$= \lim_{x \to \infty} \frac{nx^{n-1}}{ke^{kx}}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$ [Applying L'Hospital's rule]

$= \lim_{x \to \infty} \frac{n(n-1)x^{n-2}}{k^2 e^{kx}}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$ [Applying L'Hospital's rule]

Applying L'Hospital's rule $(n-2)$ times in the above expression,

$$l = \lim_{x \to \infty} \frac{n(n-1)(n-2)\ldots 2 \cdot 1}{k^n e^{kx}}$$

$$= \lim_{x \to \infty} \frac{n!}{k^n e^{kx}}$$

$$= 0 \qquad \left[\because \lim_{x \to \infty} e^{kx} = \infty\right]$$

Example 10

Prove that $\lim_{x \to \infty} \frac{1^2 + 2^2 + 3^2 + \ldots + x^2}{x^3} = \frac{1}{3}$.

Solution

Let $l = \lim_{x \to \infty} \frac{1^2 + 2^2 + 3^2 + \ldots + x^2}{x^3}$

$= \lim_{x \to \infty} \frac{x(x+1)(2x+1)}{6x^3}$ $\left[\because \sum n^2 = \frac{n(n+1)(2n+1)}{6}\right]$

$$= \lim_{x \to \infty} \frac{2x^3 + 3x^2 + x}{6x^3}$$

$$= \lim_{x \to \infty} \frac{2 + \frac{3}{x} + \frac{1}{x^2}}{6}$$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

Example 11

Prove that $\lim_{x \to \infty} \frac{e^x}{\left[\left(1+\frac{1}{x}\right)^x\right]^x} = e^{\frac{1}{2}}$.

Solution

Let $$l = \lim_{x \to \infty} \frac{e^x}{\left[\left(1+\frac{1}{x}\right)^x\right]^x} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x \to \infty} \frac{e^x}{\left(1+\frac{1}{x}\right)^{x^2}}$$

Taking logarithm on both sides,

$$\log l = \lim_{x \to \infty} \left[\log e^x - \log\left(1+\frac{1}{x}\right)^{x^2}\right]$$

$$= \lim_{x \to \infty} \left[x - x^2 \log\left(1+\frac{1}{x}\right)\right]$$

$$= \lim_{x \to \infty} x^2 \left[\frac{1}{x} - \log\left(1+\frac{1}{x}\right)\right]$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x} - \log\left(1+\frac{1}{x}\right)}{\frac{1}{x^2}} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to \infty} \frac{-\frac{1}{x^2} - \frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{2}{x^3}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x \to \infty} \frac{1 - \dfrac{1}{1 + \dfrac{1}{x}}}{\dfrac{2}{x}}$$

$$= \frac{1}{2} \lim_{x \to \infty} \frac{1}{1 + \dfrac{1}{x}}$$

$$= \frac{1}{2}$$

Hence, $l = e^{\frac{1}{2}}$

EXERCISE 1.2

1. Prove that $\lim_{x \to 0} \dfrac{\log x}{\cot x} = 0.$
2. Prove that $\lim_{x \to 1} \dfrac{\log(1-x)}{\cot \pi x} = 0.$
3. Prove that $\lim_{x \to 0} \dfrac{\log_{\sin x} \cos x}{\log_{\sin\frac{x}{2}} \cos\dfrac{x}{2}} = 4.$
4. Prove that $\lim_{x \to 0} \log_{\sin x} \sin 2x = 1.$
5. Prove that $\lim_{x \to \infty} \dfrac{\log(1+e^{3x})}{x} = 3.$
6. Prove that $\lim_{x \to 0} \dfrac{\log x^2}{\cot x^2} = 0.$ [Hint: Put $x^2 = y$]
7. Prove that $\lim_{x \to \infty} \dfrac{x^m}{e^x} = 0 \ (m > 0)$.
8. Prove that $\lim_{n \to \infty} \dfrac{\left(\frac{1}{e}\right) + \left(\frac{1}{e}\right)^2 + \left(\frac{1}{e}\right)^3 + \cdots\cdots + \left(\frac{1}{e}\right)^n}{n} = 0.$
9. Prove that $\lim_{x \to 0} \log_x \sin 2x = \dfrac{1}{2}$.

1.5 TYPE 3 : 0 × ∞ FORM

To solve the problems of the type

$$\lim_{x \to a} [f(x) \cdot g(x)], \text{ when } \lim_{x \to a} f(x) = 0, \ \lim_{x \to a} g(x) = \infty \text{ (i.e. } 0 \times \infty \text{ form)}$$

We write $\lim_{x\to a}\left[f(x)\cdot g(x)\right] = \lim_{x\to a}\frac{f(x)}{\frac{1}{g(x)}}$ or $\lim_{x\to a}\frac{g(x)}{\frac{1}{f(x)}}$.

These new forms are of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ respectively, which can be solved using L'Hospital's rule.

Example 1

Prove that $\lim_{x\to 0} x\log x = 0$.

Solution

Let

$$l = \lim_{x\to 0} x\log x \qquad [0\times\infty \text{ form}]$$

$$= \lim_{x\to 0}\frac{\log x}{\frac{1}{x}} \qquad \left[\frac{\infty}{\infty}\text{ form}\right]$$

$$l = \lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}(-x)$$

$$= 0$$

Example 2

Prove that $\lim_{x\to 0}\sin x\log x = 0$.

Solution

Let

$$l = \lim_{x\to 0}\sin x\log x \qquad [0\times\infty \text{ form}]$$

$$= \lim_{x\to 0}\frac{\log x}{\operatorname{cosec} x} \qquad \left[\frac{\infty}{\infty}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{\frac{1}{x}}{-\operatorname{cosec} x\cot x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= -\lim_{x\to 0}\sin x\cdot\frac{\tan x}{x}$$

$$= -\lim_{x\to 0}\sin x\cdot\lim_{x\to 0}\frac{\tan x}{x} \qquad \left[\because \lim_{x\to 0}\frac{\tan x}{x} = 1\right]$$

$$= 0$$

Example 3

Prove that $\lim_{x\to+\infty} 2^x \cdot \sin\left(\frac{a}{2^x}\right) = a.$

Solution

Let $$l = \lim_{x\to\infty} 2^x \cdot \sin\left(\frac{a}{2^x}\right)$$

Putting $2^x = \frac{1}{t},\ t = \frac{1}{2^x},$

When $x \to \infty,\ 2^x \to \infty,\quad t \to 0$

$$l = \lim_{t\to 0} \frac{\sin at}{t}$$

$$= \lim_{t\to 0} \frac{a \sin at}{at}$$

$$= a \cdot 1 \qquad \left[\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right]$$

$$= a$$

Example 4

Evaluate $\lim_{x\to a} \log\left(2 - \frac{x}{a}\right) \cot(x-a).$

Solution

Let $$l = \lim_{x\to a} \log\left(2 - \frac{x}{a}\right) \cot(x-a) \qquad [0 \times \infty \text{ form}]$$

$$= \lim_{x\to a} \frac{\log\left(2 - \frac{x}{a}\right)}{\tan(x-a)} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to a} \frac{\frac{1}{\left(2 - \frac{x}{a}\right)}\left(-\frac{1}{a}\right)}{\sec^2(x-a)} \qquad [\text{Applying L'Hospital's rule}]$$

$$= -\frac{1}{a}$$

Example 5

Prove that $\lim_{x\to\infty}\left(a^{\frac{1}{x}}-1\right)x=\log a$. **[Winter 2013]**

Solution

Let $l=\lim_{x\to\infty}\left(a^{\frac{1}{x}}-1\right)\cdot x$ $[0\times\infty \text{ form}]$

$$=\lim_{x\to\infty}\frac{\left(a^{\frac{1}{x}}-1\right)}{\frac{1}{x}} \qquad \left[\frac{0}{0}\text{ form}\right]$$

Putting $\frac{1}{x}=t$, when $x\to\infty$, $t\to 0$

$$l=\lim_{t\to 0}\frac{a^t-1}{t} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{t\to 0}\frac{a^t\log a}{1}$$ [Applying L'Hospital's rule]

$$=\log a$$

Example 6

Prove that $\lim_{x\to 1}\tan^2\left(\frac{\pi x}{2}\right)(1+\sec\ \pi x)=-2$.

Solution

Let $l=\lim_{x\to 1}\tan^2\left(\frac{\pi x}{2}\right)(1+\sec\pi x)$ $[\infty\times 0 \text{ form}]$

$$=\lim_{x\to 1}\frac{1+\sec\pi x}{\cot^2\left(\frac{\pi x}{2}\right)} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{x\to 1}\frac{\pi\sec\pi x\tan\pi x}{2\cot\left(\frac{\pi x}{2}\right)\left(-\operatorname{cosec}^2\frac{\pi x}{2}\right)\frac{\pi}{2}}$$ [Applying L'Hospital's rule]

$$=-\left(\lim_{x\to 1}\frac{\sec\pi x}{\operatorname{cosec}^2\frac{\pi x}{2}}\right)\left(\lim_{x\to 1}\frac{\tan\pi x}{\cot\frac{\pi x}{2}}\right)$$

$$=-\left(\frac{\sec\pi}{\operatorname{cosec}^2\frac{\pi}{2}}\right)\lim_{x\to 1}\frac{\tan\pi x}{\cot\frac{\pi x}{2}} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= -(-1)\lim_{x\to 1}\frac{\pi\sec^2\pi x}{\left(-\operatorname{cosec}^2\frac{\pi x}{2}\right)\frac{\pi}{2}}$$ [Applying L'Hospital's rule]

$$= -2\,\frac{\sec^2\pi}{\operatorname{cosec}^2\frac{\pi}{2}}$$

$$= -2$$

Example 7

Evaluate $\lim\limits_{x\to a}\sqrt{\dfrac{a+x}{a-x}}\cdot\tan^{-1}\sqrt{a^2-x^2}$.

Solution

Let $$l = \lim_{x\to a}\sqrt{\frac{a+x}{a-x}}\cdot\tan^{-1}\sqrt{a^2-x^2}$$ [$\infty\times 0$ form]

$$= \lim_{x\to a}\frac{\sqrt{a+x}}{\sqrt{a-x}}\cdot\frac{\sqrt{a+x}}{\sqrt{a+x}}\tan^{-1}\sqrt{a^2-x^2}$$

$$= \lim_{x\to a}(a+x)\frac{\tan^{-1}\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}$$

$$= \lim_{x\to a}(a+x)\cdot\lim_{x\to a}\frac{\tan^{-1}\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}$$

$$= 2a\lim_{x\to a}\frac{\tan^{-1}\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}$$

Let $\sqrt{a^2-x^2} = \alpha$

When $x\to a, \alpha\to 0$

$\therefore$ $$l = 2a\cdot\lim_{\alpha\to 0}\frac{\tan^{-1}\alpha}{\alpha}$$

$$= 2a \qquad \left[\because \lim_{\alpha\to 0}\frac{\tan^{-1}\alpha}{\alpha} = 1\right]$$

Example 8

Evaluate $\lim\limits_{x\to a}\sin^{-1}\sqrt{\dfrac{a-x}{a+x}}\operatorname{cosec}\sqrt{a^2-x^2} = \dfrac{1}{2a}$.

Solution

Let $$l = \lim_{x \to a} \sin^{-1}\sqrt{\frac{a-x}{a+x}} \operatorname{cosec}\sqrt{a^2-x^2} \qquad [0 \times \infty \text{ form}]$$

$$= \lim_{x \to a} \frac{\sin^{-1}\sqrt{\frac{a-x}{a+x}}}{\sin\sqrt{a^2-x^2}}$$

Here, applying L'Hospital's rule will make the expression complicated, so we rearrange the terms to apply the limits directly.

Let $\sqrt{\frac{a-x}{a+x}} = \alpha, \sqrt{a^2-x^2} = \beta$

When $x \to a, \alpha \to 0$ and $\beta \to 0$

$\therefore$ $$l = \lim_{\alpha \to 0} \sin^{-1}\alpha \cdot \lim_{\beta \to 0} \frac{1}{\sin\beta}$$

$$= \left[\lim_{\alpha \to 0}\left(\frac{\sin^{-1}\alpha}{\alpha}\right)\cdot\alpha\right]\left[\lim_{\beta \to 0}\left(\frac{\beta}{\sin\beta}\right)\cdot\frac{1}{\beta}\right]$$

$$= \lim_{\alpha \to 0}\alpha \cdot \lim_{\beta \to 0}\frac{1}{\beta} \qquad \left[\because \lim_{\alpha \to 0}\left(\frac{\sin^{-1}\alpha}{\alpha}\right) = 1 \text{ and } \lim_{\beta \to 0}\left(\frac{\sin\beta}{\beta}\right) = 1\right]$$

$$= \lim_{x \to a}\sqrt{\frac{a-x}{a+x}} \cdot \frac{1}{\sqrt{a^2-x^2}} \qquad [\text{Resubstituting } \alpha \text{ and } \beta]$$

$$= \lim_{x \to a}\sqrt{\frac{a-x}{a+x}} \cdot \frac{1}{\sqrt{a+x}\sqrt{a-x}}$$

$$= \lim_{x \to a}\frac{1}{a+x}$$

$$= \frac{1}{2a}$$

Example 9

Evaluate $\lim_{x \to 0} x^m (\log x)^n$, *where m and n are positive integers.*

Solution

Let $$l = \lim_{x \to 0} x^m (\log x)^n \qquad [0 \times \infty \text{ form}]$$

$$= \lim_{x \to 0}\frac{(\log x)^n}{\frac{1}{x^m}} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{n(\log x)^{n-1}\cdot\frac{1}{x}}{-m(x)^{-m-1}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{(-1)^1 n(\log x)^{n-1}}{m(x)^{-m}} \qquad \left[\frac{\infty}{\infty}\text{ form}\right]$$

$$= \lim_{x\to 0} \frac{(-1)^1 n(n-1)(\log x)^{n-2}\cdot\frac{1}{x}}{m(-m)^1(x)^{-m-1}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{(-1)^2 n(n-1)(\log x)^{n-2}}{m^2(x)^{-m}} \qquad \left[\frac{\infty}{\infty}\text{ form}\right]$$

Applying L'Hospital's rule $(n - 2)$ times in the above expression,

$$l = \lim_{x\to 0} \frac{(-1)^n n!(\log x)^0}{m^n (x)^{-m}}$$

$$= \lim_{x\to 0} \frac{(-1)^n n!}{m^n}\cdot x^m$$

$$= 0$$

EXERCISE 1.3

1. Prove that $\lim_{x\to 0} x\log x = 0.$

2. Prove that $\lim_{x\to\infty} x^2 e^{-x} = 0.$

3. Prove that $\lim_{x\to\infty} x^2\left(1 - e^{-\frac{2gy}{x^2}}\right) = 2gy.$

4. Prove that $\lim_{x\to 0} \tan x \log x = 0.$

5. Prove that $\lim_{x\to 1} (x^2 - 1)\tan\left(\frac{\pi x}{2}\right) = -\frac{4}{\pi}.$

6. Prove that $\lim_{x\to 1} (1 + \sec \pi x)\tan\frac{\pi x}{2} = 0.$

7. Prove that $\lim_{x\to 1} \log(1 - x)\cot\left(\frac{\pi x}{2}\right) = 0.$

8. Prove that $\lim_{x\to 0} \log\left(\frac{1+x}{1-x}\right)\cot x = 2.$

9. Prove that $\lim_{x\to 2} \sqrt{\frac{2+x}{2-x}}\tan^{-1}\sqrt{4-x^2} = 4.$

1.6 TYPE 4 : ∞ – ∞ FORM

To evaluate the limits of the type $\lim_{x\to a}[f(x)-g(x)]$, when $\lim_{x\to a} f(x)=\infty$ and, $\lim_{x\to a} g(x)=\infty$ [i.e., (∞ – ∞) form], we reduce the expression in the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by taking LCM or by rearranging the terms and then apply L'Hospital's rule.

Example 1

Prove that $\lim_{x\to\infty}(\cosh^{-1}x-\log x)=\log 2$.

Solution

Let $l=\lim_{x\to\infty}(\cosh^{-1}x-\log x)$ [∞ – ∞ form]

$$=\lim_{x\to\infty}\left[\log\left(x+\sqrt{x^2-1}\right)-\log x\right]$$

$$=\lim_{x\to\infty}\log\left(\frac{x+\sqrt{x^2-1}}{x}\right)$$

$$=\lim_{x\to\infty}\log\left(1+\sqrt{1-\frac{1}{x^2}}\right)$$

$$=\log 2$$

Example 2

Evaluate $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right]$.

Solution

Let $l=\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right]$ [∞ – ∞ form]

$$=\lim_{x\to 0}\left[\frac{x-\log(1+x)}{x^2}\right] \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{x\to 0}\left[\frac{1-\frac{1}{1+x}}{2x}\right] \qquad \left[\frac{0}{0}\text{ form}\right]$$ [Applying L'Hospital's rule]

$$= \lim_{x \to 0} \left[\frac{\left(\frac{1}{1+x} \right)^2}{2} \right] \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{1}{2}$$

Example 3

Evaluate $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\log x} \right)$. **[Winter 2016]**

Solution

Let

$$l = \lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\log x} \right) \qquad [\infty - \infty \text{ form}]$$

$$= \lim_{x \to 1} \left[\frac{x \log x - x + 1}{\log x (x-1)} \right] \qquad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \to 1} \left[\frac{1 + \log x - 1}{\frac{x-1}{x} + \log x} \right] \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x \to 1} \frac{\log x}{\frac{x-1}{x} + \log x} \qquad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \to 1} \frac{\frac{1}{x}}{\frac{1}{x^2} + \frac{1}{x}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{1}{2}$$

Example 4

Evaluate $\lim_{x \to 2} \left[\frac{1}{x-2} - \frac{1}{\log(x-1)} \right]$.

Solution

Let

$$l = \lim_{x \to 2} \left[\frac{1}{x-2} - \frac{1}{\log(x-1)} \right] \qquad [\infty - \infty \text{ form}]$$

$$= \lim_{x\to 2} \frac{\log(x-1)-(x-2)}{(x-2)\log(x-1)} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 2} \frac{\frac{1}{x-1}-1}{\frac{x-2}{x-1}+\log(x-1)} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 2} \frac{1-(x-1)}{(x-2)+(x-1)\log(x-1)} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 2} \frac{-1}{1+(x-1)\cdot\frac{1}{(x-1)}+\log(x-1)} \qquad \text{[Applying L'Hospital's rule]}$$

$$= -\frac{1}{2}$$

Example 5

Prove that $\lim_{x\to 0}\left(\frac{a}{x}-\cot\frac{x}{a}\right)=0.$

Solution

Let $l = \lim_{x\to 0}\left(\frac{a}{x}-\cot\frac{x}{a}\right)$ $[\infty-\infty \text{ form}]$

Putting $\frac{x}{a}=y$, when $x\to 0$, $y\to 0$

$$l = \lim_{y\to 0}\left(\frac{1}{y}-\cot y\right)$$

$$= \lim_{y\to 0}\left(\frac{1}{y}-\frac{1}{\tan y}\right) \qquad [\infty-\infty \text{ form}]$$

$$= \lim_{y\to 0}\left(\frac{\tan y - y}{y\tan y}\right) \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{y\to 0}\left(\frac{\tan y - y}{y^2}\right)\cdot\lim_{y\to 0}\frac{1}{\left(\frac{\tan y}{y}\right)}$$

$$= \lim_{y\to 0}\frac{\tan y - y}{y^2}\cdot 1 \quad \left[\frac{0}{0}\text{ form}\right] \quad \left[\because \lim_{y\to 0}\frac{\tan y}{y}=1\right]$$

$$= \lim_{y\to 0}\frac{\sec^2 y - 1}{2y} \quad \left[\frac{0}{0}\text{ form}\right] \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{y\to 0} \frac{2\sec y \cdot \sec y \tan y}{2}$$ [Applying L'Hospital's rule]

$$= 0$$

Example 6

Prove that $\lim_{x\to 0}\left[\frac{\pi}{4x} - \frac{\pi}{2x(e^{\pi x}+1)}\right] = \frac{\pi}{4}.$

Solution

Let

$$l = \frac{\pi}{2}\lim_{x\to 0}\left[\frac{1}{2x} - \frac{1}{x(e^{\pi x}+1)}\right] \qquad [\infty - \infty \text{ form}]$$

$$= \frac{\pi}{2}\lim_{x\to 0}\frac{e^{\pi x}+1-2}{2x(e^{\pi x}+1)} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \frac{\pi}{2}\lim_{x\to 0}\frac{\pi e^{\pi x}}{2\left[(e^{\pi x}+1)+x(\pi e^{\pi x})\right]} \qquad [\text{Applying L'Hospital's rule}]$$

$$= \frac{\pi^2}{4}\cdot\frac{e^0}{(e^0+1)}$$

$$= \frac{\pi^2}{8}$$

Example 7

Prove that $\lim_{x\to\infty}\left(x+\frac{1}{2}\right)\left[\log\left(x+\frac{1}{2}\right) - \log x\right] = \frac{1}{2}.$

Solution

Let

$$l = \lim_{x\to\infty}\left(x+\frac{1}{2}\right)\left[\log\left(x+\frac{1}{2}\right) - \log x\right] \qquad [\infty - \infty \text{ form}]$$

$$= \lim_{x\to\infty}\left(x+\frac{1}{2}\right)\log\left(\frac{x+\frac{1}{2}}{x}\right)$$

$$= \lim_{x\to\infty}\left[x\log\left(1+\frac{1}{2x}\right) + \frac{1}{2}\log\left(1+\frac{1}{2x}\right)\right]$$

$$= \lim_{x\to\infty}\frac{1}{2}\log\left(1+\frac{1}{2x}\right)^{2x} + \frac{1}{2}\lim_{x\to\infty}\log\left(1+\frac{1}{2x}\right)$$

$$= \frac{1}{2}\log e + \frac{1}{2}\log 1 \qquad \left[\because \lim_{x\to\infty}\left(1+\frac{1}{ax}\right)^{ax} = e\right]$$

$$= \frac{1}{2}$$

Example 8

Evaluate $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$. **[Winter 2014; Summer 2015]**

Solution

Let $l=\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$ $[\infty-\infty \text{ form}]$

$=\lim_{x\to 0}\left(\frac{\sin^2 x-x^2}{x^2\sin^2 x}\right)$ $\left[\frac{0}{0}\text{ form}\right]$

$=\lim_{x\to 0}\left(\frac{\sin^2 x-x^2}{x^4}\right)$ $\left[\because \lim_{x\to 0}\frac{\sin^2 x}{x^2}=1\right]$

$=\lim_{x\to 0}\left(\frac{2\sin x\cos x-2x}{4x^3}\right)$ [Applying L'Hospital's rule]

$=\lim_{x\to 0}\left(\frac{\sin 2x-2x}{4x^3}\right)$

$=\lim_{x\to 0}\left(\frac{2\cos 2x-2}{12x^2}\right)$ [Applying L'Hospital's rule]

$=\lim_{x\to 0}\left(\frac{-4\sin 2x}{24x}\right)$ [Applying L'Hospital's rule]

$=\lim_{x\to 0}\left(\frac{-8\cos 2x}{24}\right)$ [Applying L'Hospital's rule]

$=\frac{-8}{24}$

$=-\frac{1}{3}$

Example 9

Evaluate $\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)$.

Solution

Let $l=\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)$ $[\infty-\infty \text{ form}]$

$$= \lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\tan^2 x}\right)$$

$$= \lim_{x\to 0}\frac{\tan^2 x - x^2}{x^2\tan^2 x} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{\tan^2 x - x^2}{x^4\cdot\dfrac{\tan^2 x}{x^2}}$$

$$= \lim_{x\to 0}\left(\frac{\tan^2 x - x^2}{x^4}\right)\cdot \lim_{x\to 0}\left(\frac{x^2}{\tan^2 x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\tan^2 x - x^2}{x^4}\right)\cdot 1 \qquad \left[\because \lim_{x\to 0}\frac{\tan x}{x}=1\right]$$

$$= \lim_{x\to 0}\frac{\tan^2 x - x^2}{x^4} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{2\tan x\cdot\sec^2 x - 2x}{4x^3} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{2\tan x\left(1+\tan^2 x\right)-2x}{4x^3}$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{\tan x + \tan^3 x - x}{x^3}$$

$$= \frac{1}{2}\lim_{x\to 0}\left(\frac{\tan x - x}{x^3}\right)+\frac{1}{2}\lim_{x\to 0}\frac{\tan^3 x}{x^3}$$

$$= \frac{1}{2}\lim_{x\to 0}\left(\frac{\tan x - x}{x^3}\right)+\frac{1}{2} \qquad \left[\because \lim_{x\to 0}\frac{\tan x}{x}=1\right]$$

$$= \frac{1}{2}+\frac{1}{2}\lim_{x\to 0}\left(\frac{\sec^2 x - 1}{3x^2}\right) \qquad \left[\frac{0}{0}\text{ form}\right] \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{1}{2}+\frac{1}{2}\lim_{x\to 0}\frac{2\sec^2 x\cdot\tan x}{6x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{1}{2}+\frac{1}{6}\lim_{x\to 0}\sec^2 x\cdot\lim_{x\to 0}\frac{\tan x}{x}$$

$$= \frac{1}{2}+\frac{1}{6}$$

$$= \frac{2}{3}$$

Example 10

If $\lim\limits_{x\to 0}\left(\frac{a\cot x}{x}+\frac{b}{x^2}\right)=\frac{1}{3}$, find a and b.

Solution

$$\frac{1}{3}=\lim_{x\to 0}\left(\frac{a\cot x}{x}+\frac{b}{x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{a}{x\tan x}+\frac{b}{x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{ax+b\tan x}{x^2\tan x}\right) \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{x\to 0}\frac{(ax+b\tan x)}{(x^2\cdot x)\left(\frac{\tan x}{x}\right)}$$

$$=\lim_{x\to 0}\frac{ax+b\tan x}{x^3}\cdot\lim_{x\to 0}\left(\frac{x}{\tan x}\right)$$

$$=\lim_{x\to 0}\frac{ax+b\tan x}{x^3}\cdot 1 \qquad \left[\frac{0}{0}\text{ form}\right] \qquad \left[\because \lim_{x\to 0}\frac{x}{\tan x}=1\right]$$

$$=\lim_{x\to 0}\left(\frac{a+b\sec^2 x}{3x^2}\right) \qquad \text{[Applying L'Hospital's rule]}$$

$$=\frac{a+b\sec 0}{0}$$

$$=\frac{a+b}{0}$$

$$a+b=0,\ a=-b \qquad \text{... (1)}$$

Thus, $$\frac{1}{3}=\lim_{x\to 0}\frac{-b+b\sec^2 x}{3x^2} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{x\to 0}\frac{b\cdot 2\sec x\sec x\tan x}{6x} \qquad \text{[Applying L'Hospital's rule]}$$

$$=\left(\lim_{x\to 0}\frac{b}{3}\sec^2 x\right)\left(\lim_{x\to 0}\frac{\tan x}{x}\right)$$

$$=\frac{b}{3} \qquad \left[\because \lim_{x\to 0}\frac{\tan x}{x}=1\right]$$

$$b=1$$

From Eq. (1), $a=-1$

Hence, $a=-1,\ b=1$

Example 11

Evaluate $\lim\limits_{x\to a}\left[\dfrac{f'(x)}{f(x)-f(a)}-\dfrac{1}{x-a}\right]$.

Solution

Let $l=\lim\limits_{x\to a}\left[\dfrac{f'(x)}{f(x)-f(a)}-\dfrac{1}{x-a}\right]$ [∞ − ∞ form]

$$=\lim_{x\to a}\frac{(x-a)f'(x)-[f(x)-f(a)]}{[f(x)-f(a)](x-a)} \quad \left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{x\to a}\frac{f'(x)+(x-a)f''(x)-f'(x)}{[f(x)-f(a)]+(x-a)f'(x)} \quad \left[\frac{0}{0}\text{ form}\right] \quad \text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to a}\frac{f''(x)+(x-a)f'''(x)}{f'(x)+f'(x)+(x-a)f''(x)} \quad \text{[Applying L'Hospital's rule]}$$

$$=\frac{f''(a)}{2f'(a)}$$

EXERCISE 1.4

1. Prove that $\lim\limits_{x\to 0}\left(\cot x-\dfrac{1}{x}\right)=0$.

2. Prove that $\lim\limits_{x\to a}\left[\dfrac{1}{x-a}-\cot(x-a)\right]=0$.

3. Prove that $\lim\limits_{x\to \frac{\pi}{2}}(\sec x-\tan x)=0$.

4. Prove that $\lim\limits_{x\to \frac{\pi}{2}}\left(\tan x-\dfrac{2x\sec x}{\pi}\right)=\dfrac{2}{\pi}$.

5. Prove that $\lim\limits_{x\to 0}\left[\dfrac{1}{x-a}-\dfrac{1}{\log(x+1-a)}\right]=-\dfrac{1}{2}$.

6. Prove that $\lim\limits_{x\to 3}\left[\dfrac{1}{x-3}-\dfrac{1}{\log(x-2)}\right]=-\dfrac{1}{2}$.

7. Evaluate $\lim_{x \to \infty}\left[x - x^2 \log\left(1+\frac{1}{x}\right)\right]$

$\left[\begin{array}{l}\text{Hint: Put } x = \frac{1}{y} \\ \text{Ans.: } \frac{1}{2}\end{array}\right]$

8. Prove that $\lim_{x \to 0}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \frac{1}{2}$.

1.7 TYPE 5: 1^∞, ∞^0, 0^0 FORMS

To evaluate the limits of the type $\lim_{x \to a}[f(x)]^{g(x)}$, which takes any one of the form 1^∞, ∞^0, 0^0 for $f(x) > 0$, we proceed as follows:

Let $$l = \lim_{x \to a}[f(x)]^{g(x)} \qquad \text{where } f(x) > 0$$

$$\log l = \lim_{x \to a}[g(x) \cdot \log f(x)]$$

which takes the form $\infty \times 0$, i.e., type 3 form.

Example 1

Prove that $\lim_{x \to 0}(a^x + x)^{\frac{1}{x}} = ae.$ **[Summer 2014]**

Solution

Let
$$l = \lim_{x \to 0}(a^x + x)^{\frac{1}{x}} \qquad [1^\infty \text{ form}]$$

$$\log l = \lim_{x \to 0}\frac{1}{x} \cdot \log(a^x + x) \qquad [\infty \times 0 \text{ form}]$$

$$= \lim_{x \to 0}\frac{\log(a^x + x)}{x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0}\frac{\frac{1}{a^x + x}(a^x \log a + 1)}{1} \qquad [\text{Applying L'Hospital's rule}]$$

$$= \frac{a^0 \log a + 1}{a^0 + 0}$$

$$= \frac{\log_e a + \log_e e}{1}$$

$= \log ae$

Hence, $l = ae$

Example 2

Prove that $\lim_{x\to 0}(e^{3x} - 5x)^{\frac{1}{x}} = e^{-2}$.

Solution

Let $l = \lim_{x\to 0}(e^{3x} - 5x)^{\frac{1}{x}}$ [1^∞ form]

$$\log l = \lim_{x\to 0}\frac{1}{x}\log(e^{3x} - 5x)$$

$$= \lim_{x\to 0}\frac{\log(e^{3x} - 5x)}{x} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{\frac{1}{(e^{3x} - 5x)}\cdot(3e^{3x} - 5)}{1} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{3e^0 - 5}{e^0}$$

$$= -2$$

Hence, $l = e^{-2}$

Example 3

Evaluate $\lim_{x\to e}(\log x)^{\frac{1}{1-\log x}}$.

Solution

Let $l = \lim_{x\to e}(\log x)^{\frac{1}{1-\log x}}$ [1^∞ form]

$$\log l = \lim_{x\to e}\frac{1}{1-\log x}\log(\log x) \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to e}\frac{\frac{1}{\log x}\cdot\frac{1}{x}}{-\frac{1}{x}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= -1$$

Hence, $l = e^{-1} = \frac{1}{e}$

Example 4

Prove that $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{3x}} = (abc)^{\frac{1}{9}}$. **[Summer 2015]**

Solution

Let $$l = \lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{3x}} \qquad [1^\infty \text{ form}]$$

$$\log l = \lim_{x\to 0}\frac{1}{3x}\log\left(\frac{a^x+b^x+c^x}{3}\right)$$

$$= \lim_{x\to 0}\frac{\log\left(\frac{a^x+b^x+c^x}{3}\right)}{3x} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{\left(\frac{3}{a^x+b^x+c^x}\right)\cdot\frac{(a^x\log a+b^x\log b+c^x\log c)}{3}}{3}$$

[Applying L'Hospital's rule]

$$= \left(\frac{1}{a^0+b^0+c^0}\right)\frac{(a^0\log a+b^0\log b+c^0\log c)}{3}$$

$$= \frac{1}{9}\log abc$$

$$= \log(abc)^{\frac{1}{9}}$$

Hence, $$l = (abc)^{\frac{1}{9}}$$

Example 5

Evaluate $\lim_{x\to 0}\left(\frac{1^x+2^x+3^x}{3}\right)^{\frac{1}{x}}$ **[Summer 2017]**

Solution

Let $$l = \lim_{x\to 0}\left(\frac{1^x+2^x+3^x}{3}\right)^{\frac{1}{x}} \qquad [1^\infty \text{ form}]$$

$$\log l = \lim_{x\to 0}\frac{1}{x}\log\left(\frac{1^x+2^x+3^x}{3}\right)$$

$$= \lim_{x\to 0} \frac{\log\left(\frac{1^x + 2^x + 3^x}{3}\right)}{x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{3}{1^x + 2^x + 3^x} \cdot \frac{(1^x \log 1 + 2^x \log 2 + 3^x \log 3)}{3}$$

[Applying L'Hospital's rule]

$$= \frac{1}{3}(\log 2 + \log 3)$$

$$= \frac{1}{3}\log(6)$$

$$= \log(6)^{\frac{1}{3}}$$

Hence, $l = (6)^{\frac{1}{3}}$

Example 6

Prove that $\lim_{x\to\infty}\left(\frac{a^{\frac{1}{x}} + b^{\frac{1}{x}} + c^{\frac{1}{x}} + d^{\frac{1}{x}}}{4}\right)^x = (abcd)^{\frac{1}{4}}$.

Solution

Let $l = \lim_{x\to\infty}\left(\frac{a^{\frac{1}{x}} + b^{\frac{1}{x}} + c^{\frac{1}{x}} + d^{\frac{1}{x}}}{4}\right)^x$

Putting $\frac{1}{x} = y$, when $x \to \infty$, $y \to 0$

$$l = \lim_{y\to 0}\left(\frac{a^y + b^y + c^y + d^y}{4}\right)^{\frac{1}{y}} \qquad [1^\infty \text{ form}]$$

$$\log l = \lim_{y\to 0} \frac{1}{y} \log\left(\frac{a^y + b^y + c^y + d^y}{4}\right)$$

$$= \lim_{y\to 0} \frac{\log\left(\frac{a^y + b^y + c^y + d^y}{4}\right)}{y} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{y\to 0}\left(\frac{4}{a^y + b^y + c^y + d^y}\right)\left(\frac{a^y \log a + b^y \log b + c^y \log c + d^y \log d}{4}\right)$$

[Applying L'Hospital's rule]

$$= \frac{\log a + \log b + \log c + \log d}{4}$$

$$= \frac{1}{4}\log(abcd)$$

$$= \log(abcd)^{\frac{1}{4}}$$

Hence, $l = (abcd)^{\frac{1}{4}}$

Example 7

Prove that $\lim_{x\to\infty}\left(\frac{ax+1}{ax-1}\right)^x = e^{\frac{2}{a}}$.

Solution

Let $l = \lim_{x\to\infty}\left(\frac{ax+1}{ax-1}\right)^x$

$$= \lim_{x\to\infty}\left(\frac{1+\frac{1}{ax}}{1-\frac{1}{ax}}\right)^x \quad [1^\infty \text{ form}]$$

$$\log l = \lim_{x\to\infty} x \log\left(\frac{1+\frac{1}{ax}}{1-\frac{1}{ax}}\right)$$

$$= \lim_{x\to\infty}\frac{ax}{a}\left[\log\left(1+\frac{1}{ax}\right) - \log\left(1-\frac{1}{ax}\right)\right]$$

$$= \lim_{x\to\infty}\frac{1}{a}\left[\log\left(1+\frac{1}{ax}\right)^{ax} + \log\left(1-\frac{1}{ax}\right)^{-ax}\right]$$

$$= \frac{1}{a}(\log e + \log e) \qquad \left[\because \lim_{x\to\infty}\left(1+\frac{1}{ax}\right)^{ax} = e\right]$$

$$= \frac{1}{a}(1+1)$$

$$= \frac{2}{a}$$

Hence, $l = e^{\frac{2}{a}}$

Example 8

Evaluate $\lim_{x\to a}\left[\frac{1}{2}\left(\sqrt{\frac{a}{x}}+\sqrt{\frac{x}{a}}\right)\right]^{\frac{1}{x-a}}$.

Solution

Let $l=\lim_{x\to a}\left[\frac{1}{2}\left(\sqrt{\frac{a}{x}}+\sqrt{\frac{x}{a}}\right)\right]^{\frac{1}{x-a}}$ [1^∞ form]

$$\log l=\lim_{x\to a}\frac{1}{x-a}\log\left[\frac{1}{2}\left(\sqrt{\frac{a}{x}}+\sqrt{\frac{x}{a}}\right)\right]$$

$$=\lim_{x\to a}\frac{\log\frac{a+x}{2\sqrt{ax}}}{x-a}$$

$$=\lim_{x\to a}\frac{\log(a+x)-\log 2\sqrt{ax}}{x-a}$$

$$=\lim_{x\to a}\frac{\log(a+x)-\log 2\sqrt{a}-\frac{1}{2}\log x}{x-a}$$ $\left[\frac{0}{0}\text{ form}\right]$

$$=\lim_{x\to a}\frac{\frac{1}{a+x}-\frac{1}{2x}}{1}$$ [Applying L'Hospital's rule]

$$=\frac{1}{2a}-\frac{1}{2a}$$
$$=0$$

Hence, $l=e^0$
$$=1$$

Example 9

Prove that $\lim_{x\to 0}(\cos 2x)^{\left(\frac{3}{x^2}\right)}=e^{-6}$.

Solution

Let $l=\lim_{x\to 0}(\cos 2x)^{\left(\frac{3}{x^2}\right)}$ [1^∞ form]

$$\log l = \lim_{x\to 0} \frac{3}{x^2}\cdot \log(\cos 2x)$$

$$= \lim_{x\to 0} \frac{3\log(\cos 2x)}{x^2} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0} \frac{3}{\cos 2x}\cdot\frac{(-2\sin 2x)}{2x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} -6\left(\frac{\tan 2x}{2x}\right)$$

$$= -6$$

Hence, $l = e^{-6}$

Example 10

Prove that $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{3}}$.

Solution

Let $l = \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$ $\qquad [1^\infty \text{ form}] \qquad \left[\because \lim_{x\to 0}\frac{\tan x}{x} = 1\right]$

$$\log l = \lim_{x\to 0}\frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$$

$$= \lim_{x\to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x^2} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{x}{\tan x}\left(\frac{x\sec^2 x - \tan x}{x^2}\right)\frac{1}{2x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{x\sec^2 x - \tan x}{2x^3} \qquad \left[\frac{0}{0}\text{ form}\right]\left[\because \lim_{x\to 0}\frac{x}{\tan x} = 1\right]$$

$$= \lim_{x\to 0}\frac{\sec^2 x + x\cdot 2\sec^2 x\tan x - \sec^2 x}{6x^2}$$

$$= \lim_{x\to 0}\frac{\sec^2 x}{3}\cdot\frac{\tan x}{x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{1}{3}$$

Hence, $l = e^{\frac{1}{3}}$

Example 11

Prove that $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}} = e^{\frac{2}{\pi}}$.

Solution

Let $$l = \lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}} \qquad [1^\infty \text{ form}]$$

$$\log l = \lim_{x\to a}\tan\left(\frac{\pi x}{2a}\right)\log\left(2-\frac{x}{a}\right)$$

$$= \lim_{x\to a}\frac{\log\left(2-\frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to a}\frac{\frac{1}{\left(2-\frac{x}{a}\right)}\left(-\frac{1}{a}\right)}{\left(-\operatorname{cosec}^2\frac{\pi x}{2a}\right)\left(\frac{\pi}{2a}\right)} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{2}{\pi}$$

Hence, $$l = e^{\frac{2}{\pi}}$$

Example 12

Evaluate $\lim_{x\to 0}(\cos x)^{\cot x}$. **[Winter 2013]**

Solution

Let $$l = \lim_{x\to 0}(\cos x)^{\cot x} \qquad [1^\infty \text{ form}]$$

$$\log l = \lim_{x\to 0}\cot x\cdot\log(\cos x)$$

$$= \lim_{x\to 0}\frac{\log(\cos x)}{\tan x} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{\frac{1}{\cos x}\cdot(-\sin x)}{\sec^2 x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= 0$$

Hence, $l = e^0 = 1$

Example 13

Evaluate $\lim_{x \to \frac{\pi}{2}} (\operatorname{cosec} x)^{\tan^2 x}$.

Solution

Let $l = \lim_{x \to \frac{\pi}{2}} (\operatorname{cosec} x)^{\tan^2 x}$ $[1^\infty \text{ form}]$

$$\log l = \lim_{x \to \frac{\pi}{2}} \tan^2 x \cdot \log(\operatorname{cosec} x)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\log(\operatorname{cosec} x)}{\cot^2 x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\operatorname{cosec} x}(-\operatorname{cosec} x \cot x)}{2\cot x(-\operatorname{cosec}^2 x)} \qquad [\text{Applying L'Hospital's rule}]$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\sin^2 x}{2}$$

$$= \frac{1}{2}$$

Hence, $l = e^{\frac{1}{2}}$

Example 14

Prove that $\lim_{x \to 0} \left(\frac{\sinh x}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{6}}$.

Solution

Let $l = \lim_{x \to 0} \left(\frac{\sinh x}{x}\right)^{\frac{1}{x^2}}$ $[1^\infty \text{ form}]$ $\left[\because \lim_{x \to 0} \frac{\sinh x}{x} = 1\right]$

$$\log l = \lim_{x \to 0} \frac{1}{x^2} \log\left(\frac{\sinh x}{x}\right)$$

$$= \lim_{x\to 0} \frac{\log\left(\frac{\sinh x}{x}\right)}{x^2} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0} \frac{x}{\sinh x}\left(\frac{x\cosh x - \sinh x}{x^2}\right)\cdot\frac{1}{2x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{x\cosh x - \sinh x}{2x^3} \qquad \left[\frac{0}{0}\text{ form}\right]\left[\because \lim_{x\to 0}\frac{x}{\sinh x} = 1\right]$$

$$= \lim_{x\to 0} \frac{x\sinh x + \cosh x - \cosh x}{6x^2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{1}{6}\cdot\frac{\sinh x}{x}$$

$$= \frac{1}{6} \qquad \left[\because \lim_{x\to 0}\frac{\sinh x}{x} = 1\right]$$

Hence, $l = e^{\frac{1}{6}}$

Example 15

Evaluate $\lim_{x\to 0}\left(\sin^2\frac{\pi}{2-ax}\right)^{\sec^2\frac{\pi}{2-bx}}$.

Solution

Let $l = \lim_{x\to 0}\left(\sin^2\frac{\pi}{2-ax}\right)^{\sec^2\frac{\pi}{2-bx}}$ $\qquad [1^\infty \text{ form}]$

$$\log l = \lim_{x\to 0} \sec^2\frac{\pi}{2-bx}\cdot\log\left(\sin^2\frac{\pi}{2-ax}\right)$$

$$= \lim_{x\to 0} \frac{\log\left(\sin^2\frac{\pi}{2-ax}\right)}{\cos^2\frac{\pi}{2-bx}}$$

$$= \lim_{x\to 0} \frac{\frac{1}{\sin^2\frac{\pi}{2-ax}}\cdot 2\sin\left(\frac{\pi}{2-ax}\right)\cdot\cos\left(\frac{\pi}{2-ax}\right)\cdot\left[-\frac{\pi}{(2-ax)^2}(-a)\right]}{2\cos\left(\frac{\pi}{2-bx}\right)\left[-\sin\left(\frac{\pi}{2-bx}\right)\right]\left[-\frac{\pi}{(2-bx)^2}(-b)\right]}$$

[Applying L'Hospital's rule]

$$= \lim_{x\to 0} \frac{\cot\left(\frac{\pi}{2-ax}\right)}{-\sin\left(\frac{2\pi}{2-bx}\right)} \cdot \lim_{x\to 0} \frac{2a(2-bx)^2}{b(2-ax)^2} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= -\frac{2a}{b} \lim_{x\to 0} \frac{-\operatorname{cosec}^2\left(\frac{\pi}{2-ax}\right)\cdot\left[-\frac{\pi}{(2-ax)^2}(-a)\right]}{\cos\left(\frac{2\pi}{2-bx}\right)\cdot\left[-\frac{2\pi}{(2-bx)^2}(-b)\right]} \qquad \text{[Applying L'Hospital's rule]}$$

$$= -\frac{2a}{b} \cdot \left(\frac{-\frac{\pi a}{4}}{\frac{-2\pi b}{4}}\right)$$

$$= -\frac{a^2}{b^2}$$

Hence, $l = e^{-\frac{a^2}{b^2}}$

Example 16

Evaluate $\lim_{n\to\infty}\left(\cos\frac{\theta}{n}\right)^{n^2}$.

Solution

Let
$$l = \lim_{n\to\infty}\left(\cos\frac{\theta}{n}\right)^{n^2} \qquad [1^\infty \text{ form}]$$

$$\log l = \lim_{n\to\infty} n^2 \log\left(\cos\frac{\theta}{n}\right)$$

Putting $\frac{1}{n} = x$,

when $n\to\infty, x\to 0$

$$\log l = \lim_{x\to 0}\frac{\log(\cos\theta x)}{x^2} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\frac{\frac{1}{\cos\theta x}(-\theta\sin\theta x)}{2x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= -\frac{\theta}{2}\lim_{x\to 0}\frac{1}{\cos\theta x}\cdot\frac{\theta\sin\theta x}{\theta x}$$

$$= -\frac{\theta^2}{2} \qquad \left[\because \lim_{x\to 0}\frac{\sin\theta x}{\theta x} = 1\right]$$

Hence, $$l = e^{-\frac{\theta^2}{2}}$$

Example 17

Evaluate $\lim\limits_{x\to\infty}\left(x\sin\frac{1}{x}\right)^{x^2}$.

Solution

Let $$l = \lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)^{x^2} \qquad [1^\infty \text{ form}]$$

$$\log l = \lim_{x\to\infty} x^2 \log\left(x\sin\frac{1}{x}\right)$$

$$= \lim_{x\to\infty}\frac{\log\left(x\sin\frac{1}{x}\right)}{\frac{1}{x^2}}$$

Let $\frac{1}{x} = y$,

when $x\to\infty, y\to 0$

$$\log l = \lim_{y\to 0}\frac{\log\left(\frac{1}{y}\sin y\right)}{y^2} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{y\to 0}\frac{\frac{y}{\sin y}\left(\frac{\cos y}{y} - \frac{\sin y}{y^2}\right)}{2y} \qquad [\text{Applying L'Hospital's rule}]$$

$$= \lim_{y\to 0}\frac{y\cot y - 1}{2y^2}$$

$$= \lim_{y\to 0}\frac{y - \tan y}{2y^2\tan y} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{y\to 0}\frac{1-\sec^2 y}{4y\tan y + 2y^2\sec^2 y} \qquad [\text{Applying L'Hospital's rule}]$$

$$= \lim_{y\to 0}\frac{-\tan^2 y}{4y\tan y + 2y^2\sec^2 y}$$

$$= -\frac{1}{2}\lim_{y\to 0}\frac{\frac{\tan^2 y}{y^2}}{\frac{2\tan y}{y}+\sec^2 y}$$

$$= -\frac{1}{2}\cdot\frac{1}{2+1} \qquad \left[\because \lim_{y\to 0}\frac{\tan y}{y}=1\right]$$

$$= -\frac{1}{6}$$

Example 18

Evaluate $\lim_{x\to 0}(\cot x)^{\sin x}$.

Solution

Let $l = \lim_{x\to 0}(\cot x)^{\sin x}$ $\quad [\infty^0 \text{ form}]$

$$\log l = \lim_{x\to 0}\sin x\cdot\log(\cot x) \qquad [0\times\infty \text{ form}]$$

$$= \lim_{x\to 0}\frac{\log(\cot x)}{\operatorname{cosec} x} \qquad \left[\frac{\infty}{\infty}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{\frac{1}{\cot x}(-\operatorname{cosec}^2 x)}{-\operatorname{cosec} x\cot x} \qquad [\text{Applying L'Hospital's rule}]$$

$$= \lim_{x\to 0}\frac{\operatorname{cosec} x}{\cot^2 x}$$

$$= \lim_{x\to 0}\frac{1}{\sin x}\cdot\frac{\sin^2 x}{\cos^2 x}$$

$$= \lim_{x\to 0}\frac{\sin x}{\cos^2 x}$$

$$= 0$$

Hence, $l = e^0 = 1$

Example 19

Evaluate: $\lim_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}$. **[Summer 2016]**

Solution

Let $$l = \lim_{x\to 0}\left(\frac{1}{x}\right)^{\tan x} \qquad [\infty^0 \text{ form}]$$

$$\log l = \lim_{x\to 0} \tan x \log\left(\frac{1}{x}\right)$$

$$= \lim_{x\to 0} -\frac{\log x}{\cot x} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{\left(-\frac{1}{x}\right)}{-\operatorname{cosec}^2 x} \qquad [\text{Applying L'Hospital's rule}]$$

$$= \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{1}{\sin^2 x}}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x^2}\cdot x$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \cdot \lim_{x\to 0} x$$

$$= 1\cdot 0$$

$$= 0$$

Hence, $$l = e^0 = 1$$

Example 20

Prove that $\lim_{x\to 0}\left(\frac{1}{x}\right)^{1-\cos x} = 1$.

Solution

Let $$l = \lim_{x \to 0} \left(\frac{1}{x} \right)^{1 - \cos x} \qquad [\infty^0 \text{ form}]$$

$$\log l = \lim_{x \to 0} (1 - \cos x) \log \left(\frac{1}{x} \right)$$

$$= \lim_{x \to 0} \left(2 \sin^2 \frac{x}{2} \right) (-\log x)$$

$$= \lim_{x \to 0} \frac{2 \left(\sin \frac{x}{2} \right)^2 \left(\frac{x}{2} \right)^2}{\left(\frac{x}{2} \right)^2} (-\log x)$$

$$= \lim_{x \to 0} \frac{x^2 (-\log x)}{2} \qquad \left[\because \lim_{x \to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right) = 1 \right]$$

$$= \lim_{x \to 0} \frac{1}{2} \frac{(-\log x)}{\left(\frac{1}{x^2} \right)} \qquad \left[\frac{\infty}{\infty} \text{ form} \right]$$

$$= \frac{1}{2} \lim_{x \to 0} \left(\frac{-\frac{1}{x}}{-\frac{2}{x^3}} \right) \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{1}{2} \lim_{x \to 0} \left(\frac{x^2}{2} \right)$$

$$= 0$$

Hence, $$l = e^0 = 1$$

Example 21

Prove that $\lim_{x \to \infty} e^{\frac{\sinh^{-1} x}{\cosh^{-1} x}} = e.$

Solution

Let $$l = \lim_{x \to \infty} e^{\frac{\sinh^{-1} x}{\cosh^{-1} x}}$$

$$= \lim_{x \to \infty} \left(e^{\sinh^{-1} x} \right)^{\frac{1}{\cosh^{-1} x}} \qquad [\infty^0 \text{ form}]$$

$$\log l = \lim_{x\to\infty} \frac{\sinh^{-1} x}{\cosh^{-1} x} \cdot \log e$$

$$= \lim_{x\to\infty} \frac{\log(x+\sqrt{x^2+1})}{\log(x+\sqrt{x^2-1})} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{1}{2\sqrt{x^2+1}}\cdot 2x\right)}{\frac{1}{x+\sqrt{x^2-1}}\left(1+\frac{1}{2\sqrt{x^2-1}}\cdot 2x\right)} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2-1}}{\sqrt{x^2+1}}$$

$$= \lim_{x\to\infty} \sqrt{\frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}}}$$

$$= 1$$

Hence, $l = e^1 = e$

Example 22

Prove that $\lim_{x\to 0} \frac{e^x}{\left[\left(1+\frac{1}{x}\right)^x\right]^x} = 1$.

Solution

Let $l = \lim_{x\to 0}\left[\left(1+\frac{1}{x}\right)^x\right]^x$

$$= \lim_{x\to 0}\left(1+\frac{1}{x}\right)^{x^2} \qquad [\infty^0 \text{ form}]$$

$$\log l = \lim_{x\to 0} x^2 \log\left(1+\frac{1}{x}\right)$$

$$= \lim_{x\to 0} \frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x^2}} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{2}{x^2}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{x}{2\left(1+\frac{1}{x}\right)}$$

$$= \lim_{x\to 0} \frac{x^2}{2(x+1)}$$

$$= 0$$

$\therefore$ $l = e^0$

$= 1$

Hence, $\lim_{x\to 0} \frac{e^x}{\left[\left(1+\frac{1}{x}\right)^x\right]^x} = \frac{e^0}{1} = 1$

Example 23

Prove that $\lim_{x\to\infty} \left(\frac{1}{x}\right)^{\frac{1}{x}} = 1$.

Solution

Let $l = \lim_{x\to\infty} \left(\frac{1}{x}\right)^{\frac{1}{x}}$ $[0^0 \text{ form}]$

$$\log l = \lim_{x\to\infty} \frac{1}{x} \log\left(\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{-\log x}{x} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= -\lim_{x\to\infty} \frac{\frac{1}{x}}{1} \qquad \text{[Applying L'Hospital's rule]}$$

$$= 0$$

Hence, $l = e^0 = 1$

Example 24

Prove that $\lim_{x\to 1}(1-x^2)^{\frac{1}{\log(1-x)}} = e$. **[Winter 2013]**

Solution

Let $l = \lim_{x\to 1}(1-x^2)^{\frac{1}{\log(1-x)}}$ [0^0 form]

$$\log l = \lim_{x\to 1}\frac{1}{\log(1-x)}\log(1-x^2) \qquad \left[\frac{\infty}{\infty}\text{ form}\right]$$

$$= \lim_{x\to 1}\frac{-\frac{2x}{(1-x^2)}}{\frac{1}{(1-x)}(-1)} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 1}\frac{2x(1-x)}{(1-x)(1+x)}$$

$$= \lim_{x\to 1}\frac{2x}{1+x}$$

$$= 1$$

Hence, $l = e$

Example 25

Evaluate $\lim_{x\to\frac{\pi}{2}}(\cos x)^{\frac{\pi}{2}-x}$. **[Summer 2016]**

Solution

Let $l = \lim_{x\to\frac{\pi}{2}}(\cos x)^{\frac{\pi}{2}-x}$ [0° form]

$$\log l = \lim_{x\to\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)\log\cos x \qquad [0\times\infty\text{ form}]$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{\log\cos x}{\frac{1}{\left(\frac{\pi}{2}-x\right)}} \qquad \left[\frac{\infty}{\infty}\text{ form}\right]$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{-\frac{\sin x}{\cos x}}{-\frac{1}{\left(\frac{\pi}{2}-x\right)^2}(-1)}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{-\tan x}{\dfrac{1}{\left(\dfrac{\pi}{2}-x\right)^2}}$$

$$= \lim_{x\to\frac{\pi}{2}} -\left(\frac{\pi}{2}-x\right)^2 \tan x$$

$$= \lim_{x\to\frac{\pi}{2}} -\frac{\left(\dfrac{\pi}{2}-x\right)^2}{\cot x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{-2\left(\dfrac{\pi}{2}-x\right)(-1)}{-\text{cosec}^2 x}$$

$$= \lim_{x\to\frac{\pi}{2}} 2\left(\frac{\pi}{2}-x\right)\sin^2 x$$

$$= 0$$

Hence, $l = e^0 = 1$

Example 26

Evaluate $\lim_{x\to 0} \dfrac{1-x^x}{x\log x}$.

Solution

Let $l = \lim_{x\to 0} \dfrac{1-x^x}{x\log x}$

$$= \lim_{x\to 0} \frac{1-e^{x\log x}}{x\log x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{-e^{x\log x}\left(x\cdot\dfrac{1}{x}+\log x\right)}{x\cdot\dfrac{1}{x}+\log x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\left(-e^{x\log x}\right)$$

$$= \lim_{x\to 0}\left(-x^x\right)$$

$$= -\lim_{x\to 0} x^x$$

Let $L = \lim_{x\to 0} x^x \qquad \left[0^0 \text{ form}\right]$

$$\log L = \lim_{x\to 0} \log(x^x)$$
$$= \lim_{x\to 0} x \log x$$
$$= \lim_{x\to 0} \frac{\log x}{\frac{1}{x}} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$
$$= \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \qquad \text{[Applying L'Hospital's rule]}$$
$$= \lim_{x\to 0}(-x)$$
$$= 0$$
$$L = e^0 = 1$$

Hence, $l = -1$

Example 27

Prove that $\lim_{x\to 0} \frac{1 - x^{\sin x}}{x \log x} = -1.$

Solution

Let $l_1 = \lim_{x\to 0} x^{\sin x}$ $\qquad [0^0 \text{ form}]$

$$\log l_1 = \lim_{x\to 0} \sin x \cdot \log x$$
$$= \lim_{x\to 0} \frac{\log x}{\operatorname{cosec} x} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$
$$= \lim_{x\to 0} \frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x} \qquad \text{[Applying L'Hospital's rule]}$$
$$= \lim_{x\to 0}\left(-\frac{\sin^2 x}{x\cos x}\right)$$
$$= -\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \cdot \frac{x}{\cos x}$$
$$= 0 \qquad \left[\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right]$$
$$\log l_1 = 0$$
$$l_1 = e^0 = 1$$

$\therefore \lim_{x\to 0} x^{\sin x} = 1$ ...(1)

Let $l_2 = \lim_{x \to 0} x \log x$ [$0 \times \infty$ form]

$$= \lim_{x \to 0} \frac{\log x}{\frac{1}{x}} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x \to 0} (-x)$$

$$= 0 \qquad ...(2)$$

Let $l = \lim_{x \to 0} \frac{1 - x^{\sin x}}{x \log x}$ $\left[\frac{0}{0} \text{ form}\right]$ [Using Eqs (1) and (2)]

$$= \lim_{x \to 0} \frac{1 - e^{\sin x \log x}}{x \log x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$\lim_{x \to 0} \frac{1 - x^{\sin x}}{x \log x} = \lim_{x \to 0} \frac{-e^{\sin x \log x}\left(\frac{\sin x}{x} + \cos x \cdot \log x\right)}{1 + \log x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x \to 0} \frac{-x^{\sin x}\left[\left(\frac{\sin x}{x}\right) \cdot \frac{1}{\log x} + \cos x\right]}{\frac{1}{\log x} + 1} \qquad ...(3)$$

[Dividing numerator and denominator by $\log x$]

$$= -\frac{1\left(\frac{1}{-\infty} + \cos 0\right)}{\frac{1}{-\infty} + 1} \qquad \left[\text{Using Eq. (1) and } \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= -1$$

EXERCISE 1.5

1. Prove that $\lim_{x \to 0}\left(\frac{a_1^x + a_2^x + \ldots + a_n^x}{n}\right)^{\frac{1}{x}} = (a_1 a_2 \ldots a_n)^{\frac{1}{n}}$.

2. Prove that $\lim_{x \to \infty}\left(\frac{1^{\frac{1}{x}} + 2^{\frac{1}{x}} + 3^{\frac{1}{x}} + 4^{\frac{1}{x}}}{4}\right)^{4x} = 24$.

3. Prove that $\lim\limits_{x \to 0}(1+\sin x)^{\frac{1}{2x}} = e^{\frac{1}{2}}$.

4. Prove that $\lim\limits_{x \to 1}(x)^{\frac{1}{1-x}} = \frac{1}{e}$.

5. Prove that $\lim\limits_{x \to \frac{\pi}{2}}(\sin x)^{\tan^2 x} = \left(\frac{1}{\sqrt{e}}\right)$.

6. Prove that $\lim\limits_{x \to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = e^{-\frac{1}{6}}$.

7. Prove that $\lim\limits_{x \to \infty}\left(1+\frac{a}{x}\right)^{x} = e^{a}$.

8. Prove that $\lim\limits_{x \to \infty}\left(\frac{x+1}{x-1}\right)^{x} = e^{2}$.

9. Prove that $\lim\limits_{x \to \infty}\left(\frac{2x+1}{2x-1}\right)^{x} = e$.

10. Prove that $\lim\limits_{x \to 0}(1+\sin x)^{\operatorname{cosec} x} = e$.

11. Prove that $\lim\limits_{x \to 0}(1+\sin x)^{\cot x} = e$.

12. Prove that $\lim\limits_{x \to 0}(1+\tan x)^{\cot x} = e$.

13. Prove that $\lim\limits_{x \to 0}(1-\tan x)^{\frac{1}{x}} = \frac{1}{e}$.

14. Prove that $\lim\limits_{x \to 0}(\cos x)^{\frac{1}{x^2}} = \frac{1}{\sqrt{e}}$.

15. Prove that $\lim\limits_{x \to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}} = 0$.

16. Prove that $\lim\limits_{x \to 0}(\cos ax)^{\operatorname{cosec}^2 bx} = e^{-\frac{a^2}{2b^2}}$.

17. Prove that $\lim\limits_{x \to 0}(\cos x)^{\cot^2 x} = \frac{1}{\sqrt{e}}$.

18. Prove that $\lim\limits_{x \to 0}(\operatorname{cosec} x)^{\sin x} = 1$.

19. Prove that $\lim\limits_{x \to \infty}\left(1+\frac{2}{x}\right)^{x} = e^{2}$.

20. Prove that $\lim\limits_{x\to a}\left(2-\dfrac{x}{a}\right)^{\cot(x-a)}=e^{-\frac{1}{a}}$.

21. Prove that $\lim\limits_{x\to\frac{\pi}{4}}(\tan x)^{\cot 2x}=1$.

22. Prove that $\lim\limits_{x\to\frac{\pi}{2}}(\sec x)^{\cot x}=1$.

23. Prove that $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\right)^{2\sin x}=1$.

24. Prove that $\lim\limits_{x\to 0}(\sin x)^{\tan x}=1$.

25. Prove that $\lim\limits_{x\to 1}(1-x^n)^{\frac{1}{\log(1-x)}}=e$.

26. Prove that $\lim\limits_{x\to 0}x^{\tan\left(\frac{\pi x}{2}\right)}=e$.

27. Prove that $\lim\limits_{x\to 0}(e^{3x}-5x)^{\frac{1}{x}}=e^{-2}$.

28. Prove that $\lim\limits_{x\to 0}(\cos 2x)^{\left(\frac{3}{x^2}\right)}=e^{-6}$.

29. Prove that $\lim\limits_{x\to\frac{\pi}{2}}(\cos x)^{\cos^2 x}=1$.

Points to Remember

L'Hospital's Rule

If $\lim\limits_{x\to a}f(x)=0$, $\lim\limits_{x\to a}g(x)=0$, then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. The value of $\lim\limits_{x\to\infty}\dfrac{\log x}{x^n}$, $n>0$ is

(a) ∞ (b) $-\infty$ (c) 1 (d) 0

2. The value of $\lim_{x\to 0}\frac{\tan x}{x}$ is

(a) 0 (b) 1 (c) π (d) ∞

3. If $\lim_{x\to 0}\frac{(1-a)e^x}{x}$ exists and is finite then it is equal to

(a) 1 (b) 0 (c) –1 (d) $\frac{1}{2}$

4. The integer p for which $\lim_{x\to 0}\frac{px+\sin x}{x^2}$ is finite is

(a) 0 (b) –1 (c) 1 (d) 2

5. The value of $\lim_{x\to 0}(\cosh^{-1} x-\log x)$ is

(a) log 1 (b) log 2 (c) log 3 (d) 0

6. The value of $\lim_{x\to \infty} x^2\, e^{-x}$ is

(a) 1 (b) e^1 (c) e^2 (d) 0

7. The value of $\lim_{x\to \frac{\pi}{2}}(\cos x)^{\cos x}$ is

(a) 0 (b) 1 (c) –1 (d) 2

8. The value of $\lim_{x\to 0}\log_{\tan x}\tan 2x$ is

(a) 0 (b) 1 (c) –1 (d) 2

9. The value of $\lim_{x\to 0}(1-x^x)$ is

(a) 0 (b) 1 (c) –1 (d) $\frac{1}{2}$

10. The value of $\lim_{x\to \infty}\frac{1+2+3+\ldots+x}{x^2}$ is

(a) 0 (b) 1 (c) –1 (d) $\frac{1}{2}$

11. The value of $\lim_{x\to \infty}\left(\frac{1}{x}\right)^{\frac{1}{x}}$ is

(a) 0 (b) 1 (c) 2 (d) 3

12. The value of $\lim_{x\to 0}\sin x\log x$ is

(a) –1 (b) 0 (c) 1 (d) 5

13. The value of $\lim_{x\to 0}(\cos x)^{\cot x}$ is **[Summer 2016]**

(a) 0 (b) –1 (c) 1 (d) 2

14. The value of $\lim_{x\to 0}(e^{3x}-5x)^{\frac{1}{x}}$ is

(a) 0 (b) 1 (c) e^{-2} (d) e^2

15. If $\lim_{x\to 0}\frac{\sin 2x + p\sin x}{x^3}$ is finite then value of p is

(a) -2 (b) 1 (c) -1 (d) 2

16. The value of $\lim_{x\to 0}\frac{\tan x}{x}$ is **[Winter 2013]**

(a) 0 (b) 1 (c) π (d) ∞

17. The value of $\lim_{x\to 0}\frac{|x|}{x}$ is **[Winter 2013]**

(a) 0 (b) ± 1 (c) π (d) ∞

18. The value of $\lim_{x\to 0}\frac{x^2 - x - 2}{x^2 - 4}$ is **[Winter 2014]**

(a) 4 (b) $\frac{3}{4}$ (c) 0 (d) $-\frac{3}{4}$

19. The value of $\lim_{x\to 0} x^x$ is **[Winter 2015]**

(a) 1 (b) -1 (c) e (d) $\frac{1}{e}$

20. The value of $\lim_{n\to\infty}\frac{1-n^2}{\Sigma n}$ is **[Summer 2014]**

(a) -2 (b) 1 (c) 2 (d) 0

21. The value of $\lim_{x\to\infty} x^{\frac{1}{x}}$ is **[Summer 2015]**

(a) ∞ (b) $-\infty$ (c) 1 (d) 0

22. The value of $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ is **[Winter 2015]**

(a) 1 (b) -1 (c) e (d) $\frac{1}{e}$

23. The value of $\lim_{x\to\infty}\frac{\sin x}{x}$ is **[Summer 2016]**

(a) 1 (b) -1 (c) 0 (d) none of these

24. The value of $\lim_{x\to 6}\frac{\sin(x-6)}{x-6}$ is **[Winter 2016]**

(a) 0 (b) 1 (c) -1 (d) 0.5

25. The value of $\lim_{x\to 0}\dfrac{\sin 2x}{x}$ is **[Summer 2017]**

(a) 2 (b) 1 (c) –1 (d) $\dfrac{1}{2}$

26. The value of $\lim_{x\to \frac{\pi}{2}}\dfrac{\cos x}{x-\dfrac{\pi}{2}}$ is **[Summer 2017]**

(a) 0 (b) 1 (c) –1 (d) $\dfrac{\pi}{2}$

Answers

1. (d)	2. (b)	3. (b)	4. (a)	5. (b)	6. (d)	7. (a)	8. (b)	9. (a)
10. (d)	11. (b)	12. (b)	13. (c)	14. (c)	15. (a)	16. (b)	17. (b)	18. (b)
19. (a)	20. (a)	21. (c)	22. (c)	23. (c)	24. (b)	25. (a)	26. (c)	

CHAPTER 2

Improper Integrals

Chapter Outline

2.1 INTRODUCTION

The definition of a definite integral $\int_a^b f(x)\,dx$ requires the interval $[a, b]$ be finite. The fundamental theorem of calculus requires that $f(x)$ be continuous on $[a, b]$ or at least bounded. In this chapter, we will study a method of evaluating integrals that fail these requirements either because their limits of integration are infinite, or because a finite number of discontinuities exist on the interval $[a, b]$. Integrals that fail either of these requirements are known as improper integrals. Improper integrals cannot be computed using a normal Riemann integral.

2.2 IMPROPER INTEGRALS

The integral $\int_a^b f(x)\,dx$ is called an improper integral if

(i) one or both limits of integration are infinite
(ii) function $f(x)$ becomes infinite at a point within or at the end points of the interval of integration.

Improper integrals are classified into three kinds.

2.3 IMPROPER INTEGRALS OF THE FIRST KIND

It is a definite integral in which one or both limits of integration are infinite, for example, $\int_0^\infty e^{-x}\,dx$ is an improper integral of the first kind since the upper limit of integration is infinite. These integrals are evaluated as follows:

(i) If $f(x)$ is continuous on $[a, \infty)$ then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx \qquad \ldots (1)$$

(ii) If $f(x)$ is continuous on $(-\infty, b]$ then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx \qquad \ldots (2)$$

(iii) If $f(x)$ is continuous on $(-\infty, \infty)$ then

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{\substack{a\to-\infty\\ b\to\infty}}\int_a^b f(x)\,dx$$

$$= \lim_{a\to-\infty}\int_a^0 f(x)\,dx + \lim_{b\to\infty}\int_0^b f(x)\,dx \qquad \ldots (3)$$

The improper integral is said to converge (or exist) when the limit in RHS of (1), (2) and (3) exist (or finite). Otherwise, it is said to diverge.

Example 1

Evaluate $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$.

Solution

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{b\to\infty}\left|2\sqrt{x}\right|_1^b$$

$$= \lim_{b\to\infty}(2\sqrt{b}-2)$$

$$= \infty$$

Example 2

Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$.

Solution

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx\\ &= \lim_{b\to\infty}\left|-\frac{1}{x}\right|_1^b\\ &= \lim_{b\to\infty}\left(-\frac{1}{b}+1\right)\\ &= 1\end{aligned}$$

Example 3

Evaluate $\int_0^\infty \frac{dx}{x^2+1}$. **[Winter 2014]**

Solution

$$\begin{aligned}\int_0^\infty \frac{dx}{x^2+1} &= \lim_{b\to\infty}\int_0^b \frac{dx}{x^2+1}\\ &= \lim_{b\to\infty}\left|\tan^{-1} x\right|_0^b\\ &= \lim_{b\to\infty}\left[\tan^{-1}(b)-\tan^{-1} 0\right]\\ &= \lim_{b\to\infty}\tan^{-1}(b)\\ &= \tan^{-1}(\infty)\\ &= \frac{\pi}{2}\end{aligned}$$

Example 4

Evaluate $\int_{-\infty}^0 x\sin x\,dx$.

Solution

$$\begin{aligned}\int_{-\infty}^0 x\sin x\,dx &= \lim_{a\to-\infty}\int_a^0 x\sin x\,dx\\ &= \lim_{a\to-\infty}\left|-x\cos x+\sin x\right|_a^0\\ &= \lim_{a\to-\infty}(a\cos a-\sin a)\\ &= -\infty \qquad [\because \sin a \text{ and } \cos a \text{ oscillate between } \pm 1]\end{aligned}$$

Example 5

Evaluate $\int_{-\infty}^{0} e^{2x}\,dx$.

Solution

$$\int_{-\infty}^{0} e^{2x}\,dx = \lim_{a\to-\infty}\int_{a}^{0} e^{2x}\,dx$$

$$= \lim_{a\to-\infty}\left|\frac{e^{2x}}{2}\right|_{a}^{0}$$

$$= \lim_{a\to-\infty}\left(\frac{1}{2}-\frac{1}{2}e^{2a}\right)$$

$$= \frac{1}{2}-0$$

$$= \frac{1}{2}$$

Example 6

Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

Solution

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty}\int_{a}^{0} \frac{1}{1+x^2}\,dx + \lim_{b\to\infty}\int_{0}^{b} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to-\infty}\left|\tan^{-1}x\right|_{a}^{0} + \lim_{b\to\infty}\left|\tan^{-1}x\right|_{0}^{b}$$

$$= \lim_{a\to-\infty}(0-\tan^{-1}a) + \lim_{b\to\infty}(\tan^{-1}b-0)$$

$$= \frac{\pi}{2}+\frac{\pi}{2}$$

$$= \pi$$

Example 7

Evaluate $\int_{-\infty}^{\infty} e^{x}\,dx$.

Solution

$$\int_{-\infty}^{\infty} e^x \, dx = \lim_{a\to-\infty}\int_a^0 e^x \, dx + \lim_{b\to\infty}\int_0^b e^x \, dx$$

$$= \lim_{a\to-\infty}\left| e^x \right|_a^0 + \lim_{b\to\infty}\left| e^x \right|_0^b$$

$$= \lim_{a\to-\infty}(1-e^a) + \lim_{b\to\infty}(e^b-1)$$

$$= (1-0) + \lim_{b\to\infty}(e^b-1)$$

$$= \infty$$

Example 8

Evaluate $\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\, dx$.

Solution

$$\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\, dx = \int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+1}\, dx$$

$$= \lim_{a\to-\infty}\int_a^0 \frac{e^x}{e^{2x}+1}\, dx + \lim_{b\to\infty}\int_0^b \frac{e^x}{e^{2x}+1}\, dx$$

Putting $u = e^x$, $du = e^x\, dx$,

$$\int \frac{e^x}{e^{2x}+1}\, dx = \int \frac{du}{u^2+1} = \tan^{-1} u = \tan^{-1} e^x$$

$$\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\, dx = \lim_{a\to-\infty}\left|\tan^{-1} e^x\right|_a^0 + \lim_{b\to\infty}\left|\tan^{-1} e^x\right|_0^b$$

$$= \lim_{a\to-\infty}\left(\frac{\pi}{4}-\tan^{-1} e^a\right) + \lim_{b\to\infty}\left(\tan^{-1} e^b - \frac{\pi}{4}\right)$$

$$= \left(\frac{\pi}{4}-0\right) + \left(\frac{\pi}{2}-\frac{\pi}{4}\right)$$

$$= \frac{\pi}{2}$$

Example 9

Evaluate $\int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2}\, dx$.

Solution

$$\int_{-\infty}^{\infty}\frac{x}{(1+x^2)^2}\,dx=\lim_{a\to-\infty}\int_a^0\frac{x}{(1+x^2)^2}\,dx+\lim_{b\to\infty}\int_0^b\frac{x}{(1+x^2)^2}\,dx$$

$$=\lim_{a\to-\infty}\int_a^0(1+x^2)^{-2}\frac{2x}{2}\,dx+\lim_{b\to\infty}\int_0^b(1+x^2)^{-2}\frac{2x}{2}\,dx$$

$$=\lim_{a\to-\infty}\left|-\frac{1}{2(1+x^2)}\right|_a^0+\lim_{b\to\infty}\left|-\frac{1}{2(1+x^2)}\right|_0^b$$

$$\left[\because\int[f(x)]^n f'(x)\,dx=\frac{[f(x)]^{n+1}}{n+1}\right]$$

$$=\lim_{a\to-\infty}\left[-\frac{1}{2}+\frac{1}{2(1+a^2)}\right]+\lim_{b\to\infty}\left[-\frac{1}{2(1+b^2)}+\frac{1}{2}\right]$$

$$=-\frac{1}{2}+\frac{1}{2}$$

$$=0$$

Example 10

Evaluate $\int_0^{\infty}\frac{dv}{(1+v^2)(1+\tan^{-1}v)}$. **[Winter 2016]**

Solution

$$\int_0^{\infty}\frac{dv}{(1+v^2)(1+\tan^{-1}v)}=\lim_{b\to\infty}\int_0^b\frac{\frac{1}{1+v^2}}{1+\tan^{-1}v}\,dv$$

$$=\lim_{b\to\infty}\left|\log\left|1+\tan^{-1}v\right|\right|_0^b\qquad\left[\because\int\frac{f'(v)}{f(v)}\,dv=\log|f(v)|\right]$$

$$=\lim_{b\to\infty}\left[\log\left|1+\tan^{-1}b\right|-\log 1\right]$$

$$=\log(1+\tan^{-1}\infty)-0$$

$$=\log\left(1+\frac{\pi}{2}\right)$$

Example 11

Evaluate $\int_2^{\infty} \frac{x+3}{(x-1)(x^2+1)}\,dx$.

Solution

$$\int_2^{\infty} \frac{x+3}{(x-1)(x^2+1)}\,dx = \lim_{b\to\infty}\int_2^b \left[\frac{2}{x-1}+\frac{(-2x-1)}{x^2+1}\right]dx \quad \text{[Using partial fraction expansion]}$$

$$= \lim_{b\to\infty}\left[\int_2^b \frac{2}{x-1}\,dx - \int_2^b \frac{2x}{x^2+1}\,dx - \int_2^b \frac{1}{x^2+1}\,dx\right]$$

$$= \lim_{b\to\infty}\left[\left|2\log(x-1)-\log(x^2+1)-\tan^{-1}x\right|_2^b\right]$$

$$\left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|\right]$$

$$= \lim_{b\to\infty}\left[\left|\log\frac{(x-1)^2}{x^2+1}-\tan^{-1}x\right|_2^b\right]$$

$$= \lim_{b\to\infty}\left[\log\frac{(b-1)^2}{b^2+1}-\tan^{-1}b-\log\frac{1}{5}+\tan^{-1}2\right]$$

$$= \lim_{b\to\infty}\left[\log\frac{\left(1-\frac{1}{b}\right)^2}{1+\frac{1}{b^2}}-\tan^{-1}b+\log 5+\tan^{-1}2\right]$$

$$= \log 1-\tan^{-1}\infty+\log 5+\tan^{-1}2$$

$$= 0-\frac{\pi}{2}+\log 5+\tan^{-1}2$$

$$= -\frac{\pi}{2}+\log 5+\tan^{-1}2$$

Example 12

Prove that p-integral $\int_1^{\infty} \frac{1}{x^p}\,dx$ *converges when* $p > 1$ *and diverges when* $p \le 1$.

Solution

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx$$

$$= \lim_{b\to\infty}\left|\frac{x^{-p+1}}{-p+1}\right|_1^b, \; p \neq 1$$

$$= -\lim_{b\to\infty}\left[\frac{b^{-(p-1)}-1}{p-1}\right], \; p \neq 1$$

Case (I) When $p > 1$

$$\int_1^\infty \frac{1}{x^p}\,dx = -\lim_{b\to\infty}\left[\frac{\frac{1}{b^{p-1}}-1}{p-1}\right]$$

$$= -\left[\frac{\frac{1}{\infty}-1}{p-1}\right]$$

$$= \frac{1}{p-1} \quad \text{[finite]}$$

Case (II) When $p < 1$

$$\int_1^\infty \frac{1}{x^p}\,dx = -\lim_{b\to\infty}\left[\frac{b^{1-p}-1}{p-1}\right]$$

$$= -\left[\frac{\infty-1}{p-1}\right]$$

$$= -\infty$$

Case (III) When $p = 1$

$$\int_1^\infty \frac{1}{x}\,dx = \left|\log x\right|_1^\infty = \infty$$

Hence, p integral $\int_1^\infty \frac{1}{x^p}\,dx$ converges when $p > 1$ and diverges when $p \leq 1$.

EXERCISE 2.1

Evaluate the following improper integrals:

1. $\int_1^\infty \frac{1}{x}\,dx$ **[Ans.: ∞]**

2. $\int_0^{\infty} \frac{1}{1+x^2}\,dx$ $\left[\text{Ans.: } \frac{\pi}{2}\right]$

3. $\int_{-\infty}^{-1} \frac{2}{x^5}\,dx$ $\left[\text{Ans.: } -\frac{1}{2}\right]$

4. $\int_2^{\infty} \frac{1}{x\log^2 x}$ $\left[\text{Ans.: } \frac{1}{\log 2}\right]$

5. $\int_0^{\infty} e^{-x}\sin x\,dx$ $\left[\text{Ans.: } \frac{1}{2}\right]$

6. $\int_0^{\infty} x^2 e^{-x}\,dx$ **[Ans.: 2]**

7. $\int_{-\infty}^{\infty} |x|\, e^{-x^2}\,dx$ **[Ans.: 1]**

8. $\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}}\,dx$ $\left[\text{Ans.: } \frac{\pi}{2}\right]$

9. $\int_{-\infty}^{\infty} \frac{1}{x^2+2x+5}\,dx$ $\left[\text{Ans.: } \frac{\pi}{2}\right]$

10. $\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$ $\left[\text{Ans.: } \frac{2}{e}\right]$

2.4 IMPROPER INTEGRALS OF THE SECOND KIND

It is a definite integral in which integrand become infinite (or unbounded or discontinuous) at one or more points within or at the end points of the interval of integration, for example,

(i) $\int_0^1 \frac{1}{x}\,dx$ is an improper integral of the second kind as $\frac{1}{x}$ is not continuous at $x = 0$.

(ii) $\int_{-2}^{2} \frac{1}{x^2-1}\,dx$ is an improper integral of the second kind because $\frac{1}{x^2-1}$ is not continuous at $x = -1$ and $x = 1$.

These integrals are evaluated as follows:

(i) If $f(x)$ is unbounded at $x = a$ then

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0} \int_a^b f(x)\,dx \qquad \ldots (1)$$

(ii) If $f(x)$ is unbounded at $x = b$ then

$$\int_a^b f(x)\,dx = \lim_{c\to b}\int_a^c f(x)\,dx \qquad \dots (2)$$

(iii) If $f(x)$ is unbounded at $x = a$ and $x = b$ then

$$\int_a^b f(x)\,dx = \lim_{c_1\to a}\int_{c_1}^{0} f(x)\,dx + \lim_{c_2\to b}\int_0^{c_2} f(x)\,dx \qquad \dots (3)$$

The improper integral is said to converge (or exist) when the limit in RHS of (1), (2) and (3) exist (or finite). Otherwise, it is said to diverge.

Example 1

Evaluate $\int_0^3 \frac{1}{\sqrt{3-x}}\,dx$. **[Summer 2014, 2017]**

Solution

The integrand $\frac{1}{\sqrt{3-x}}$ is unbounded at $x = 3$.

$$\begin{aligned}\int_0^3 \frac{1}{\sqrt{3-x}}\,dx &= \lim_{c\to 3}\int_0^c \frac{1}{\sqrt{3-x}}\,dx\\ &= \lim_{c\to 3}\left|-2\sqrt{3-x}\right|_0^c\\ &= \lim_{c\to 3}(-2\sqrt{3-c}+2\sqrt{3})\\ &= 2\sqrt{3}\end{aligned}$$

Example 2

Check the convergence of $\int_0^5 \frac{1}{x^2}\,dx$. **[Summer 2016]**

Solution

The integrand $\frac{1}{x^2}$ is unbounded at $x = 0$.

$$\int_0^5 \frac{1}{x^2}\,dx = \lim_{c\to 0}\int_c^5 \frac{1}{x^2}\,dx$$

$$= \lim_{c\to 0}\left|-\frac{1}{x}\right|_c^5$$

$$= \lim_{c\to 0}\left[-\frac{1}{5}-\left(-\frac{1}{c}\right)\right]$$

$$= \infty$$

Hence, the integrand is divergent.

Example 3

Check the convergence of $\int_0^1 \frac{dx}{1-x}$. *If convergent, then evaluate the same.* **[Winter 2016]**

Solution

The integrand $\frac{1}{1-x}$ is unbounded at $x = 1$.

$$\int_0^1 \frac{dx}{1-x} = \lim_{c\to 1}\int_0^c \frac{1}{1-x}dx$$

$$= \lim_{c\to 1}\left|-\log(1-x)\right|_0^c$$

$$= \lim_{c\to 1}[-\log(1-c)+\log 1]$$

$$= -\log(1-1)$$

$$= \infty$$

Hence, the integrand is divergent.

Example 4

Check the convergence of $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$. **[Summer 2015]**

Solution

The integrand $\frac{1}{\sqrt{9-x^2}}$ is unbounded at $x = 3$.

$$\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \lim_{c\to 3}\int_0^c \frac{dx}{\sqrt{9-x^2}}$$

$$= \lim_{c\to 3} \left| \sin^{-1}\left(\frac{x}{3}\right)\right|_0^c$$

$$= \lim_{c\to 3} \left[\sin^{-1}\left(\frac{c}{3}\right) - \sin^{-1} 0\right]$$

$$= \sin^{-1}(1) - 0$$

$$= \frac{\pi}{2} \quad \text{[finite]}$$

Hence, the integrand is convergent.

Example 5

Evaluate $\int_0^{\frac{\pi}{2}} \sec x\,dx$.

Solution

The integrand sec x is not continuous at $x = \frac{\pi}{2}$.

$$\int_0^{\frac{\pi}{2}} \sec x\,dx = \lim_{c\to\frac{\pi}{2}} \int_0^c \sec x\,dx$$

$$= \lim_{c\to\frac{\pi}{2}} \left| \log|\sec x + \tan x| \right|_0^c$$

$$= \lim_{c\to\frac{\pi}{2}} \log|\sec c + \tan c|$$

$$= \log\left|\sec\frac{\pi}{2} - \tan\frac{\pi}{2}\right|$$

$$= \infty$$

Example 6

Evaluate $\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$. **[Summer 2015]**

Solution

The integrand $\frac{1}{(x-1)^{\frac{2}{3}}}$ is unbounded at $x = 1$.

$$\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{c_1 \to 1} \int_0^{c_1} \frac{dx}{(x-1)^{\frac{2}{3}}} + \lim_{c_2 \to 1} \int_{c_2}^{3} \frac{dx}{(x-1)^{\frac{2}{3}}}$$

$$= \lim_{c_1 \to 1} \left| \frac{(x-1)^{\frac{1}{3}}}{\frac{1}{3}} \right|_0^{c_1} + \lim_{c_2 \to 1} \left| \frac{(x-1)^{\frac{1}{3}}}{\frac{1}{3}} \right|_{c_2}^{3}$$

$$= \lim_{c_1 \to 1} 3 \left[(c_1 - 1)^{\frac{1}{3}} - (-1)^{\frac{1}{3}} \right] + \lim_{c_2 \to 1} 3 \left[(3-1)^{\frac{1}{3}} - (c_2 - 1)^{\frac{1}{3}} \right]$$

$$= 3\left[0 - (-1)^{\frac{1}{3}} \right] + 3\left[2^{\frac{1}{3}} - 0 \right]$$

$$= 3\left[2^{\frac{1}{3}} + 1 \right]$$

Example 7

Evaluate $\int_{-1}^{1} \frac{1}{x^{\frac{2}{3}}} dx$.

Solution

The integrand $\frac{1}{x^{\frac{2}{3}}}$ is unbounded at $x = 0$.

$$\int_{-1}^{1} \frac{1}{x^{\frac{2}{3}}} dx = \lim_{c_1 \to 0} \int_{-1}^{c_1} \frac{1}{x^{\frac{2}{3}}} dx + \lim_{c_2 \to 0} \int_{c_2}^{1} \frac{1}{x^{\frac{2}{3}}} dx$$

$$= \lim_{c_1 \to 0} \left| 3x^{\frac{1}{3}} \right|_{-1}^{c_1} + \lim_{c_2 \to 0} \left| 3x^{\frac{1}{3}} \right|_{c_2}^{1}$$

$$= \lim_{c_1 \to 0} \left[3c_1^{\frac{1}{3}} - 3(-1)^{\frac{1}{3}} \right] + \lim_{c_2 \to 0} \left[3 - 3c_2^{\frac{1}{3}} \right]$$

$$= [0 - 3(-1)^{\frac{1}{3}}] + [3 - 0]$$

$$= 6$$

Example 8

Evaluate $\int_0^5 \frac{1}{(x-2)^2}\,dx$.

Solution

The integrand $\frac{1}{(x-2)^2}$ is unbounded at $x = 2$.

$$\int_0^5 \frac{1}{(x-2)^2}\,dx = \lim_{c_1 \to 2} \int_0^{c_1} \frac{1}{(x-2)^2}\,dx + \lim_{c_2 \to 2} \int_{c_2}^5 \frac{1}{(x-2)^2}\,dx$$

$$= \lim_{c_1 \to 2} \left| -\frac{1}{x-2} \right|_0^{c_1} + \lim_{c_2 \to 2} \left| -\frac{1}{x-2} \right|_{c_2}^5$$

$$= \lim_{c_1 \to 2} \left(-\frac{1}{c_1 - 2} - \frac{1}{2} \right) + \lim_{c_2 \to 2} \left(-\frac{1}{3} + \frac{1}{c_2 - 2} \right)$$

$$= -\infty + \infty, \quad \text{indeterminate form}$$

Hence, no conclusion can be made about the value of the integral.

Example 9

Evaluate $\int_{-a}^{a} \frac{1}{\sqrt{a^2 - x^2}}\,dx$.

Solution

The integrand $\frac{1}{\sqrt{a^2 - x^2}}$ is unbounded at $x = \pm a$.

$$\int_{-a}^{a} \frac{1}{\sqrt{a^2 - x^2}}\,dx = \lim_{c_1 \to -a} \int_{c_1}^{0} \frac{1}{\sqrt{a^2 - x^2}}\,dx + \lim_{c_2 \to a} \int_0^{c_2} \frac{1}{\sqrt{a^2 - x^2}}\,dx$$

$$= \lim_{c_1 \to -a} \left| \sin^{-1} \frac{x}{a} \right|_{c_1}^{0} + \lim_{c_2 \to a} \left| \sin^{-1} \frac{x}{a} \right|_0^{c_2}$$

$$= \lim_{c_1 \to -a}\left[\sin^{-1} 0 - \sin^{-1}\frac{c_1}{a}\right] + \lim_{c_2 \to a}\left[\sin^{-1}\frac{c_2}{a} - \sin^{-1} 0\right]$$

$$= -\sin^{-1}\left(-\frac{a}{a}\right) + \sin^{-1}\left(\frac{a}{a}\right)$$

$$= \sin^{-1} 1 + \sin^{-1} 1$$

$$= 2\sin^{-1} 1$$

$$= 2\cdot\frac{\pi}{2}$$

$$= \pi$$

Example 10

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{1-\cos x}}\,dx.$

Solution

The integrand is unbounded at $x = 0$.

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{1-\cos x}}\,dx = \lim_{\varepsilon \to 0}\int_{\varepsilon}^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{1-\cos x}}\,dx$$

$$= \lim_{\varepsilon \to 0}\int_{\varepsilon}^{\frac{\pi}{2}} (1-\cos x)^{-\frac{1}{2}} \sin x\,dx$$

$$= \lim_{\varepsilon \to 0}\left|\frac{(1-\cos x)^{\frac{1}{2}}}{\frac{1}{2}}\right|_{\varepsilon}^{\frac{\pi}{2}} \qquad \left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= \lim_{\varepsilon \to 0}\left|2(1-\cos x)^{\frac{1}{2}}\right|_{\varepsilon}^{\frac{\pi}{2}}$$

$$= \lim_{c \to 0} 2\left[1-(1-\cos c)^{\frac{1}{2}}\right]$$

$$= 2$$

2.5 IMPROPER INTEGRAL OF THE THIRD KIND

It is a definite integral in which one or both limits of integration are infinite, and the integrand become infinite at one or more points within or at the end points of the interval of integration. Thus, it is a combination of the first kind and the second kind.

For example, $\int_0^\infty \frac{1}{x^2}\,dx$ is an improper integral of the third kind as the upper limit of integration is infinite and integrand $\frac{1}{x^2}$ is infinite at $x = 0$.

Example 1

Evaluate the improper integral $\int_0^\infty \frac{1}{x^2}\,dx.$

Solution

$$\int_0^\infty \frac{1}{x^2}\,dx = \lim_{c_1 \to 0} \int_{c_1}^{1} \frac{1}{x^2}\,dx + \lim_{c_2 \to \infty} \int_{1}^{c_2} \frac{1}{x^2}\,dx$$

$$= \lim_{c_1 \to 0} \left|-\frac{1}{x}\right|_{c_1}^{1} + \lim_{c_2 \to \infty} \left|-\frac{1}{x}\right|_{1}^{c_2}$$

$$= \lim_{c_1 \to 0} \left(-1+\frac{1}{c_1}\right) + \lim_{c_2 \to \infty} \left(-\frac{1}{c_2}+1\right)$$

$$= \infty + 1$$

$$= \infty$$

EXERCISE 2.2

Evaluate the following improper integrals:

1. $\int_0^1 \frac{1}{x} dx$

[Ans.: ∞]

2. $\int_0^1 \frac{1}{\sqrt{x}} dx$

[Ans.: 2]

3. $\int_0^1 \frac{1}{x^2} dx$

[Ans.: ∞]

4. $\int_0^1 \log x dx$

[Ans.: –1]

5. $\int_0^9 \frac{1}{(x-1)^{\frac{2}{3}}} dx$

[Ans.: 9]

6. $\int_0^1 \frac{1}{1-x^4} dx$

[Ans.: ∞]

7. $\int_0^2 \frac{1}{\sqrt{x(2-x)}} dx$

[Ans.: π]

2.6 CONVERGENCE AND DIVERGENCE OF IMPROPER INTEGRALS

(i) Direct Comparison Test

(a) If $f(x)$ and $g(x)$ are two continuous functions for $x \ge a$ such that $0 \le f(x) \le g(x)$, then

$\int_a^\infty f(x)dx$ converges, if $\int_a^\infty g(x)dx$ converges and $\int_a^\infty f(x)dx \le \int_a^\infty g(x)dx$.

(b) If $f(x)$ and $g(x)$ are two continuous functions for $x \ge a$ such that $f(x) \ge g(x)$, then $\int_a^\infty f(x)dx$ diverges, if $\int_a^\infty g(x)dx$ diverges.

(ii) Limit Comparison Test

If $f(x)$ and $g(x)$ are two continuous functions for $x \geq a$ such that $f(x) > 0, g(x) > 0$ and $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = l, 0 < l < \infty$ then $\int_a^\infty f(x)\, dx$ and $\int_a^\infty g(x)\, dx$ both converge or diverge simultaneously.

Note: Convergence of improper integral of second kind can be tested by direct comparison test and limit comparison test similarly.

Example 1

Test the convergence of the improper integral $\int_1^\infty \frac{\cos x}{x^2} dx$.

Solution

$f(x) = \frac{\cos x}{x^2}$ and let $g(x) = \frac{1}{x^2}$

$$\frac{\cos x}{x^2} \leq \frac{1}{x^2} \quad \text{for} \quad x \geq 1 \qquad [\because \cos x \leq 1]$$

$$\begin{aligned}\int_1^\infty g(x)\, dx = \int_1^\infty \frac{1}{x^2} dx &= \lim_{b \to \infty} \int_1^b \frac{1}{x^2} dx \\ &= \lim_{b \to \infty} \left| -\frac{1}{x} \right|_1^b \\ &= \lim_{b \to \infty} \left(-\frac{1}{b} + 1 \right) \\ &= 1\end{aligned}$$

Thus, $\int_1^\infty \frac{1}{x^2} dx$ is convergent.

By direct comparison test, $\int_1^\infty \frac{\cos x}{x^2}$ is convergent.

Example 2

Test the convergence of the improper integral $\int_1^\infty e^{-x^2} dx$.

Solution

$$f(x) = e^{-x^2} \text{ and let } g(x) = e^{-x}$$

$$e^{-x^2} \leq e^{-x} \quad \text{for} \quad x \geq 1 \quad \begin{bmatrix} \because x^2 \geq x \\ -x^2 \leq -x \end{bmatrix}$$

$$\int_1^{\infty} g(x)\,dx = \int_1^{\infty} e^{-x}\,dx$$
$$= \lim_{b\to\infty}\int_1^{b} e^{-x}\,dx$$
$$= \lim_{b\to\infty}(e^{-1}-e^{-b})$$
$$= e^{-1}$$
$$= \frac{1}{e}$$

Thus, $\int_1^{\infty} e^{-x}\,dx$ is convergent.

By direct comparison text, $\int_1^{\infty} e^{-x^2}\,dx$ is convergent.

Example 3

Test the convergence of the improper integral $\int_4^{\infty}\frac{3x+5}{x^4+7}\,dx$.

Solution

$$f(x)=\frac{3x+5}{x^4+7} \text{ and let } g(x)=\frac{1}{x^3}$$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{\frac{3x+5}{x^4+7}}{\frac{1}{x^3}}$$
$$= \lim_{x\to\infty}\frac{3x^4+5x^3}{x^4+7}$$
$$= \lim_{x\to\infty}\frac{3+\frac{5}{x}}{1+\frac{7}{x^4}}$$
$$= 3$$

$$\int_4^{\infty} g(x)\,dx = \int_4^{\infty}\frac{1}{x^3}\,dx$$
$$= \lim_{b\to\infty}\int_4^{b}\frac{1}{x^3}\,dx$$

$$= \lim_{b\to\infty}\left|-\frac{1}{2x^2}\right|_4^b$$

$$= 0 + \frac{1}{2(4^2)}$$

$$= \frac{1}{32}$$

Thus, $\int_4^\infty \frac{1}{x^3}\,dx$ is convergent.

By limit comparison test, $\int_4^\infty \frac{3x+5}{x^4+7}$ is convergent.

Example 4

Test the convergence of the improper integral $\int_0^1 \frac{1}{x^2+\sqrt{x}}\,dx.$

Solution

$f(x) = \frac{1}{x^2+\sqrt{x}}$ and let $g(x) = \frac{1}{\sqrt{x}}$

$$\frac{1}{x^2+\sqrt{x}} < \frac{1}{\sqrt{x}} \quad 0 < x \le 1 \quad [\because x^2+\sqrt{x} > \sqrt{x}]$$

$$\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{\varepsilon\to 0}\left|2\sqrt{x}\right|_\varepsilon^1$$

$$= \lim_{\varepsilon\to 0}\left(2-2\sqrt{\varepsilon}\right)$$

$$= 2$$

Thus, $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is convergent.

By direct comparison test, $\int_0^1 \frac{1}{x^2+\sqrt{x}}\,dx$ is convergent.

Example 5

Test the convergence of the improper integral $\int_0^1 \frac{1-e^{-x}}{x^3}\,dx.$

Solution

$f(x) = \dfrac{1-e^{-x}}{x^3}$ and let $g(x) = \dfrac{1}{x^3}$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{\dfrac{1-e^{-x}}{x^3}}{\dfrac{1}{x^3}} = \lim_{x\to\infty}\left(1-e^{-x}\right) = 1$$

$$\begin{aligned}\int_0^1 g(x)\,dx &= \int_0^1 \frac{1}{x^3}\,dx\\ &= \lim_{\epsilon\to 0}\int_\epsilon^1 \frac{1}{x^3}\,dx\\ &= \lim_{\epsilon\to 0}\left|-\frac{1}{2x^2}\right|_\epsilon^1\\ &= \lim_{\epsilon\to 0}\left(-\frac{1}{2}+\frac{1}{2\epsilon^2}\right)\\ &= \infty\end{aligned}$$

Thus $\int_0^1 \frac{1}{x^3}\,dx$ is divergent.

By limit comparison test, $\int_0^1 \frac{1-e^{-x}}{x^3}\,dx$ is divergent.

EXERCISE 2.3

Test the convergence of the following improper integrals.

1. $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ **[Ans.:** convergent**]**

2. $\int_1^\infty \frac{\sin^2\left(\frac{1}{x}\right)}{\sqrt{x}}\,dx$ **[Ans.:** convergent**]**

3. $\int_2^\infty \frac{x^2-1}{\sqrt{x^6+16}}\,dx$ **[Ans.:** divergent**]**

4. $\int_1^\infty \frac{x}{3x^4+5x^2+1}\,dx$ **[Ans.:** convergent**]**

5. $\int_1^\infty \frac{\log x\,dx}{x+e^{-x}}$ **[Ans.:** divergent**]**

6. $\int_{-\infty}^{\infty} \frac{2+\sin x}{x^2+1}\, dx$

[Ans.: convergent]

7. $\int_{0}^{1} \frac{1}{(x+1)\sqrt{1-x^2}}\, dx$

[Ans.: convergent]

8. $\int_{0}^{\frac{\pi}{2}} \log \sin x\, dx$

[Ans.: convergent]

9. $\int_{0}^{\frac{\pi}{2}} \frac{e^{-x}\cos x}{x}\, dx$

[Ans.: divergent]

10. $\int_{1}^{\frac{\pi}{2}} \frac{\tan x}{x^{\frac{3}{2}}}\, dx$

[Ans.: divergent]

Points to Remember

Improper Integrals of the First Kind

It is a definite integral in which one or both limits of integration are infinite.

Improper Integrals of the Second Kind

It is a definite integral in which integrand become infinite (or unbounded or discontinuous) at one or more points within or at the end points of the interval of integration.

Improper Integral of the Third Kind

It is a definite integral in which one or both limits of integration are infinite, and the integrand become infinite at one or more points within or at the end points of the interval of integration.

Convergence and Divergence of Improper Integrals

(i) Direct Comparison Test

(a) If $f(x)$ and $g(x)$ are two continuous functions for $x \ge a$ such that $0 \le f(x) \le g(x)$, then

$\int_{a}^{\infty} f(x)dx$ converges, if $\int_{a}^{\infty} g(x)dx$ converges and $\int_{a}^{\infty} f(x)dx \le \int_{a}^{\infty} g(x)dx$.

(b) If $f(x)$ and $g(x)$ are two continuous functions for $x \geq a$ such that $f(x) \geq g(x)$, then $\int_a^\infty f(x)\mathrm{d}x$ diverges, if $\int_a^\infty g(x)\mathrm{d}x$ diverges.

(ii) Limit Comparison Test

If $f(x)$ and $g(x)$ are two continuous functions for $x \geq a$ such that $f(x) > 0, g(x) > 0$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = l, 0 < l < \infty$ then $\int_a^\infty f(x)\,\mathrm{d}x$ and $\int_a^\infty g(x)\,\mathrm{d}x$ both converge or diverge simultaneously.

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. The value of $\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}\mathrm{d}x$ is

(a) 0 (b) ∞ (c) 3 (d) 1

2. The value of $\int_{0}^{\infty} \frac{1}{x^2}\mathrm{d}x$ is

(a) 0 (b) 1 (c) ∞ (d) 2

3. The value of $\int_{0}^{\infty} (1+2x)e^{-x}\mathrm{d}x$ is

(a) 0 (b) 1 (c) 3 (d) ∞

4. The value of $\int_{-5}^{1} \frac{1}{10+2z}\mathrm{d}z$ is

(a) 0 (b) 1 (c) 3 (d) ∞

5. The value of $\int_{-\infty}^{1} \sqrt{6-y}\,\mathrm{d}y$ is

(a) 0 (b) 1 (c) 2 (d) ∞

6. The value of $\int_{2}^{\infty} \frac{9}{(1-3z)^4}\mathrm{d}z$ is

(a) 0 (b) $\frac{3}{125}$ (c) $\frac{1}{125}$ (d) ∞

7. The value of $\int_{-\infty}^{\infty} \frac{6x^3}{(x^4+1)^2}\,dx$ is

(a) 0 (b) 1 (c) 6 (d) ∞

8. The value of $\int_{1}^{\infty} \frac{1}{x^2+x-6}\,dx$ is

(a) 0 (b) 1 (c) 6 (d) ∞

9. The value of $\int_{-\infty}^{0} \frac{e^{\frac{1}{x}}}{x^2}\,dx$ is

(a) 0 (b) 1 (c) 2 (d) ∞

10. The value of $\int_{1}^{\pi} \sec^2 x\,dx$ is

(a) 0 (b) 1 (c) π (d) ∞

11. The value of $\int_{0}^{\infty} \frac{dx}{x^2+4}$ is

(a) 0 (b) 1 (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

12. The value of $\int_{0}^{3} \frac{dx}{\sqrt{9-x^2}}$ is

(a) 0 (b) 1 (c) $\frac{\pi}{2}$ (d) ∞

13. The value of $\int_{0}^{\infty} x^2 e^{-x}\,dx$ is

(a) 0 (b) 2 (c) –2 (d) ∞

14. The value of $\int_{-\infty}^{0} 2^{5x}\,dx$ is

(a) 0 (b) $\frac{1}{5\log 2}$ (c) $\frac{1}{\log 2}$ (d) $\frac{2}{5\log 2}$

15. The value of $\int_{-\infty}^{\infty} \frac{dx}{25+4x^2}$ is

(a) $\frac{\pi}{10}$ (b) $\frac{\pi}{5}$ (c) $\frac{\pi}{2}$ (d) π

16. The value of $\int_1^{\infty} \frac{1}{x^2} \mathrm{d}x$ is **[Winter 2015]**

(a) 1 (b) 0 (c) –1 (d) does not exist

17. The value of $\int_0^{\infty} \frac{1}{1+x^2} \mathrm{d}x$ is **[Winter 2016]**

(a) π (b) $\frac{\pi}{2}$ (c) 0 (d) 1

18. The value of $\int_0^{\infty} e^{-x} \cos 2x \, \mathrm{d}x$ is **[Summer 2017]**

(a) 0 (b) $-\frac{1}{5}$ (c) $\frac{1}{5}$ (d) $\frac{2}{5}$

Answers

1. (b)	2. (c)	3. (c)	4. (d)	5. (d)	6. (c)	7. (a)	8. (d)	9. (b)
10. (d)	11. (d)	12. (c)	13. (b)	14. (b)	15. (a)	16. (a)	17. (b)	18. (c)

CHAPTER 3

Gamma and Beta Functions

Chapter Outline

3.1 INTRODUCTION

There are some special functions which have importance in mathematical analysis, functional analysis, physics or other applications. In this chapter, we will study two special functions, *gamma* and *beta functions*. The beta function is also called the *Euler integral of the first kind*. The gamma function is an extension of the factorial function to real and complex numbers and is also known as *Euler integral of the second kind*. Gamma function is a component in various probability distribution functions. It also appears in various areas such as asymptotic series, definite integration, number theory, etc.

3.2 GAMMA FUNCTION

Gamma function is defined by the improper integral $\int_0^\infty e^{-x} x^{n-1} \mathrm{d}x$, $n > 0$ and is denoted by $\overline{|n}$.

Hence,
$$\overline{|n} = \int_0^\infty e^{-x} x^{n-1} \mathrm{d}x, \; n > 0$$

Alternate form of gamma function

$$\overline{|n} = 2\int_0^\infty e^{-x^2} x^{2n-1} \mathrm{d}x$$

Proof: By definition,

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} \mathrm{d}x$$

Let $x = t^2$, $\mathrm{d}x = 2t\,\mathrm{d}t$

$$\Gamma n = \int_0^\infty e^{-t^2} \cdot t^{2n-2} \cdot 2t\,\mathrm{d}t$$

$$= 2\int_0^\infty e^{-t^2} \cdot t^{2n-1}\,\mathrm{d}t$$

Changing the variable t to x,

$$\Gamma n = 2\int_0^\infty e^{-x^2} \cdot x^{2n-1} \mathrm{d}x$$

3.3 PROPERTIES OF GAMMA FUNCTION

Property 1: $\boldsymbol{\Gamma(n+1) = n\,\Gamma n}$

Proof: $\Gamma(n+1) = \int_0^\infty e^{-x} x^n \mathrm{d}x$

Integrating by parts,

$$\Gamma(n+1) = \left|-e^{-x} x^n\right|_0^\infty - \int_0^\infty (-e^{-x})\, n x^{n-1} \mathrm{d}x$$

$$= n\int_0^\infty e^{-x} x^{n-1} \mathrm{d}x$$

$$= n\,\Gamma n$$

Hence,

$$\Gamma(n+1) = n\,\Gamma n$$

This is known as recurrence or reduction formula for Gamma function.

Note:

(i) $\Gamma(n+1) = n!$ if n is a positive integer

(ii) $\Gamma(n+1) = n\,\Gamma n$ if n is a positive real number

(iii) $\Gamma n = \dfrac{\Gamma(n+1)}{n}$ if n is a negative fraction

(iv) $\Gamma n\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

Property 2: $\boldsymbol{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$

Proof: By alternate form of Gamma function,

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} x^{2\cdot\left(\frac{1}{2}\right)-1} \mathrm{d}x = 2\int_0^\infty e^{-x^2} \mathrm{d}x$$

$$\Gamma\left(\frac{1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} \mathrm{d}x \cdot 2\int_0^\infty e^{-y^2} \mathrm{d}y$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \mathrm{d}x\,\mathrm{d}y$$

Changing to polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$

$$\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$$

Limits of x $\quad x = 0 \quad$ to $\quad x \to \infty$

Limits of y $\quad y = 0 \quad$ to $\quad y \to \infty$

This shows that the region of integration is the first quadrant.

Draw an elementary radius vector in the region which starts from the pole and extends up to ∞.

Limits of r $\quad r = 0 \quad$ to $\quad r \to \infty$

Limits of θ $\quad \theta = 0 \quad$ to $\quad \theta = \dfrac{\pi}{2}$

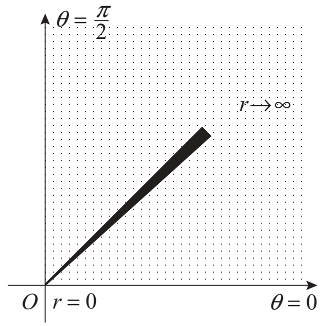


Fig. 3.1

$$\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right) = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}\cdot r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 4\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\cdot\int_0^{\infty}\left(-\frac{1}{2}\right)e^{-r^2}(-2r)\,\mathrm{d}r$$

$$= \frac{4}{-2}\left|\theta\right|_0^{\frac{\pi}{2}}\left|e^{-r^2}\right|_0^{\infty} \qquad \left[\because \int e^{f(r)}\cdot f'(r)\mathrm{d}r = e^{f(r)}\right]$$

$$= -2\cdot\frac{\pi}{2}(0-1)$$

$$= \pi$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Example 1

Find the value of $\Gamma\left(-\dfrac{5}{2}\right)$.

Solution

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}} = -\frac{2}{5}\Gamma\left(-\frac{3}{2}\right)$$

$$= -\frac{2}{5}\cdot\frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = \frac{4}{15}\Gamma\left(-\frac{1}{2}\right)$$

$$= \frac{4}{15}\cdot\frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -\frac{8}{15}\Gamma\left(\frac{1}{2}\right) = -\frac{8\sqrt{\pi}}{15}$$

Example 2

Given $\Gamma\left(\frac{8}{5}\right) = 0.8935$, *find the value of* $\Gamma\left(-\frac{12}{5}\right)$.

Solution:

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\Gamma\left(-\frac{12}{5}\right) = \frac{\Gamma\left(-\frac{12}{5}+1\right)}{-\frac{12}{5}} = -\frac{5}{12}\cdot\frac{\Gamma\left(-\frac{7}{5}+1\right)}{-\frac{7}{5}} = \frac{25}{84}\cdot\frac{\Gamma\left(-\frac{2}{5}+1\right)}{-\frac{2}{5}}$$

$$= -\frac{125}{168}\cdot\frac{\Gamma\left(\frac{3}{5}+1\right)}{\frac{3}{5}} = -\frac{625}{504}\Gamma\left(\frac{8}{5}\right) = -\frac{625}{504}(0.8935) = -1.108$$

Example 3

Evaluate $\int_0^\infty e^{-x^3}\,\mathrm{d}x$.

Solution

Let $x^3 = t,\ x = t^{\frac{1}{3}},\ \mathrm{d}x = \frac{1}{3}t^{-\frac{2}{3}}\mathrm{d}t$

When $x = 0,\quad t = 0$

When $x \to \infty,\quad t \to \infty$

$$\int_0^\infty e^{-x^3}\,\mathrm{d}x = \int_0^\infty e^{-t}\cdot\frac{1}{3}t^{-\frac{2}{3}}\,\mathrm{d}t = \frac{1}{3}\int_0^\infty e^{-t}\,t^{\frac{1}{3}-1}\,\mathrm{d}t = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)$$

Example 4

Evaluate $\int_0^\infty e^{-\sqrt{x}}x^{\frac{1}{4}}\,\mathrm{d}x$.

Solution

Let $\sqrt{x} = t,\ x = t^2,\ \mathrm{d}x = 2t\,\mathrm{d}t$

When $x = 0,\quad t = 0$

When $x \to \infty,\quad t \to \infty$

$$\int_0^\infty e^{-\sqrt{x}}x^{\frac{1}{4}}\,\mathrm{d}x = \int_0^\infty e^{-t}\left(t^2\right)^{\frac{1}{4}}2t\,\mathrm{d}t$$

$$= 2\int_0^\infty e^{-t}\,t^{\frac{3}{2}}\,\mathrm{d}t = 2\Gamma\left(\frac{5}{2}\right) = 2\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}\sqrt{\pi}$$

Example 5

Evaluate $\int_0^\infty (x^2+4)e^{-2x^2}\,\mathrm{d}x$.

Solution

Let $\quad 2x^2 = t,\ x = \left(\frac{t}{2}\right)^{\frac{1}{2}},\ \mathrm{d}x = \frac{1}{\sqrt{2}}\cdot\frac{1}{2}t^{-\frac{1}{2}}\mathrm{d}t = \frac{t^{-\frac{1}{2}}}{2\sqrt{2}}\mathrm{d}t$

When $\quad x = 0, \quad t = 0$

When $\quad x \to \infty, \quad t \to \infty$

$$\int_0^\infty (x^2+4)\,e^{-2x^2}\,\mathrm{d}x = \int_0^\infty \left(\frac{t}{2}+4\right)e^{-t}\cdot\frac{t^{-\frac{1}{2}}}{2\sqrt{2}}\mathrm{d}t$$

$$= \frac{1}{4\sqrt{2}}\int_0^\infty e^{-t}t^{\frac{1}{2}}\,\mathrm{d}t + \frac{2}{\sqrt{2}}\int_0^\infty e^{-t}t^{-\frac{1}{2}}\,\mathrm{d}t$$

$$= \frac{1}{4\sqrt{2}}\,\overline{\left|\frac{3}{2}\right.} + \frac{2}{\sqrt{2}}\,\overline{\left|\frac{1}{2}\right.} = \frac{1}{4\sqrt{2}}\cdot\frac{1}{2}\,\overline{\left|\frac{1}{2}\right.} + \frac{2}{\sqrt{2}}\,\overline{\left|\frac{1}{2}\right.}$$

$$= \frac{1}{8\sqrt{2}}\sqrt{\pi} + \frac{2}{\sqrt{2}}\sqrt{\pi} = \frac{17\sqrt{\pi}}{8\sqrt{2}}$$

Example 6

Evaluate $\int_0^\infty x^n e^{-\sqrt{ax}}\mathrm{d}x$.

Solution

Let $\quad \sqrt{ax} = t,\ x = \frac{t^2}{a},\ \mathrm{d}x = \frac{2t}{a}\mathrm{d}t$

When $\quad x = 0, \quad t = 0$

When $\quad x \to \infty, \quad t \to \infty$

$$\int_0^\infty x^n e^{-\sqrt{ax}}\mathrm{d}x = \int_0^\infty \left(\frac{t^2}{a}\right)^n e^{-t}\cdot\frac{2t}{a}\mathrm{d}t$$

$$= \frac{2}{a^{n+1}}\int_0^\infty e^{-t}t^{2n+1}\mathrm{d}t$$

$$= \frac{2}{a^{n+1}}\,\overline{|2n+2}$$

Example 7

Evaluate $\int_0^\infty \sqrt{x}\,e^{-x^2}\mathrm{d}x\cdot\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\mathrm{d}x$.

Solution

Let $x^2 = t,\ x = t^{\frac{1}{2}},\ \mathrm{d}x = \frac{1}{2}t^{-\frac{1}{2}}\mathrm{d}t$

When $x = 0, \quad t = 0$

When $x \to \infty, \quad t \to \infty$

$$\int_0^\infty \sqrt{x}\, e^{-x^2}\mathrm{d}x \cdot \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\mathrm{d}x = \int_0^\infty t^{\frac{1}{4}} e^{-t} \cdot \frac{1}{2}t^{-\frac{1}{2}}\mathrm{d}t \cdot \int_0^\infty \frac{e^{-t}}{t^{\frac{1}{4}}} \cdot \frac{1}{2}t^{-\frac{1}{2}}\mathrm{d}t$$

$$= \frac{1}{4}\int_0^\infty e^{-t}t^{-\frac{1}{4}}\mathrm{d}t \cdot \int_0^\infty e^{-t}t^{-\frac{3}{4}}\mathrm{d}t$$

$$= \frac{1}{4}\left\lceil\frac{3}{4} \cdot \left\lceil\frac{1}{4}\right.\right. = \frac{1}{4}\left\lceil 1-\frac{1}{4}\right. \left\lceil\frac{1}{4}\right.$$

$$= \frac{1}{4} \cdot \frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{4} \cdot \pi\sqrt{2} = \frac{\pi}{2\sqrt{2}}$$

Example 8

Evaluate $\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}}\mathrm{d}x \cdot \int_0^\infty x^4 e^{-x^6}\mathrm{d}x.$

Solution

Let $x^3 = t,\ x = t^{\frac{1}{3}},\ \mathrm{d}x = \frac{1}{3}t^{-\frac{2}{3}}\mathrm{d}t$

When $x = 0, \quad t = 0$

When $x \to \infty, \quad t \to \infty$

$$\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}}\mathrm{d}x \cdot \int_0^\infty x^4 e^{-x^6}\mathrm{d}x = \int_0^\infty \frac{e^{-t}}{t^{\frac{1}{6}}} \cdot \frac{1}{3}t^{-\frac{2}{3}}\mathrm{d}t \cdot \int_0^\infty t^{\frac{4}{3}}e^{-t^2} \cdot \frac{1}{3}t^{-\frac{2}{3}}\mathrm{d}t$$

$$= \frac{1}{9}\int_0^\infty e^{-t}t^{-\frac{5}{6}}\mathrm{d}t \cdot \int_0^\infty e^{-t^2}t^{\frac{2}{3}}\mathrm{d}t$$

$$= \frac{1}{9}\left\lceil\frac{1}{6}\right. \cdot \frac{1}{2} \cdot 2\int_0^\infty e^{-t^2}t^{2\left(\frac{5}{6}\right)-1}\mathrm{d}t$$

$$= \frac{1}{9}\left\lceil\frac{1}{6}\right. \cdot \frac{1}{2}\left\lceil\frac{5}{6}\right. \qquad \left[\because 2\int_0^\infty e^{-x^2}x^{2n-1}\mathrm{d}x = \lceil n\right]$$

$$= \frac{1}{18}\left\lceil\frac{1}{6}\right. \left\lceil 1-\frac{1}{6}\right. = \frac{1}{18} \cdot \frac{\pi}{\sin\frac{\pi}{6}} = \frac{\pi}{9}$$

Example 9

Evaluate $\int_0^1 (\log x)^5 \, dx$.

Solution

Let $\log x = -t, x = e^{-t}, dx = -e^{-t} \, dt$

When $x = 0, \quad t \to \infty$

When $x = 1, \quad t = 0$

$$\int_0^1 (\log x)^5 \, dx = \int_\infty^0 (-t)^5 (-e^{-t}) \, dt$$

$$= -\int_0^\infty e^{-t} t^5 dt$$

$$= -\overline{|6} = -120$$

Example 10

Evaluate $\int_0^1 x^3 \log\left(\frac{1}{x}\right)^4 dx$.

Solution

$$\int_0^1 x^3 \log\left(\frac{1}{x}\right)^4 dx = \int_0^1 x^3 \cdot 4\log\left(\frac{1}{x}\right) dx$$

$$= 4\int_0^1 x^3 \log\left(\frac{1}{x}\right) dx$$

Let $\log\left(\frac{1}{x}\right) = t, \ \frac{1}{x} = e^t, \ x = e^{-t}, \ dx = -e^{-t} dt$

When $x = 0, \quad t \to \infty$

When $x = 1, \quad t = 0$

$$\int_0^1 x^3 \log\left(\frac{1}{x}\right)^4 dx = 4\int_\infty^0 e^{-3t} t (-e^{-t}) \, dt$$

$$= 4\int_0^\infty e^{-4t} t^{2-1} dt$$

$$= 4 \cdot \frac{\overline{|2}}{(4)^2} \qquad \left[\because \int_0^\infty e^{-kx} x^{n-1} dx = \frac{\overline{|n}}{k^n}\right]$$

$$= \frac{1}{4}$$

Example 11

Evaluate $\int_0^1 \frac{dx}{\sqrt{x \log\left(\frac{1}{x}\right)}}$.

Solution

$$\int_0^1 \frac{dx}{\sqrt{x\log\left(\frac{1}{x}\right)}} = \int_0^1 x^{-\frac{1}{2}}\left[\log\left(\frac{1}{x}\right)\right]^{-\frac{1}{2}} dx \qquad \ldots (1)$$

Let $\log\left(\frac{1}{x}\right) = t,\ \frac{1}{x} = e^t,\ x = e^{-t},\ dx = -e^{-t}\,dt$

When $x = 0, \quad t \to \infty$

When $x = 1, \quad t = 0$

$$\int_0^1 \frac{1}{\sqrt{x\log\left(\frac{1}{x}\right)}} dx = \int_\infty^0 (e^{-t})^{-\frac{1}{2}} \cdot t^{-\frac{1}{2}}(-e^{-t})\,dt$$

$$= \int_0^\infty e^{-\frac{t}{2}} t^{\frac{1}{2}-1} dt$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^{\frac{1}{2}}} \qquad \left[\because \int_0^\infty e^{-kx}x^{n-1}dx = \frac{\Gamma(n)}{k^n}\right]$$

$$= \sqrt{2\pi}$$

Example 12

Evaluate $\int_0^\infty \frac{x^a}{a^x}\,dx$.

Solution

Let $a^x = e^t,\ x\log a = t,\ dx = \frac{1}{\log a}dt$

When $x = 0, \quad t = 0$

When $x \to \infty, \quad t \to \infty$

$$\int_0^\infty \frac{x^a}{a^x}\,dx = \int_0^\infty \left(\frac{t}{\log a}\right)^a \cdot \frac{1}{e^t} \cdot \frac{1}{\log a}dt$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^\infty e^{-t}t^a dt$$

$$= \frac{1}{(\log a)^{a+1}} \overline{|a+1}$$

$$= \frac{\overline{|a+1}}{(\log a)^{a+1}}$$

Example 13

Evaluate $\int_0^\infty 3^{-4x^2} \mathrm{d}x$.

Solution

Let $3^{-4x^2} = e^{-t}, -4x^2 \log 3 = -t \log e, 4x^2 \log 3 = t$

$$x = \frac{\sqrt{t}}{2\sqrt{\log 3}}, \ \mathrm{d}x = \frac{1}{2\sqrt{\log 3}} \cdot \frac{1}{2\sqrt{t}} \mathrm{d}t$$

When $x = 0, \quad t = 0$

When $x \to \infty, \quad t \to \infty$

$$\int_0^\infty 3^{-4x^2} \mathrm{d}x = \int_0^\infty e^{-t} \cdot \frac{1}{4\sqrt{\log 3}} \cdot \frac{1}{\sqrt{t}} \mathrm{d}t$$

$$= \frac{1}{4\sqrt{\log 3}} \int_0^\infty e^{-t} t^{-\frac{1}{2}} \mathrm{d}t$$

$$= \frac{1}{4\sqrt{\log 3}} \overline{|\tfrac{1}{2}} = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$$

Example 14

Prove that $\int_0^\infty xe^{-ax} \sin bx \, \mathrm{d}x = \frac{2ab}{(a^2+b^2)^2}$.

Solution

$$\int_0^\infty xe^{-ax} \sin bx \, \mathrm{d}x = \int_0^\infty xe^{-ax} \ [\text{Imaginary part of } e^{ibx}] \mathrm{d}x$$

$$= \text{Im. part} \int_0^\infty xe^{-ax} \cdot e^{ibx} \mathrm{d}x$$

$$= \text{Im. part} \int_0^\infty e^{-(a-ib)x} \cdot x \, \mathrm{d}x$$

$$= \text{Im. part} \ \frac{\overline{|2}}{(a-ib)^2} \qquad \left[\because \int_0^\infty e^{-kx} x^{n-1} \mathrm{d}x = \frac{\overline{|n}}{k^n}\right]$$

$$= \text{Im. part} \ \frac{1}{(a^2-b^2)-2iab}$$

$$= \text{Im. part}\left[\frac{(a^2-b^2)+2iab}{(a^2-b^2)^2+4a^2b^2}\right] = \frac{2ab}{(a^2+b^2)^2}$$

EXERCISE 3.1

1. Evaluate the following integrals:

(i) $\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx$ (ii) $\int_0^\infty e^{-\frac{x^2}{4}}\, dx$ (iii) $\int_0^\infty \frac{e^{-\sqrt{x}}}{x^{\frac{7}{4}}}\, dx$

(iv) $\int_0^1 (x \log x)^4\, dx$ (v) $\int_0^1 \sqrt{\log \frac{1}{x}}\, dx$ (vi) $\int_0^1 \frac{dx}{\sqrt{-\log x}}$

(vii) $\int_0^1 x^4 \left(\log \frac{1}{x}\right)^3 dx$ (viii) $\int_0^1 \sqrt[3]{x \log \frac{1}{x}}\, dx$ (ix) $\int_0^\infty 5^{-4x^2}\, dx.$

$\left[\textbf{Ans.:}\right.$ (i) $\frac{315}{16}\sqrt{\pi}$ (ii) $\sqrt{\pi}$ (iii) $\frac{8}{3}\sqrt{\pi}$ (iv) $\frac{4!}{5^5}$ (v) $\frac{\sqrt{\pi}}{2}$ (vi) $\sqrt{\pi}$ (vii) $\frac{6}{625}$ (viii) $\left(\frac{3}{4}\right)^{\frac{4}{3}} \Gamma\left(\frac{4}{3}\right)$ (ix) $\left.\frac{\sqrt{\pi}}{4\sqrt{\log 5}}\right]$

2. Prove that

$$\int_0^\infty x^m e^{-ax^n}\, dx = \frac{1}{na^{\frac{(m+1)}{n}}} \Gamma\left(\frac{m+1}{n}\right).$$

3. Prove that

$$\int_0^\infty x^2 e^{-x^4}\, dx \cdot \int_0^\infty e^{-x^4}\, dx = \frac{\pi}{8\sqrt{2}}.$$

4. Prove that

$$\int_0^\infty \sqrt{x}\, e^{-x^2}\, dx \cdot \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx = \frac{\pi}{2\sqrt{2}}.$$

5. Prove that

$$\int_0^\infty x e^{-x^8}\, dx \cdot \int_0^\infty x^2 e^{-x^4}\, dx = \frac{\pi}{16\sqrt{2}}.$$

6. Prove that

$$\int_0^1 x^m (\log x)^n \, dx = \frac{(-1)^n \overline{|n+1}}{(m+1)^{n+1}}.$$

7. Prove that

$$\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx = \frac{\overline{|n+1}}{(m+1)^{n+1}}.$$

8. Prove that

$$\int_0^\infty x^{m-1} \cos ax \, dx = \frac{\overline{|m}}{a^m} \cos\left(\frac{m\pi}{2}\right).$$

9. Prove that

$$\int_0^\infty x^{n-1} e^{-ax} \sin bx \, dx = \frac{\overline{|n}}{\left(a^2 + b^2\right)^{\frac{n}{2}}} \sin\left(n \tan^{-1} \frac{b}{a}\right).$$

3.4 BETA FUNCTION

Beta function $B(m, n)$ is defined by

$$B\,(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, \quad m > 0, n > 0.$$

$B(m, n)$ is also known as Euler's integral of first kind.

3.4.1 Trigonometric Form of Beta Function

$$B(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1} x \cos^{2n-1} x \, dx$$

Proof: $\qquad B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

Let $\qquad x = \sin^2\theta, \qquad dx = 2\sin\theta\cos\theta\, d\theta$

When $\qquad x = 0, \qquad \theta = 0$

When $\qquad x = 1, \qquad \theta = \frac{\pi}{2}$

$$B(m, n) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Changing the variable θ to x,

$$B(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1} x \cos^{2n-1} x \, dx$$

Corollary: Putting $2m - 1 = p, \quad 2n - 1 = q$

$$m = \frac{p+1}{2}, \qquad n = \frac{q+1}{2}$$

$$B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^p x \cos^q x \, dx$$

3.5 PROPERTIES OF BETA FUNCTIONS

1. *Symmetry*

$$B(m, n) = B(n, m)$$

Proof: $$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

Let $1 - x = t, \quad -dx = dt$

When $x = 0, \quad t = 1$

When $x = 1, \quad t = 0$

$$B(m, n) = \int_1^0 (1-t)^{m-1} t^{n-1}(-dt) = \int_0^1 t^{n-1}(1-t)^{m-1} dt$$

$$= B(n, m).$$

2. *Relation between Beta and Gamma Functions*

$$B(m, n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$$

Proof: By alternate form of Gamma function,

$$\Gamma m \, \Gamma n = 2\int_0^\infty e^{-x^2} x^{2m-1} dx \cdot 2\int_0^\infty e^{-y^2} y^{2n-1} dy$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dx\, dy$$

Changing to polar coordinates $x = r\cos\theta, \ y = r\sin\theta$

$$dx\, dy = r\, dr\, d\theta$$

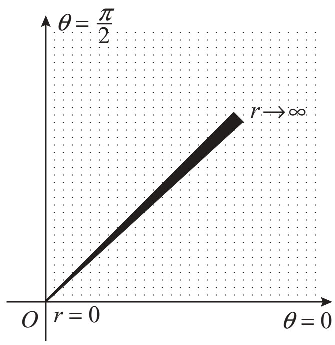


Fig. 3.2

Limits of x $\quad x = 0$ to $x \to \infty$

Limits of y $\quad y = 0$ to $y \to \infty$

This shows that the region of integration is the first quadrant.

Draw an elementary radius vector in the region which starts from pole and extends up to ∞.

Limits of r $\quad r = 0$ to $r \to \infty$

Limits of θ $\quad \theta = 0$ to $\theta = \frac{\pi}{2}$

$$\Gamma m \, \Gamma n = 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} (r\cos\theta)^{2m-1}(r\sin\theta)^{2n-1} r\, dr\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} (\cos\theta)^{2m-1}(\sin\theta)^{2n-1} d\theta \cdot \int_0^{\frac{\pi}{2}} e^{-r^2} r^{2(m+n)-1} dr$$

$$= 4 \cdot \frac{1}{2} B(m,n) \cdot \frac{1}{2} \Gamma(m+n)$$

$$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

3. ***Duplication Formula***

$$\boldsymbol{\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}}}$$

Proof: $B(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Putting $n = m$,

$$B(m, m) = 2\int_0^{\frac{\pi}{2}} (\sin\theta \cos\theta)^{2m-1} \, d\theta$$

$$\frac{\Gamma(m)\,\Gamma(m)}{\Gamma(2m)} = \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} (\sin 2\theta)^{2m-1} \, d\theta$$

Let $2\theta = t$, $d\theta = \frac{1}{2} dt$

When $\theta = 0$, $t = 0$

When $\theta = \frac{\pi}{2}$, $t = \pi$

$$\frac{\Gamma(m) \cdot \Gamma(m)}{\Gamma(2m)} = \frac{2}{2^{2m-1}} \int_0^{\pi} (\sin t)^{2m-1} \cdot \frac{1}{2} dt$$

$$= \frac{1}{2^{2m-1}} \cdot 2\int_0^{\frac{\pi}{2}} (\sin t)^{2m-1} (\cos t)^0 dt \qquad \left[\because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \text{ if } f(2a-x) = f(x)\right]$$

$$= \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} (\sin t)^{2m-1} (\cos t)^{2\left(\frac{1}{2}\right)-1} dt$$

$$= \frac{1}{2^{2m-1}} B\left(m, \frac{1}{2}\right)$$

$$= \frac{1}{2^{2m-1}} \frac{\Gamma(m)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\frac{\Gamma(m)\,\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \frac{\Gamma(m)\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}}.$$

Example 1

Find the value of

(i) $B\left(\frac{3}{2}, \frac{1}{2}\right)$ (ii) $B\left(\frac{4}{3}, \frac{5}{3}\right)$

Solution

(i)

$$B\left(\frac{3}{2}, \frac{1}{2}\right)=\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}$$

$$=\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{1}=\frac{1}{2}\sqrt{\pi}\cdot\sqrt{\pi}$$

$$=\frac{\pi}{2}$$

(ii)

$$B\left(\frac{4}{3},\frac{5}{3}\right)=\frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)}{\Gamma(3)}$$

$$=\frac{1}{2}\Gamma\left(\frac{1}{3}+1\right)\cdot\Gamma\left(\frac{2}{3}+1\right)=\frac{1}{2}\cdot\frac{1}{3}\Gamma\left(\frac{1}{3}\right)\cdot\frac{2}{3}\Gamma\left(\frac{2}{3}\right)$$

$$=\frac{1}{9}\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)=\frac{1}{9}\cdot\frac{\pi}{\sin\frac{\pi}{3}} \qquad \left[\because \Gamma(n)\,\Gamma(1-n)=\frac{\pi}{\sin n\pi}\right]$$

$$=\frac{1}{9}\cdot\frac{2\pi}{\sqrt{3}}=\frac{2\pi}{9\sqrt{3}}$$

Example 2

If $B(n, 3)=\frac{1}{60}$ *and n is a positive integer, find the value of n.*

Solution

$$B(n, 3)=\frac{1}{60}$$

$$\frac{\Gamma(n)\,\Gamma(3)}{\Gamma(n+3)}=\frac{1}{60}$$

$$\frac{\Gamma(n)\cdot 2}{(n+2)(n+1)n\,\Gamma(n)}=\frac{1}{60}$$

$$n^3 + 3n^2 + 2n = 120$$
$$n^3 + 3n^2 + 2n - 120 = 0$$
$$n = 4, -3.5$$

But n is a positive integer.

Hence, $n = 4$.

Example 3

Prove that $B(n, n) = \dfrac{\sqrt{\pi}}{2^{2n-1}} \cdot \dfrac{\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)}$.

Solution

$$B(n, n) = \frac{\Gamma(n)\,\Gamma(n)}{\Gamma(2n)} = \frac{\Gamma(n)}{\Gamma(2n)} \cdot \frac{\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)}$$

$$= \frac{\Gamma(n)}{\Gamma(2n)} \cdot \frac{1}{\Gamma\left(n+\frac{1}{2}\right)} \cdot \frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}} \qquad \text{[By Duplication formula]}$$

$$= \frac{\sqrt{\pi}}{2^{2n-1}} \cdot \frac{\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)}$$

Example 4

Prove that $B(n, n).\ B\left(n+\frac{1}{2},\ n+\frac{1}{2}\right) = \dfrac{\pi}{n} 2^{1-4n}$.

Solution

$$B(n, n)\, B\left(n+\frac{1}{2},\ n+\frac{1}{2}\right) = \frac{\Gamma(n)\,\Gamma(n)}{\Gamma(2n)} \cdot \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n+1)}$$

$$= \frac{\left(\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)\right)^2}{\Gamma(2n) \cdot 2n\,\Gamma(2n)} = \frac{1}{2n}\left(\frac{\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n)}\right)^2$$

$$= \frac{1}{2n}\left(\frac{\sqrt{\pi}}{2^{2n-1}}\right)^2 = \frac{\pi}{n} 2^{1-4n}$$

Example 5

Prove that $\Gamma(n)\,\Gamma\left(\dfrac{1-n}{2}\right) = \dfrac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}{2^{1-p}\cos\frac{n\pi}{2}}$.

Solution

We know that

$$\Gamma n\,\Gamma 1-n = \frac{\pi}{\sin n\pi}$$

Replacing n by $\frac{n+1}{2}$,

$$\Gamma\frac{n+1}{2}\,\Gamma\left(1-\frac{n+1}{2}\right) = \frac{\pi}{\sin\left(\frac{n+1}{2}\right)\pi}$$

$$\Gamma\frac{n+1}{2}\,\Gamma\frac{1-n}{2} = \frac{\pi}{\sin\left(\frac{\pi}{2}+\frac{n\pi}{2}\right)}$$

$$\Gamma\frac{n}{2}\,\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\Gamma\frac{1-n}{2} = \frac{\pi\,\Gamma\frac{n}{2}}{\cos\frac{n\pi}{2}}$$

$$\frac{\sqrt{\pi}\,\Gamma n}{2^{n-1}}\,\Gamma\frac{1-n}{2} = \frac{\pi\,\Gamma\frac{n}{2}}{\cos\frac{n\pi}{2}}$$

$$\Gamma n\,\Gamma\frac{1-n}{2} = \frac{\sqrt{\pi}\,\Gamma\frac{n}{2}}{2^{1-n}\cos\frac{n\pi}{2}}$$

EXERCISE 3.2

1. Find the value of

(i) $B\left(\frac{5}{2}, \frac{3}{2}\right)$ (ii) $B\left(\frac{1}{2}, \frac{2}{3}\right)$

$\left[\text{Ans.: } (i)\ \frac{\pi}{16} \quad (ii)\ \frac{2\pi}{\sqrt{3}}\right]$

2. If $B(n, 2) = \frac{1}{42}$ and n is a positive integer, find the value of n.

[Ans.: $n = 6$]

3. Prove that

$$B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{1}{2^{2n}}\cdot\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}\sqrt{\pi}.$$

4. Prove that

$B(m, n) = B(m, n+1) + B(m+1, n)$.

5. Prove that $\Gamma\left(\frac{3}{2}-n\right)\Gamma\left(\frac{3}{2}+n\right)$

$= \left(\frac{1}{4}-n^2\right)\pi \sec n\pi, \ (-1 < 2n < 1)$.

Problems Based on Definition of Beta Function

Example 1

Evaluate $\int_0^1 x^3 \left(1-\sqrt{x}\right)^5 \mathrm{d}x$.

Solution

Let $\sqrt{x} = t$, $x = t^2$, $\mathrm{d}x = 2t\,\mathrm{d}t$

When $x = 0, \quad t = 0$

When $x = 1, \quad t = 1$

$$\int_0^1 x^3 \left(1-\sqrt{x}\right)^5 \mathrm{d}x = \int_0^1 t^6 (1-t)^5\, 2t\,\mathrm{d}t$$

$$= 2\int_0^1 t^7 (1-t)^5\,\mathrm{d}t = 2B(8, 6)$$

$$= 2\frac{\Gamma 8\, \Gamma 6}{\Gamma 14} = 2\frac{7!\,5!}{13!} = \frac{1}{5148}$$

Example 2

Evaluate $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\mathrm{d}x \cdot \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^4}}$.

Solution

Let $x^4 = t$, $x = t^{\frac{1}{4}}$, $\mathrm{d}x = \frac{1}{4}t^{-\frac{3}{4}}\mathrm{d}t$

When $x = 0, \quad t = 0$

When $x = 1, \quad t = 1$

$$\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\mathrm{d}x \cdot \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^4}} = \int_0^1 \frac{t^{\frac{1}{2}}}{\sqrt{1-t}} \cdot \frac{1}{4}t^{-\frac{3}{4}}\,\mathrm{d}t \cdot \int_0^1 \frac{1}{\sqrt{1-t}} \cdot \frac{1}{4}t^{-\frac{3}{4}}\,\mathrm{d}t$$

$$= \frac{1}{16}\int_0^1 t^{-\frac{1}{4}} (1-t)^{-\frac{1}{2}}\,\mathrm{d}t \cdot \int_0^1 t^{-\frac{3}{4}} (1-t)^{-\frac{1}{2}}\,\mathrm{d}t$$

$$= \frac{1}{16} B\left(\frac{3}{4}, \frac{1}{2}\right) \cdot B\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{16} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)} \cdot \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{1}{16} \frac{\sqrt{\pi}}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} \cdot \Gamma\left(\frac{1}{4}\right)\sqrt{\pi} = \frac{\pi}{4}$$

Example 3

Evaluate $\int_0^1 \sqrt{1-y^4}\, dy.$

Solution

Let $y^4 = t,\ y = t^{\frac{1}{4}},\ dy = \frac{1}{4} t^{-\frac{3}{4}} dt$

When $y = 0, \quad t = 0$

When $y = 1, \quad t = 1$

$$\int_0^1 \sqrt{1-y^4}\, dy = \int_0^1 (1-t)^{\frac{1}{2}} \cdot \frac{1}{4} t^{-\frac{3}{4}}\, dt$$

$$= \frac{1}{4} B\left(\frac{3}{2}, \frac{1}{4}\right) = \frac{1}{4} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{7}{4}\right)}$$

$$= \frac{1}{4} \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\frac{3}{4}\Gamma\left(\frac{3}{4}\right)} = \frac{1}{6}\sqrt{\pi} \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}$$

$$= \frac{\sqrt{\pi}}{6} \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)} = \frac{\sqrt{\pi}}{6} \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\frac{\pi}{\sin\frac{\pi}{4}}}$$

$$= \frac{\sqrt{\pi}}{6} \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\pi\sqrt{2}} = \frac{1}{6\sqrt{2\pi}} \left(\Gamma\left(\frac{1}{4}\right)\right)^2$$

Example 4

Evaluate $\int_0^2 y^4 \left(8-y^3\right)^{-\frac{1}{3}} dy.$

Solution

Let $y^3 = 8t,\ y = 2t^{\frac{1}{3}},\ \mathrm{d}y = 2\cdot\frac{1}{3}t^{-\frac{2}{3}}\mathrm{d}t$

When $\qquad y = 0, \qquad t = 0$

When $\qquad y = 2, \qquad t = 1$

$$\int_0^2 y^4\left(8-y^3\right)^{-\frac{1}{3}}\mathrm{d}y = \int_0^1\left(2t^{\frac{1}{3}}\right)^4(8-8t)^{-\frac{1}{3}}\frac{2}{3}t^{-\frac{2}{3}}\mathrm{d}t$$

$$= \frac{16}{3}\int_0^1 t^{\frac{2}{3}}(1-t)^{-\frac{1}{3}}\,\mathrm{d}t \;= \frac{16}{3}B\left(\frac{5}{3},\frac{2}{3}\right)$$

$$= \frac{16}{3}\,\frac{\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{7}{3}\right)} = \frac{16}{3}\,\frac{\frac{2}{3}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\frac{4}{3}\cdot\frac{1}{3}\Gamma\left(\frac{1}{3}\right)} = 8\,\frac{\left(\Gamma\left(\frac{2}{3}\right)\right)^2}{\Gamma\left(\frac{1}{3}\right)}$$

Example 5

Evaluate $\displaystyle\int_0^{2a} x^2\sqrt{2ax-x^2}\,\mathrm{d}x.$

Solution

$$\int_0^{2a} x^2\sqrt{2ax-x^2}\,\mathrm{d}x = \int_0^{2a} x^{\frac{5}{2}}\sqrt{2a-x}\,\mathrm{d}x$$

Let $\quad x = 2at,\ \mathrm{d}x = 2a\,\mathrm{d}t$

When $\qquad x = 0, \qquad t = 0$

When $\qquad x = 2a, \qquad t = 1$

$$\int_0^{2a} x^2\sqrt{2ax-x^2}\,\mathrm{d}x = \int_0^1 (2at)^{\frac{5}{2}}\sqrt{2a-2at}\cdot 2a\,\mathrm{d}t$$

$$= 16a^4\int_0^1 t^{\frac{5}{2}}(1-t)^{\frac{1}{2}}\,\mathrm{d}t \;= 16a^4\,B\left(\frac{7}{2},\frac{3}{2}\right)$$

$$= 16a^4\,\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(5)} = \frac{16a^4}{24}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{15\pi a^4}{24}$$

Example 6

Evaluate $\displaystyle\int_0^3 \frac{x^{\frac{3}{2}}}{\sqrt{3-x}}\,\mathrm{d}x\cdot\int_0^1\frac{\mathrm{d}x}{\sqrt{1-x^{\frac{1}{4}}}}.$

Solution

$$I_1 = \int_0^3 \frac{x^{\frac{3}{2}}}{\sqrt{3-x}}\,dx$$

Let $x = 3t, \quad dx = 3\,dt$

When $\quad x = 0, \quad t = 0$

When $\quad x = 3, \quad t = 1$

$$I_1 = \int_0^1 \frac{(3t)^{\frac{3}{2}}}{\sqrt{3-3t}}\cdot 3\,dt$$

$$= 9\int_0^1 t^{\frac{3}{2}}(1-t)^{-\frac{1}{2}}\,dt = 9\,B\left(\frac{5}{2}, \frac{1}{2}\right)$$

$$= 9\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)} = \frac{9}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{27\pi}{8}$$

$$I_2 = \int_0^1 \frac{dx}{\sqrt{1-x^{\frac{1}{4}}}}$$

Let $x^{\frac{1}{4}} = t,\ x = t^4,\ dx = 4t^3\,dt$

When $\quad x = 0, \quad t = 0$

When $\quad x = 1, \quad t = 1$

$$I_2 = \int_0^1 \frac{4t^3}{\sqrt{1-t}}\,dt$$

$$= 4\int_0^1 t^3(1-t)^{-\frac{1}{2}}\,dt = 4\,B\left(4, \frac{1}{2}\right)$$

$$= 4\frac{\Gamma(4)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{9}{2}\right)} = 4\frac{3!\,\Gamma\left(\frac{1}{2}\right)}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)} = \frac{128}{35}$$

Hence,
$$\int_0^3 \frac{x^{\frac{3}{2}}}{\sqrt{3-x}}\,dx\cdot\int_0^1 \frac{dx}{\sqrt{1-x^{\frac{1}{4}}}} = \frac{27\pi}{8}\cdot\frac{128}{35} = \frac{432\pi}{35}$$

Example 7

Prove that $\displaystyle\int_0^a \frac{dx}{(a^n - x^n)^{\frac{1}{n}}} = \frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi}{n}\right).$

Solution

Let $x^n = a^n t,\ x = at^{\frac{1}{n}},\ dx = \dfrac{a}{n}t^{\frac{1}{n}-1}\,dt$

When $x = 0, \quad t = 0$

When $x = a, \quad t = 1$

$$\int_0^a \frac{dx}{(a^n - x^n)^{\frac{1}{n}}} = \int_0^1 \frac{1}{(a^n - a^n t)^{\frac{1}{n}}} \cdot \frac{a}{n} t^{\frac{1}{n}-1} dt$$

$$= \frac{1}{n}\int_0^1 (1-t)^{-\frac{1}{n}} t^{\frac{1}{n}-1} dt = \frac{1}{n} B\left(-\frac{1}{n}+1, \frac{1}{n}\right)$$

$$= \frac{1}{n} \cdot \frac{\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(\frac{1}{n}\right)}{\Gamma(1)} = \frac{1}{n} \cdot \frac{\pi}{\sin\frac{\pi}{n}} \qquad \left[\because \Gamma(1-n)\,\Gamma(n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi}{n}\right)$$

Example 8

Prove that $\int_a^b (x-a)^m (b-x)^n dx = (b-a)^{m+n+1} B(m+1, n+1)$ *and hence, deduce that* $\int_5^9 \sqrt[4]{(x-5)(9-x)}\, dx = \dfrac{2\left(\Gamma\frac{1}{4}\right)^2}{3\sqrt{\pi}}$.

Solution

Let $(x - a) = (b - a)\, t, \qquad dx = (b - a)\, dt$

When $x = a, \quad t = 0$

When $x = b, \quad t = 1$

$$\int_a^b (x-a)^m (b-x)^n dx = \int_0^1 [(b-a)t]^m [b - \{a + (b-a)t\}]^n (b-a)dt$$

$$= (b-a)^{m+n+1}\int_0^1 t^m (1-t)^n dt$$

$$= (b-a)^{m+n+1} B(m+1, n+1)$$

Putting $a = 5$, $b = 9$, $m = \frac{1}{4}$, $n = \frac{1}{4}$ in the above integral,

$$\int_5^9 (x-5)^{\frac{1}{4}}(9-x)^{\frac{1}{4}} dx = (9-5)^{\frac{1}{4}+\frac{1}{4}+1} B\left(\frac{1}{4}+1, \frac{1}{4}+1\right)$$

$$= 2^3 \frac{\Gamma\frac{5}{4}\,\Gamma\frac{5}{4}}{\Gamma\frac{5}{2}} = 8\frac{\left(\frac{1}{4}\,\Gamma\frac{1}{4}\right)^2}{\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}} = \frac{2\left(\Gamma\frac{1}{4}\right)^2}{3\sqrt{\pi}}$$

Example 9

Prove that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx = \frac{B(m,n)}{(a+b)^m\, a^n}$ *and hence, evaluate* $\int_0^1 \frac{x^2-2x^3+x^4}{(1+x)^6}\,dx.$

Solution

Let $x = \frac{at}{a+b-bt}$, $\quad dx = \frac{a(a+b-bt)-at(-b)}{(a+b-bt)^2}dt = \frac{a(a+b)}{(a+b-bt)^2}dt$

When $x = 0, \quad t = 0$

When $x = 1, \quad t = 1$

Also, $1-x = 1-\frac{at}{a+b-bt} = \frac{(a+b)(1-t)}{a+b-bt}$

$$a+bx = a+\frac{bat}{a+b-bt} = \frac{a(a+b)}{a+b-bt}$$

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}dx = \int_0^1 \frac{\left(\frac{at}{a+b-bt}\right)^{m-1}\left[\frac{(a+b)(1-t)}{a+b-bt}\right]^{n-1}}{\left[\frac{a(a+b)}{a+b-bt}\right]^{m+n}}\cdot\frac{a(a+b)}{(a+b-bt)^2}dt$$

$$= \frac{1}{(a+b)^m\, a^n}\int_0^1 t^{m-1}(1-t)^{n-1}dt = \frac{1}{(a+b)^m\, a^n}B(m,\, n)$$

Putting $a = 1, b = 1, m = 3, n = 3$ in the above integral

$$\int_0^1 \frac{x^2(1-x)^2}{(1+x)^6}dx = \frac{1}{(1+1)^3\cdot 1^3}B(3,\, 3)$$

$$\int_0^1 \frac{x^2-2x^3+x^4}{(1+x)^6}dx = \frac{1}{8}\frac{\Gamma 3\,\Gamma 3}{\Gamma 6} = \frac{1}{8}\cdot\frac{4}{120} = \frac{1}{240}$$

Example 10

Prove that $\int_0^1 \frac{(1-x^4)^{\frac{3}{4}}}{(1+x^4)^2}dx = \frac{1}{4}\cdot\frac{1}{2^{\frac{1}{4}}}B\left(\frac{1}{4},\ \frac{7}{4}\right).$

Solution

Let $x^4 = t,\ x = t^{\frac{1}{4}},\ dx = \frac{1}{4}t^{-\frac{3}{4}}dt$

When $x = 0, \quad t = 0$

When $x = 1, \quad t = 1$

$$\int_0^1 \frac{(1-x^4)^{\frac{3}{4}}}{(1+x^4)^2}\,dx = \int_0^1 \frac{(1-t)^{\frac{3}{4}}}{(1+t)^2}\cdot\frac{1}{4}t^{-\frac{3}{4}}dt = \frac{1}{4}\int_0^1 \frac{t^{-\frac{3}{4}}(1-t)^{\frac{3}{4}}}{(1+t)^2}\,dt$$

Let $t = \dfrac{u}{2-u}$, $\quad dt = \dfrac{(2-u)-u(-1)}{(2-u)^2}du = \dfrac{2}{(2-u)^2}du$

When $\quad t = 0, \quad u = 0$

When $\quad t = 1, \quad u = 1$

$$\int_0^1 \frac{(1-x^4)^{\frac{3}{4}}}{(1+x^4)^2}\,dx = \frac{1}{4}\int_0^1 \frac{\left(\frac{u}{2-u}\right)^{-\frac{3}{4}}\left(1-\frac{u}{2-u}\right)^{\frac{3}{4}}}{\left(1+\frac{u}{2-u}\right)^2}\cdot\frac{2}{(2-u)^2}\,du$$

$$= \frac{2}{4}\int_0^1 \frac{u^{-\frac{3}{4}}(2-2u)^{\frac{3}{4}}}{2^2}\,du = \frac{1}{4}\cdot\frac{1}{2^{\frac{1}{4}}}\int_0^1 u^{-\frac{3}{4}}(1-u)^{\frac{3}{4}}\,du$$

$$= \frac{1}{4}\cdot\frac{1}{2^{\frac{1}{4}}}B\left(\frac{1}{4}, \frac{7}{4}\right)$$

EXERCISE 3.3

1. Evaluate the following integrals:

(i) $\int_0^1 \sqrt{1-x^m}\,dx$ (ii) $\int_0^1 \frac{dx}{\sqrt{1-x^6}}$ (iii) $\int_0^1 \left(1-x^{\frac{1}{4}}\right)^{\frac{2}{3}} dx$

(iv) $\int_0^2 x^2(2-x)^{-\frac{1}{2}}dx$ (v) $\int_0^a x^4\sqrt{a^2-x^2}\,dx$ (vi) $\int_0^{\frac{1}{2}} x^3\sqrt{1-4x^2}\,dx$

Ans.:
(i) $\frac{1}{m}B\left(\frac{1}{m}, \frac{3}{2}\right)$ (ii) $\frac{1}{8}B\left(\frac{1}{8}, \frac{1}{2}\right)$
(iii) $\frac{128}{1155}$ (iv) $\frac{64\sqrt{2}}{15}$
(v) $\frac{\pi a^6}{32}$ (vi) $\frac{1}{120}$

2. Prove that

(i) $\int_3^7 \sqrt[4]{(x-3)(7-x)}\,dx = \dfrac{2\left(\boxed{\frac{1}{4}}\right)^2}{3\sqrt{\pi}}$ (ii) $\int_5^6 (x-5)^5(6-x)^6\,dx = \dfrac{5!\,6!}{12!}$

3. Prove that

$$\int_0^1 \frac{x^{-\frac{1}{3}}(1-x)^{-\frac{2}{3}}}{(1+2x)}\,dx = \frac{\pi}{3^{\frac{7}{6}}}.$$

4. Prove that

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(1+x)^{m+n}}\,dx = \frac{B(m,n)}{2^m}$$ and hence, evaluate $\int_0^1 \frac{x^3-2x^2+x}{(1+x)^5}\,dx.$

$\left[\textbf{Ans.:}\ \frac{1}{48}\right]$

5 Prove that

$$\int_0^1 \frac{x^{n-1}}{(1+cx)(1-x)^n}\,dx$$

$$= \frac{1}{(1+c)^n}\cdot\frac{\pi}{\sin n\pi},\quad 0<n<1.$$

Problems Based on Trigonometric Form of Beta Function

Example 1

Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\,d\theta.$

Solution

$$\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\,d\theta = \int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{1}{2}}(\sin\theta)^{-\frac{1}{2}}\,d\theta$$

$$= \frac{1}{2}B\left(\frac{\frac{1}{2}+1}{2}, \frac{-\frac{1}{2}+1}{2}\right)$$

$$= \frac{1}{2}B\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{1}{2}\frac{\left\lceil\frac{3}{4}\right.\left\lceil\frac{1}{4}\right.}{\lceil 1}$$

$$= \frac{1}{2}\left\lceil 1-\frac{1}{4}\right.\left\lceil\frac{1}{4}\right. = \frac{1}{2}\cdot\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}}$$

Example 2

Evaluate $\int_0^{\frac{\pi}{4}} \cos^3 2\theta \sin^4 4\theta\,d\theta.$

Solution

$$\int_0^{\frac{\pi}{4}} \cos^3 2\theta \sin^4 4\theta\,d\theta = \int_0^{\frac{\pi}{4}} \cos^3 2\theta (2\sin 2\theta\cos 2\theta)^4\,d\theta$$

$$= 16\int_0^{\frac{\pi}{4}} \cos^7 2\theta \sin^4 2\theta \, d\theta$$

Let $2\theta = t$, $\quad d\theta = \frac{1}{2}dt$

When $\quad \theta = 0, \quad t = 0$

When $\quad \theta = \frac{\pi}{4}, \quad t = \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{4}} \cos^3 2\theta \sin^4 4\theta \, d\theta = 16\int_0^{\frac{\pi}{2}} \sin^4 t \cdot \cos^7 t \cdot \frac{1}{2} dt$$

$$= 8 \cdot \frac{1}{2} B\left(\frac{5}{2}, 4\right) = 4 \frac{\left\lceil \frac{5}{2} \right. \left\lceil 4 \right.}{\left\lceil \frac{13}{2} \right.}$$

$$= 4 \cdot \frac{\frac{3}{2} \cdot \frac{1}{2} \left\lceil \frac{1}{2} \right. \cdot 3!}{\frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \left\lceil \frac{1}{2} \right.} = \frac{128}{1155}$$

Example 3

Evaluate $\int_0^{2\pi} \sin^2\theta (1 + \cos\theta)^4 d\theta.$

Solution

$$I = \int_0^{2\pi} \sin^2\theta \, (1 + \cos\theta)^4 d\theta$$

$$= \int_0^{2\pi} \left(2 \sin\frac{\theta}{2} \cos\frac{\theta}{2}\right)^2 \left(2\cos^2\frac{\theta}{2}\right)^4 d\theta$$

$$= 2^6 \int_0^{2\pi} \sin^2\frac{\theta}{2} \cos^{10}\frac{\theta}{2} d\theta$$

Let $\frac{\theta}{2} = t$, $d\theta = 2dt$

When $\quad \theta = 0, \quad t = 0$

When $\quad \theta = 2\pi, \quad t = \pi$

$$I = 2^6 \int_0^{\pi} \sin^2 t \cos^{10} t \cdot 2dt$$

$$= 2^7 \cdot 2\int_0^{\frac{\pi}{2}} \sin^2 t \cos^{10} t \, dt \qquad \left[\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(2a - x) = f(x)\right]$$

$$= 2^8 \cdot \frac{1}{2} B\left(\frac{3}{2}, \frac{11}{2}\right) = 2^7 \frac{\left\lceil \frac{3}{2} \right. \left\lceil \frac{11}{2} \right.}{\left\lceil 7 \right.}$$

$$= \frac{2^7}{6!} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{21\pi}{8}$$

Example 4

Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos\theta + \sin\theta)^{\frac{1}{3}} \, d\theta.$

Solution

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos\theta + \sin\theta)^{\frac{1}{3}} \, d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right)\right]^{\frac{1}{3}} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2^{\frac{1}{6}} \left(\sin\frac{\pi}{4}\cos\theta + \cos\frac{\pi}{4}\sin\theta\right)^{\frac{1}{3}} d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2^{\frac{1}{6}} \left[\sin\left(\frac{\pi}{4} + \theta\right)\right]^{\frac{1}{3}} d\theta$$

Let $\frac{\pi}{4} + \theta = t, \qquad d\theta = dt$

When $\theta = -\frac{\pi}{4}, \qquad t = 0$

When $\theta = \frac{\pi}{4}, \qquad t = \frac{\pi}{2}$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos\theta + \sin\theta)^{\frac{1}{3}} \, d\theta = 2^{\frac{1}{6}} \int_0^{\frac{\pi}{2}} (\sin t)^{\frac{1}{3}} \, dt$$

$$= 2^{\frac{1}{6}} \int_0^{\frac{\pi}{2}} (\sin t)^{\frac{1}{3}} (\cos t)^0 \, dt = \frac{2^{\frac{1}{6}}}{2} B\left(\frac{4}{6}, \frac{1}{2}\right)$$

$$= \frac{1}{2^{\frac{5}{6}}} \cdot \frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{6}\right)} = \frac{1}{2^{\frac{5}{6}}} \frac{\Gamma\left(\frac{2}{3}\right)\sqrt{\pi}}{\frac{1}{6}\Gamma\left(\frac{1}{6}\right)} = \frac{6\sqrt{\pi}}{2^{\frac{5}{6}}} \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{6}\right)}$$

Example 5

Prove that $\int_0^{\frac{\pi}{2}} \tan^n x \, dx = \frac{\pi}{2} \sec\left(\frac{n\pi}{2}\right).$

Solution

$$\int_0^{\frac{\pi}{2}} \tan^n x \, dx = \int_0^{\frac{\pi}{2}} (\sin x)^n \, (\cos x)^{-n} \, dx$$

$$= \frac{1}{2} B\left(\frac{n+1}{2}, \frac{-n+1}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{-n+1}{2}\right)}{\Gamma(1)}$$

$$= \frac{1}{2}\left\lceil\frac{n+1}{2}\right.\left\lceil 1-\frac{n+1}{2}\right.$$

$$= \frac{1}{2}\cdot\frac{\pi}{\sin\left(\frac{n+1}{2}\right)\pi} \qquad \left[\because \lceil n \lceil 1-n = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{\pi}{2}\cdot\frac{1}{\sin\left(\frac{\pi}{2}+\frac{n\pi}{2}\right)} = \frac{\pi}{2}\cdot\frac{1}{\cos\frac{n\pi}{2}} = \frac{\pi}{2}\sec\left(\frac{n\pi}{2}\right).$$

Example 6

Evaluate $\int_0^{\pi} x\sin^7 x\cos^4 x\,dx.$

Solution

$$\int_0^{\pi} x\sin^7 x\cos^4 x\,dx = \int_0^{\pi}(\pi-x)\sin^7(\pi-x)\cos^4(\pi-x)dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \pi\int_0^{\pi}\sin^7 x\cos^4 x\,dx - \int_0^{\pi} x\sin^7 x\cos^4 x\,dx$$

$$2\int_0^{\pi} x\sin^7 x\cos^4 x\,dx = \pi\int_0^{\pi}\sin^7 x\cos^4 x\,dx$$

$$= \pi\left[\int_0^{\frac{\pi}{2}}\sin^7 x\cos^4 x\,dx + \int_0^{\frac{\pi}{2}}\sin^7(\pi-x)\cos^4(\pi-x)dx\right]$$

$$\left[\because \int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx\right]$$

$$= 2\pi\int_0^{\frac{\pi}{2}}\sin^7 x\cos^4 x\,dx = 2\pi\cdot\frac{1}{2}B\left(4,\frac{5}{2}\right)$$

$$= \pi\frac{\lceil 4 \left\lceil\frac{5}{2}\right.}{\left\lceil\frac{13}{2}\right.} = \pi\frac{3!\left\lceil\frac{5}{2}\right.}{\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\left\lceil\frac{5}{2}\right.}$$

$$\int_0^{\pi} x\sin^7 x\cos^4 x\,dx = \frac{16\pi}{1155}$$

EXERCISE 3.4

1. Evaluate the following integrals:

(i) $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta$ (ii) $\int_0^{\frac{\pi}{6}} \cos^6 3\theta \sin^2 6\theta d\theta$

(iii) $\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\sqrt{3}\sin\theta + \cos\theta\right)^{\frac{1}{4}} d\theta$ (iv) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta\,(1+\sin\theta)^2 d\theta$

(v) $\int_0^{2\pi} \sin^2\theta\,(1+\cos\theta)^4 d\theta$ (vi) $\int_0^{\pi} x \sin^5 x \cos^6 x\, dx$

$$\left[\textbf{Ans.:}\ (i)\ \frac{\pi}{\sqrt{2}} \quad (ii)\ \frac{7\pi}{384} \quad (iii)\ 2^{-\frac{3}{4}}\sqrt{\pi}\frac{\Gamma\left(\frac{5}{8}\right)}{\Gamma\left(\frac{9}{8}\right)} \quad (iv)\ \frac{8}{5} \quad (v)\ \frac{21\pi}{8} \quad (iv)\ \frac{8\pi}{693}\right]$$

2. Prove that

$$\int_0^{\frac{\pi}{2}} (\sin x)^{2n} dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n(n!)}\cdot\frac{\pi}{2}.$$

3.6 BETA FUNCTION AS IMPROPER INTEGRAL

$$B(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

Proof: Let $x = \tan^2\theta$, $dx = 2\tan\theta\sec^2\theta\, d\theta$

When $x = 0,\quad \theta = 0$

When $x \to \infty,\quad \theta = \frac{\pi}{2}$

$$\begin{aligned}
\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx &= \int_0^{\frac{\pi}{2}} \frac{(\tan^2\theta)^{m-1}}{(1+\tan^2\theta)^{m+n}}\cdot 2\tan\theta\sec^2\theta\, d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \frac{(\tan\theta)^{2m-1}\sec^2\theta}{(\sec\theta)^{2m+2n}}\,d\theta \\
&= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta \\
&= B(m, n)
\end{aligned}$$

Example 1

Prove that $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \frac{1}{a^n b^m} B(m, n)$ *and hence, find the value of* $\int_0^\infty \frac{x^5}{(2+3x)^{16}}$.

Solution

Let $bx = at$, $dx = \frac{a}{b}\,dt$

When $x = 0, \quad t = 0$

When $x \to \infty, \quad t \to \infty$

$$\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \int_0^\infty \frac{\left(\frac{a}{b}t\right)^{m-1}}{(a+at)^{m+n}} \cdot \frac{a}{b}\,dt$$

$$= \frac{1}{a^n\, b^m} \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,dt = \frac{1}{a^n\, b^m} B(m, n)$$

Putting $a = 2$, $b = 3$, $m = 6$, $n = 10$ in the above integral,

$$\int_0^\infty \frac{x^5}{(2+3x)^{16}}\,dx = \frac{1}{2^{10}\cdot 3^6} B(6, 10)$$

$$= \frac{1}{2^{10}\cdot 3^6} \frac{\Gamma 6\, \Gamma 10}{\Gamma 16} = \frac{1}{2^{10}\cdot 3^6} \frac{5!\,10!}{15!}$$

Example 2

Prove that $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = 0$.

Solution

$$\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = \int_0^\infty \frac{x^8}{(1+x)^{24}}\,dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\,dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

$$= B(9, 15) - B(15, 9)$$

$$= 0$$

Example 3

Prove that $\int_0^\infty \frac{x^2}{(1+x^4)^3}\,dx = \frac{5\pi\sqrt{2}}{128}$.

Solution

Let $x^4 = t,\ x = t^{\frac{1}{4}},\ dx = \frac{1}{4}t^{-\frac{3}{4}}dt$

When $x = 0, \quad t = 0$

When $x \to \infty, \quad t \to \infty$

$$\begin{aligned}
\int_0^\infty \frac{x^2}{(1+x^4)^3}\,dx &= \int_0^\infty \frac{t^{\frac{1}{2}}}{(1+t)^3}\cdot\frac{1}{4}t^{-\frac{3}{4}}dt \\
&= \frac{1}{4}\int_0^\infty \frac{t^{-\frac{1}{4}}}{(1+t)^3}\,dt = \frac{1}{4}\int_0^\infty \frac{t^{\frac{3}{4}-1}}{(1+t)^{\frac{3}{4}+\frac{9}{4}}}\,dt \\
&= \frac{1}{4}B\left(\frac{3}{4},\frac{9}{4}\right) = \frac{1}{4}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{9}{4}\right)}{\Gamma(3)} \\
&= \frac{1}{4}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\cdot\frac{5}{4}\cdot\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}{2!} = \frac{5}{128}\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right) \\
&= \frac{5}{128}\cdot\frac{\pi}{\sin\frac{\pi}{4}} \qquad \left[\because \Gamma(1-n)\,\Gamma(n) = \frac{\pi}{\sin n\pi}\right] \\
&= \frac{5\pi\sqrt{2}}{128}
\end{aligned}$$

Example 4

Prove that $\int_0^\infty \operatorname{sech}^6 x\,dx = \frac{8}{15}$.

Solution

$$\begin{aligned}
\int_0^\infty \operatorname{sech}^6 x\,dx &= \int_0^\infty \left(\frac{2}{e^x+e^{-x}}\right)^6 dx \qquad \left[\because \cosh x = \frac{e^x+e^{-x}}{2}\right] \\
&= 2^6\cdot\frac{1}{2}\int_{-\infty}^\infty \frac{1}{(e^x+e^{-x})^6}\,dx \qquad \left[\because \int_{-a}^a f(x)dx = 2\int_0^a f(x)dx \text{ if } f(-x) = f(x)\right]
\end{aligned}$$

$$= 2^5 \int_{-\infty}^{\infty} \frac{e^{6x}}{(e^{2x}+1)^6}\, dx$$

Let $e^{2x} = t,\quad 2e^{2x}\, dx = dt,\quad dx = \frac{1}{2t}\, dt$

When $x \to -\infty,\quad t = 0$

When $x \to \infty,\quad t \to \infty$

$$\int_0^{\infty} \operatorname{sech}^6 x\, dx = 2^5 \int_0^{\infty} \frac{t^3}{(t+1)^6} \cdot \frac{1}{2t}\, dt$$

$$= 2^4 \int_0^{\infty} \frac{t^{3-1}}{(1+t)^{3+3}}\, dt = 2^4 B(3, 3)$$

$$= 16 \cdot \frac{\overline{|3}\,\overline{|3}}{\overline{|6}} = 16 \cdot \frac{2!\,2!}{5!} = \frac{8}{15}.$$

Example 5

Prove that $\displaystyle\int_0^{\infty} \frac{e^{2mx}+e^{-2mx}}{(e^x+e^{-x})^{2n}}\, dx = \frac{1}{2} B(n+m, n-m), n > m.$

Solution

$$\int_0^{\infty} \frac{e^{2mx}+e^{-2mx}}{(e^x+e^{-x})^{2n}}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{(e^{2mx}+e^{-2mx})\, e^{2nx}}{(e^{2x}+1)^{2n}}\, dx$$

$$\left[\because \int_{-a}^{a} f(x)dx = 2\int_0^{a} f(x)dx \quad \text{if } f(-x) = f(x)\right]$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{2(m+n)x}+e^{2(n-m)x}}{(1+e^{2x})^{2n}}\, dx$$

Let $e^{2x} = t,\quad 2e^{2x}\, dx = dt,\quad dx = \frac{1}{2t}\, dt$

When $x \to -\infty,\quad t = 0$

When $x \to \infty,\quad t \to \infty,$

$$\int_0^{\infty} \frac{e^{2mx}+e^{-2mx}}{(e^x+e^{-x})^{2n}}\, dx = \frac{1}{2}\int_0^{\infty} \frac{t^{m+n}+t^{n-m}}{(1+t)^{2n}} \cdot \frac{1}{2t}\, dt$$

$$= \frac{1}{4}\left[\int_0^{\infty} \frac{t^{(m+n)-1}}{(1+t)^{(m+n)+(n-m)}}\, dt + \int_0^{\infty} \frac{t^{(n-m)-1}}{(1+t)^{(n-m)+(n+m)}}\, dt\right]$$

$$= \frac{1}{4}\left[B(m+n, n-m) + B(n-m, n+m)\right]$$

$$= \frac{1}{2} B(n+m, n-m) \qquad [\because B(m,n) = B(n,m)]$$

Example 6

Prove that $\displaystyle\int_0^{\pi} \frac{\sin^{n-1}x}{(a+b\cos x)^n}\,dx = \frac{2^{n-1}}{(a^2-b^2)^{\frac{n}{2}}} B\left(\frac{n}{2},\frac{n}{2}\right)$ *and hence, deduce that* $\displaystyle\int_0^{\pi} \frac{\sqrt{\sin x}}{(5+3\cos x)^{\frac{3}{2}}}\,dx = \frac{\left(\left\lceil\frac{3}{4}\right.\right)^2}{2\sqrt{2\pi}}$.

Solution

Let $\tan\dfrac{x}{2}=t$, $\dfrac{x}{2}=\tan^{-1}t$, $dx=\dfrac{2}{1+t^2}dt$

$$\cos x = \frac{1-t^2}{1+t^2}, \qquad \sin x = \frac{2t}{1+t^2}$$

When $\quad x=0, \quad t=0$

When $\quad x=\pi, \quad t\to\infty$

$$\int_0^{\pi} \frac{\sin^{n-1}x}{(a+b\cos x)^n}\,dx = \int_0^{\infty} \frac{\left(\frac{2t}{1+t^2}\right)^{n-1}}{\left[a+b\left(\frac{1-t^2}{1+t^2}\right)\right]^n}\cdot\frac{2}{1+t^2}\,dt = 2^n\int_0^{\infty}\frac{t^{n-1}}{\left[(a+b)+(a-b)\,t^2\right]^n}\,dt$$

Let $\quad (a-b)\,t^2=(a+b)u$, $t=\dfrac{\sqrt{a+b}\cdot\sqrt{u}}{\sqrt{a-b}}$, $\quad dt=\dfrac{\sqrt{a+b}}{2\sqrt{a-b}}\cdot\dfrac{1}{u^{\frac{1}{2}}}du$

When $\quad t=0, \quad u=0$

When $\quad t\to\infty, \quad u\to\infty$

$$\int_0^{\pi} \frac{\sin^{n-1}x}{(a+b\cos x)^n}\,dx = 2^n\int_0^{\infty}\frac{\left[\frac{(a+b)u}{(a-b)}\right]^{\frac{n-1}{2}}}{\left[(a+b)+(a+b)u\right]^n}\cdot\frac{\sqrt{a+b}}{2\sqrt{a-b}}\,\frac{1}{u^{\frac{1}{2}}}\,du$$

$$= \frac{2^{n-1}}{(a+b)^{\frac{n}{2}}\,(a-b)^{\frac{n}{2}}}\int_0^{\infty}\frac{u^{\frac{n}{2}-1}}{(1+u)^{\frac{n}{2}+\frac{n}{2}}}\,du = \frac{2^{n-1}}{(a^2-b^2)^{\frac{n}{2}}}B\left(\frac{n}{2},\frac{n}{2}\right)$$

Putting $a=5$, $b=3$, $n=\dfrac{3}{2}$ in the above integral,

$$\int_0^{\pi}\frac{\sqrt{\sin x}}{(5+3\cos x)^{\frac{3}{2}}}\,dx = \frac{2^{\frac{3}{2}-1}}{(5^2-3^2)^{\frac{3}{4}}}B\left(\frac{3}{4},\frac{3}{4}\right)$$

$$= \frac{\sqrt{2}}{2^3}\frac{\left|\frac{3}{4}\right.\left|\frac{3}{4}\right.}{\left|\frac{3}{2}\right.} = \frac{\sqrt{2}\left(\left|\frac{3}{4}\right.\right)^2}{2^3\cdot\frac{1}{2}\left|\frac{1}{2}\right.}$$

$$= \frac{\left(\left|\frac{3}{4}\right.\right)^2}{2\sqrt{2\pi}}$$

Example 7

Prove that $\int_0^{\frac{\pi}{2}} \frac{\cos^{2m-1}\theta\sin^{2n-1}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{m+n}}\,d\theta = \frac{B(m,n)}{2a^{2m}b^{2n}}$.

Solution

$$\int_0^{\frac{\pi}{2}} \frac{\cos^{2m-1}\theta\sin^{2n-1}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{m+n}}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^{2m-1}\theta\sin^{2n-1}\theta\,(\cos\theta)^{-2m-2n}}{(a^2+b^2\tan^2\theta)^{m+n}}\,d\theta$$

$$= \frac{1}{a^{2(m+n)}}\int_0^{\frac{\pi}{2}} \frac{(\tan\theta)^{2n-1}\sec^2\theta}{\left(1+\frac{b^2}{a^2}\tan^2\theta\right)^{m+n}}\,d\theta$$

Let $\frac{b^2}{a^2}\tan^2\theta = t,\quad \tan\theta = \frac{a}{b}\sqrt{t},\quad \sec^2\theta\,d\theta = \frac{a}{2b}\cdot\frac{1}{\sqrt{t}}\,dt$

When $\theta = 0,\quad t = 0$

When $\theta = \frac{\pi}{2},\quad t\to\infty$

$$\int_0^{\frac{\pi}{2}} \frac{\cos^{2m-1}\theta\sin^{2n-1}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{m+n}}\,d\theta = \frac{1}{a^{2(m+n)}}\int_0^{\infty}\frac{\left(\frac{a}{b}t^{\frac{1}{2}}\right)^{2n-1}}{(1+t)^{m+n}}\cdot\frac{a}{2b}\cdot\frac{1}{\sqrt{t}}\,dt$$

$$= \frac{1}{2a^{2m}b^{2n}}\int_0^{\infty}\frac{t^{n-1}}{(1+t)^{m+n}}\,dt$$

$$= \frac{1}{2a^{2m}b^{2n}}\,B(m,n)$$

Example 8

Prove that $B(m, n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$.

Solution

We have $$B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \qquad \dots (1)$$

Consider, $$I = \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

Let $x = \frac{1}{y}$, $\quad dx = -\frac{1}{y^2}\,dy$

When $x = 1$, $\quad y = 1$

When $x \to \infty$, $\quad y = 0$

$$I = \int_1^0 \frac{\left(\frac{1}{y}\right)^{m-1}}{\left(1+\frac{1}{y}\right)^{m+n}}\left(-\frac{1}{y^2}\right)dy = \int_0^1 \frac{y^{n-1}}{(y+1)^{m+n}}\,dy$$

Substituting in Eq. (1),

$$B(m, n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

Replacing y by x,

$$B(m, n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx$$

Example 9

Prove that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = \frac{\left(\left|\frac{1}{4}\right.\right)^2}{4\sqrt{\pi}}$.

Solution

Let $\tan\frac{\theta}{2} = t$, $\frac{\theta}{2} = \tan^{-1} t$, $d\theta = \frac{2}{1+t^2}\,dt$

$$\sin\theta = \frac{2t}{1+t^2}$$

When $\theta = 0, \quad t = 0$

When $\theta = \frac{\pi}{2}, \quad t = 1$

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = \int_0^1 \frac{1}{\sqrt{1-\frac{1}{2}\left(\frac{2t}{1+t^2}\right)^2}} \cdot \frac{2}{1+t^2}\, dt$$

Let $t^4 = u,\ t = u^{\frac{1}{4}},\ dt = \frac{1}{4}u^{-\frac{3}{4}}du$ $\qquad = 2\int_0^1 \frac{1}{(1+t^4)^{\frac{1}{2}}}\, dt$

When $t = 0, \quad u = 0$

When $t = 1, \quad u = 1$

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = 2\int_0^1 \frac{1}{(1+u)^{\frac{1}{2}}} \cdot \frac{1}{4}u^{-\frac{3}{4}}\, du$$

$$= \frac{1}{2}\int_0^1 \frac{u^{\frac{1}{4}-1}}{(1+u)^{\frac{1}{2}}}\, du = \frac{1}{4}\int_0^1 \frac{u^{\frac{1}{4}-1} + u^{\frac{1}{4}-1}}{(1+u)^{\frac{1}{4}+\frac{1}{4}}}\, du$$

$$= \frac{1}{4} B\left(\frac{1}{4}, \frac{1}{4}\right)$$ [From Ex. 8]

$$= \frac{1}{4}\frac{\left|\frac{1}{4}\right.\left|\frac{1}{4}\right.}{\left|\frac{1}{2}\right.} = \frac{\left(\left|\frac{1}{4}\right.\right)^2}{4\sqrt{\pi}}$$

EXERCISE 3.5

1. Evaluate $\int_0^\infty \frac{dy}{1+y^4}$.

$\left[\textbf{Ans.:} \frac{\pi}{2\sqrt{2}}\right]$

2. Prove that

$$\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\, dx = 2B\ (m,\ n).$$

3. Prove that

$$\int_0^\infty \frac{\sqrt{x}}{(4+4x+x^2)}\, dx = \frac{\pi}{4\sqrt{2}}.$$

4. Prove that

$$\int_0^\infty \frac{dx}{(e^x+e^{-x})^n} = \frac{1}{4}\ B\left(\frac{n}{2},\frac{n}{2}\right)$$

and hence, evaluate $\int_0^\infty sech^8 x\ dx$.

$$\left[\textbf{Ans.}: \frac{16}{35}\right]$$

5. Prove that $\int_1^\infty \frac{dx}{x^{p+1}(x-1)^q} = B(p+q,\ 1-q)$, if $-p<q<1$.

Points to Remember

Gamma Function

$$\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx,\ n>0$$

$$\Gamma n = 2\int_0^\infty e^{-x^2}x^{2n-1}dx$$

Properties of Gamma Function

Property 1: $\boldsymbol{\Gamma(n+1) = n\,\Gamma n}$

(i) $\Gamma(n+1) = n!$ if n is a positive integer

(ii) $\Gamma(n+1) = n\,\Gamma n$ if n is a positive real number

(iii) $\Gamma n = \frac{\Gamma(n+1)}{n}$ if n is a negative fraction

(iv) $\Gamma n\ \Gamma(1-n) = \frac{\pi}{\sin n\pi}$

Property 2: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Beta Function

$$B\ (m,\ n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,\quad m>0,\ n>0.$$

Trigonometric Form of Beta Function

$$B(m,\ n)\ = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}x\ \cos^{2n-1}x\,dx$$

Properties of Beta Functions

Symmetry

$$B(m, n) = B(n, m)$$

Relation between Beta and Gamma Functions

$$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Duplication Formula

$$\boldsymbol{\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}}}$$

Beta Function as Improper Integral

$$B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. The value of the integral $I = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{\left(\frac{-x^2}{8}\right)} dx$ is

(a) 1 (b) π (c) 2 (d) 2π

2. Match the items in columns I and II for the following special functions

I		II	
(P)	$\beta(p, q)$	(i)	$\Gamma\left(\frac{1}{2}\right)$
(Q)	$\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$	(ii)	$\int_0^\infty \frac{y^{p-1}}{(1+y)(p+q)} dy$
(R)	$\sqrt{\pi}$	(iii)	$\beta(p, q)$
(S)	$\frac{\pi}{\sin p\pi}$	(iv)	$\Gamma(p)\,\Gamma(1-p)$

(a) P-(iv), Q-(iii), R-(i), S-(ii) (b) P-(ii), Q-(iii), R-(i), S-(iv)

(c) P-(iii), Q-(ii), R-(i), S-(iv) (d) P-(ii), Q-(iii), R-(iv), S-(i)

3. The value of $\int_0^\infty \sqrt{y}\, e^{-y^3}\, dy$ is

(a) $\frac{\sqrt{\pi}}{2}$ (b) $\frac{\sqrt{\pi}}{3}$ (c) $\sqrt{\pi}$ (d) $\frac{\sqrt{\pi}}{6}$

4. The value of $B(m+1, n)$ is

(a) $\frac{n}{m+n} B(m, n)$ (b) $\frac{n}{m+1} B(m, n)$

(c) $\frac{m}{m+n} B(m, n)$ (d) $\frac{m}{m+1} B(m, n)$

5. The value of $\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta$ is

(a) $\frac{\pi\sqrt{2}}{2}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi\sqrt{2}}{4}$ (d) $\frac{\pi}{4}$

6. The value of $\int_0^1 x^4 \left[\log\left(\frac{1}{x}\right)\right]^3 dx$ is

(a) $\frac{3}{325}$ (b) $\frac{6}{625}$ (c) $\frac{3}{625}$ (d) $\frac{6}{325}$

7. If $B(n, 2) = \frac{1}{6}$ and n is a positive integer, then the value of n is

(a) 3 (b) -2 (c) 2 (d) -3

8. The value of $\int_0^\infty \frac{t^2 dt}{1+t^4}$ is

(a) $\frac{\pi}{\sqrt{2}}$ (b) $\frac{\sqrt{\pi}}{2}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

9. The value of $B(m, m)$ is

(a) $2^{1-2m} B\left(m, \frac{1}{2}\right)$ (b) $2^{1-2m} B\left(m+1, \frac{1}{2}\right)$

(c) $2^{1-2m} B\left(m+\frac{1}{2}, 1\right)$ (d) $2^{1-2m} B\left(m, \frac{3}{2}\right)$

10. Gamma function is discontinuous for

(a) all $p < 0$ (b) any $p > 0$

(c) $p = 0$ only (d) $p = 0$ and negative integers

11. Beta function $B(p, q)$ is convergent for

(a) $p > 0, q < 0$ (b) $p > 0, q > 0$

(c) $p < 0, q > 0$ (d) $p < 0, q < 0$

Answers

1. (a)	2. (b)	3. (b)	4. (c)	5. (a)	6. (b)	7. (c)	8. (a)
9. (a)	10. (d)	11. (b)					

CHAPTER 4

Applications of Definite Integrals

Chapter Outline

4.1 Introduction
4.2 Volume using Cross-sections
4.3 Length of Plane Curves
4.4 Area of Surface of Solid of Revolution

4.1 INTRODUCTION

In this chapter, we will explore few applications of definite integrals. Applications of definite integrals in finding volume by cross-sections, length of plane curves and areas of surfaces of revolution are discussed in detail.

4.2 VOLUME USING CROSS-SECTIONS

The volume of a solid of known integrable cross-section area $A(x)$ formed by a plane perpendicular to the x-axis at any point between $x = a$ to $x = b$ is

$$V = \int_a^b A(x)\,dx$$

Note: If the cross-section $A(y)$ is perpendicular to the y-axis at any point between $y = c$ to $y = d$, then the volume of the solid is

$$V = \int_c^d A(y)\,dy$$

Example 1

Find the volume of the solid generated by rotating the plane region bounded by $y=\frac{1}{x}$, $x=1$ *and* $x = 3$ *about the x-axis.*

Solution

The volume is generated by rotating the dotted region about the x-axis.
The cross-section PQ of the generated volume is a circle of radius y perpendicular to x-axis.

Area of cross-section, $A = \pi y^2$

$$= \pi \frac{1}{x^2}$$

For the region shown, x varies from 1 to 3.

Volume,
$$V = \int_1^3 \frac{\pi}{x^2}\,dx$$

$$= \pi\left|-\frac{1}{x}\right|_1^3$$

$$= \pi\left(-\frac{1}{3}+1\right)$$

$$= \frac{2}{3}\pi$$

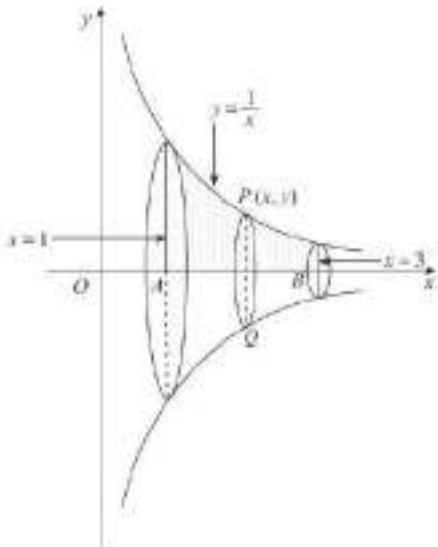


Fig. 4.1

Example 2

Find the volume of the solid that lies between planes perpendicular to the x-axis at $x = 0$ *and* $x = 4$. *The cross-sections perpendicular to the x-axis between these planes run from one side of the parabola* $x = y^2$ *to the other. The cross-sections are squares with bases in the xy-plane.*

Solution

The cross-section of the solid is a square $PQRS$ with side runs from $y=-\sqrt{x}$ to $y=\sqrt{x}$

Length $PQ = 2y = 2\sqrt{x}$

Area of cross-section, $A = (2\sqrt{x})^2 = 4x$

For the region shown, x varies from 0 to 4.

Volume, $V = \int_0^4 4x\,dx$

$$= 4\left|\frac{x^2}{2}\right|_0^4$$

$$= 2(16-0)$$

$$= 32$$

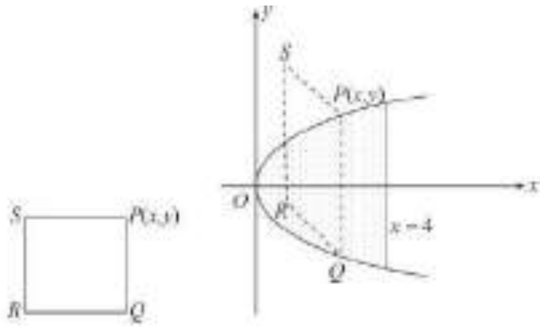


Fig. 4.2

Example 3

Find the volume of a right circular cone of base radius r and height h.

Solution

The cone is generated by rotating the line AB about the y-axis. Equation of the line AB passing through the points $A(r, 0)$ and $B(0, h)$ is

$$y - 0 = \frac{h-0}{0-r}(x-r)$$

$$x = r\left(1-\frac{y}{h}\right)$$

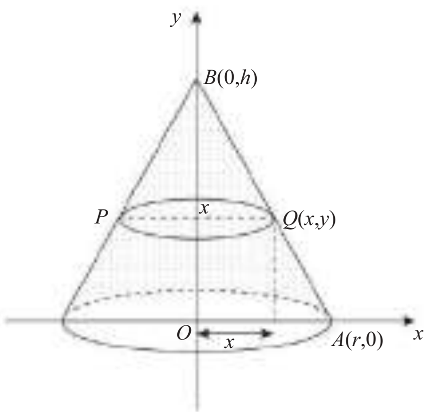


Fig. 4.3

The cross-section PQ of the cone is a circle of radius x perpendicular to the y-axis.

Area of cross-section, $A = \pi x^2$

$$= \pi r^2\left(1-\frac{y}{h}\right)^2$$

For the region shown, y varies from 0 to h.

Volume, $V = \int_0^h \pi r^2\left(1-\frac{y}{h}\right)^2 dy$

$$= \pi r^2\left|\frac{\left(1-\frac{y}{h}\right)^3}{3\left(-\frac{1}{h}\right)}\right|_0^h$$

$$= -\frac{\pi r^2 h}{3}(0-1)$$

$$= \frac{1}{3}\pi r^2 h$$

Example 4

Find the volume of the solid with a circular base of radius 5 *and whose cross-sections perpendicular to the base and parallel to the x-axis are equilateral triangles.*

Solution

Equation of the circular base of radius 5 is,

$$x^2 + y^2 = 25 \qquad \ldots(1)$$

The cross-section of the solid is an equilateral triangle *PQR* with base $2x$.

Area of cross-section,

$$\begin{aligned} A &= \frac{1}{2} \times (\text{base}) \times (\text{height}) \\ &= \frac{1}{2}(2x)(h) \\ &= \frac{1}{2}(2x)(x \tan 60^\circ) \\ &= x^2\sqrt{3} \\ &= (25 - y^2)\sqrt{3} \qquad [\text{from Eq. (1)}] \end{aligned}$$

For the region shown, y varies from -5 to 5.

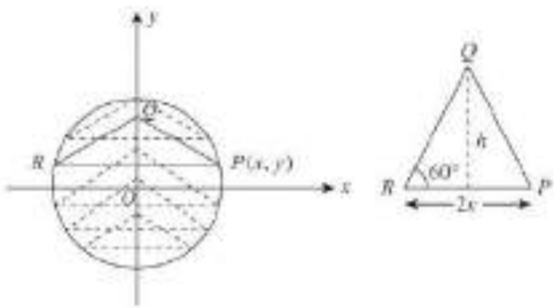


Fig. 4.4

Volume,

$$\begin{aligned} V &= \int_{-5}^{5} \sqrt{3}(25 - y^2)\,dy \\ &= 2\int_0^5 \sqrt{3}(25 - y^2)\,dy \\ &= 2\sqrt{3}\left|25y - \frac{y^3}{3}\right|_0^5 \\ &= 2\sqrt{3}\left(125 - \frac{125}{3}\right) \\ &= \frac{500\sqrt{3}}{3} \end{aligned}$$

Example 5

Use the method of slicing to find the volume of solid with semicircular base defined by $y = 5\sqrt{\cos x}$ *on the interval* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. *The cross-sections of the solid are squares perpendicular to the x-axis with base running from x-axis to the curve.* **[Winter 2015]**

Solution

The cross-section of the solid is a square with side runs from $y = 0$ (x-axis) to $y = 5\sqrt{\cos x}$.

Length of one side of the square $= 5\sqrt{\cos x}$

Area of cross-section, $A = \left(5\sqrt{\cos x}\right)^2$

$$= 25 \cos^2 x$$

In the given interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, x varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

Volume,
$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 25 \cos^2 x \, dx$$

$$= 25 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{25}{2} \left| x + \frac{\sin 2x}{2} \right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{25}{2}\left[\left(\frac{\pi}{2} + \frac{\pi}{2}\right) + 0\right] \quad [\because \sin \pi = 0]$$

$$= \frac{25}{2}\pi$$

EXERCISE 4.1

1. Find the volume of the solid that lies between planes perpendicular to the y-axis at $y = 0$ and $y = 2$. The cross-sections perpendicular to the y-axis are circular disks with diameters running from the *y*-axis to the parabola $x = \sqrt{5}y^2$.

[Ans.: 8π]

2. Find the volume of the solid that lies between the planes perpendicular to the *x*-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.

$$\left[\text{Ans.: } \frac{16}{15}\pi\right]$$

3. Show that the volume of a sphere of radius *r* is $\frac{4}{3}\pi r^3$.

4. Find the volume of the solid whose base is the region bounded between the curves $y = x$ and $y = x^2$, and whose cross-sections perpendicular to the *x*-axis are squares.

$$\left[\text{Ans.: } \frac{1}{30}\right]$$

5. Find the volume of the solid whose base is a triangle with vertices (0, 0), (2, 0) and (0, 2) and whose cross-sections perpendicular to the base and parallel to the *y*-axis are semicircles.

$$\left[\text{Ans.: } \frac{\pi}{3}\right]$$

4.3 LENGTH OF PLANE CURVES

The process of determining the length of an arc of a plane curve is known as rectification of curves.

4.3.1 Length of Arc in Cartesian Form

We know from differential calculus that for the curve $y = f(x)$,

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

The length of the arc of the curve $y = f(x)$ between $x = a$ and $x = b$ is given by,

$$s = \int_a^b \frac{ds}{dx} dx$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

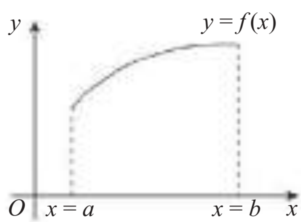


Fig. 4.5

Similarly, the length of the arc of the curve $x = f(y)$ between $y = c$ and $y = d$ is given by,

$$s = \int_c^d \frac{ds}{dy}\,dy$$

$$= \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Example 1

Show that the length of the arc of the curve $4ax = y^2 - 2a^2 \log\frac{y}{a} - a^2$ *from* $(0, a)$ *to any point* (x, y) *is given by* $\frac{y^2}{2a} - \frac{a}{2} - x$.

Solution

$$4ax = y^2 - 2a^2\log\frac{y}{a} - a^2 \qquad \ldots (1)$$

$$4a\frac{dx}{dy} = 2y - 2a^2\cdot\frac{a}{y}\cdot\frac{1}{a}$$

$$\frac{dx}{dy} = \frac{y}{2a} - \frac{a}{2y} = \frac{y^2-a^2}{2ay}$$

For the required arc, y varies from a to y.

Length of the arc, $s = \int_a^y \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

$$= \int_a^y \sqrt{1+\left(\frac{y^2-a^2}{2ay}\right)^2}\,dy$$

$$= \int_a^y \sqrt{\frac{(y^2+a^2)^2}{(2ay)^2}}\,dy$$

$$= \frac{1}{2a}\int_a^y \frac{y^2+a^2}{y}\,dy$$

$$= \frac{1}{2a}\left(\int_a^y y\,dy + a^2\int_a^y \frac{1}{y}\,dy\right)$$

$$= \frac{1}{2a}\left|\frac{y^2}{2} + a^2\log y\right|_a^y$$

$$= \frac{1}{2a}\left(\frac{y^2}{2} + a^2 \log\frac{y}{a} - \frac{a^2}{2}\right)$$

$$= \frac{1}{2a}\left[\frac{y^2}{2} + \left(\frac{y^2}{2} - 2ax - \frac{a^2}{2}\right) - \frac{a^2}{2}\right] \qquad \text{... [From Eq. (1)]}$$

$$= \frac{1}{2a}(y^2 - 2ax - a^2)$$

$$= \frac{y^2}{2a} - x - \frac{a}{2}$$

$$= \frac{y^2}{2a} - \frac{a}{2} - x$$

Example 2

Find the length of the arc of the curve $y = e^x$ from the point (0, 1) *to* (1, *e*).

Solution

$$y = e^x$$

$$\frac{dy}{dx} = e^x$$

For the required arc, x varies from 0 to 1.

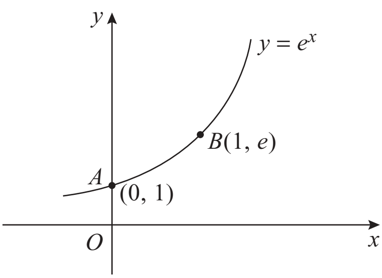


Fig. 4.6

Length of the arc AB, $s = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$= \int_0^1 \sqrt{1 + e^{2x}}\, dx$$

Putting $1 + e^{2x} = t^2$,

$$2e^{2x} dx = 2t\, dt$$

$$dx = \frac{t}{t^2 - 1} dt$$

When $x = 0, \quad t = \sqrt{2}$

When $x = 1, \quad t = \sqrt{1 + e^2}$

Length of the arc, $s = \int_{\sqrt{2}}^{\sqrt{1+e^2}} t \cdot \frac{t}{t^2 - 1} dt$

$$= \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{t^2 - 1 + 1}{t^2 - 1} dt$$

$$= \int_{\sqrt{2}}^{\sqrt{1+e^2}} \left(1 + \frac{1}{t^2 - 1}\right) dt$$

$$= \left| t + \frac{1}{2}\log\frac{t-1}{t+1} \right|_{\sqrt{2}}^{\sqrt{1+e^2}}$$

$$= \sqrt{1+e^2} - \sqrt{2} + \frac{1}{2}\left(\log\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1} - \log\frac{\sqrt{2}-1}{\sqrt{2}+1} \right)$$

$$= \sqrt{1+e^2} - \sqrt{2} + \frac{1}{2}\left[\log\left\{ \frac{\left(\sqrt{1+e^2}-1\right)^2}{1+e^2-1} \right\} - \log\left\{ \frac{\left(\sqrt{2}-1\right)^2}{2-1} \right\} \right]$$

$$= \sqrt{1+e^2} - \sqrt{2} + \log\left(\sqrt{1+e^2}-1\right) - \frac{1}{2}\log e^2 - \log\left(\sqrt{2}-1\right)$$

$$= \sqrt{1+e^2} - \sqrt{2} + \log\left(\sqrt{1+e^2}-1\right) - 1 - \log\left(\sqrt{2}-1\right)$$

$[\because \log e^2 = 2 \log e = 2]$

Example 3

Find the length of the arc of the curve $y = \log \sec x$ *from* $x = 0$ *to* $x = \frac{\pi}{3}$.

Solution

$$y = \log \sec x$$

$$\frac{dy}{dx} = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x$$

For the required arc, x varies from 0 to $\frac{\pi}{3}$.

Length of the arc, $s = \int_0^{\frac{\pi}{3}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$= \int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{3}} \sec x\, dx$$

$$= \left| \log(\sec x + \tan x) \right|_0^{\frac{\pi}{3}}$$

$$= \log\left(2+\sqrt{3}\right)$$

Example 4

Find the length of the arc of the curve $y = \log\left(\frac{e^x-1}{e^x+1}\right)$ *from* $x = 1$ *to* $x = 2$.

Solution

$$y = \log \frac{e^x - 1}{e^x + 1}$$

$$y = \log(e^x - 1) - \log(e^x + 1)$$

$$\frac{dy}{dx} = \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1} = \frac{2e^x}{e^{2x} - 1}$$

For the required arc, x varies from 1 to 2.

Length of the arc,
$$s = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \frac{4e^{2x}}{(e^{2x}-1)^2}}\, dx$$

$$= \int_1^2 \sqrt{\frac{(e^{2x}-1)^2 + 4e^{2x}}{(e^{2x}-1)^2}}\, dx$$

$$= \int_1^2 \left(\frac{e^{2x}+1}{e^{2x}-1}\right) dx$$

$$= \int_1^2 \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right) dx$$

$$= \left|\log(e^x - e^{-x})\right|_1^2$$

$$= \log(e^2 - e^{-2}) - \log(e - e^{-1})$$

$$= \log \frac{e^2 - e^{-2}}{e - e^{-1}}$$

$$= \log(e + e^{-1})$$

$$= \log\left(e + \frac{1}{e}\right)$$

Example 5

Show that the length of the arc of the curve $ay^2 = x^3$ from the origin to the point whose abscissa is b is $\frac{8a}{7}\left[\left(1 + \frac{9b}{4a}\right)^{\frac{3}{2}} - 1\right]$.

Solution

$$ay^2 = x^3$$

$$2ay\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2ay} = \frac{3x^2}{2a\sqrt{\frac{x^3}{a}}} = \frac{3}{2}\sqrt{\frac{x}{a}}$$

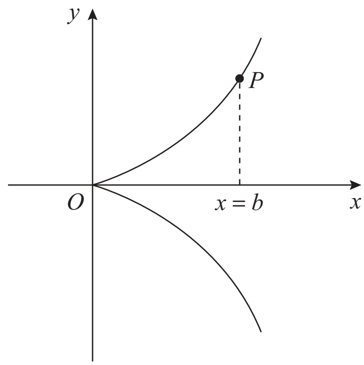


Fig. 4.7

For the arc OP, x varies from 0 to b.

Length of the arc OP,
$$s = \int_0^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^b \sqrt{1+\frac{9x}{4a}}\,dx$$

$$= \left|\frac{2}{3}\left(1+\frac{9x}{4a}\right)^{\frac{3}{2}}\frac{4a}{9}\right|_0^b$$

$$= \frac{8a}{27}\left[\left(1+\frac{9b}{4a}\right)^{\frac{3}{2}}-1\right]$$

Example 6

For the catenary $y = c\cosh\frac{x}{c}$, *prove that the length of the arc s, measured from its vertex to any point* (x, y), *is*

(i) $s = c\sinh\frac{x}{c}$ *(ii)* $s^2 = y^2 - c^2$ *(iii)* $s = c\tan\psi$

Solution

(i)
$$y = c\cosh\frac{x}{c}$$

$$\frac{dy}{dx} = \sinh\frac{x}{c}$$

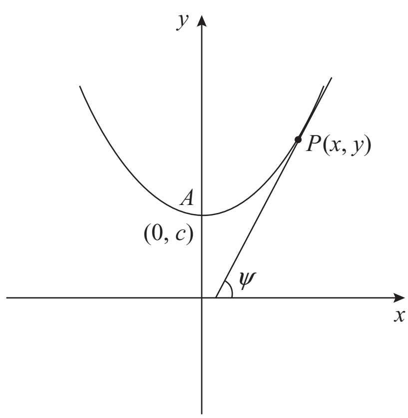


Fig. 4.8

For the arc AP, x varies from 0 to x.

Length of the arc AP,
$$s = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^x \sqrt{1+\sinh^2\frac{x}{c}}\,dx$$

$$= \int_0^x \cosh\frac{x}{c}\,\mathrm{d}x$$

$$= \left|c\sinh\frac{x}{c}\right|_0^x$$

$$= c\sinh\frac{x}{c}$$

(ii)

$$s^2 = c^2\sinh^2\frac{x}{c}$$

$$= c^2\left(\cosh^2\frac{x}{c} - 1\right)$$

$$= c^2\cosh^2\frac{x}{c} - c^2$$

$$= y^2 - c^2$$

(iii) The tangent at point $P(x, y)$ makes an angle ψ with the x-axis.

$$\tan\psi = \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$= \sinh\frac{x}{c}$$

$$= \frac{s}{c} \qquad \text{[From (i)]}$$

$$s = c\tan\psi$$

Example 7

Prove that the length of the arc of the curve $y^2 = x\left(1 - \frac{1}{3}x\right)^2$ *from the origin to the point* $P(x, y)$ *is given by* $s^2 = y^2 + \frac{4}{3}x^2$. *Hence, rectify the loop.*

Solution

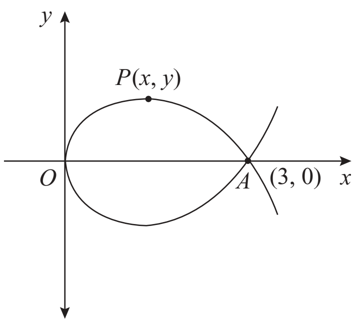


Fig. 4.9

$$y^2 = x\left(1 - \frac{1}{3}x\right)^2$$

$$y = \sqrt{x}\left(1 - \frac{x}{3}\right) = x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{3}\cdot\frac{3}{2}x^{\frac{1}{2}} = \frac{(1-x)}{2\sqrt{x}}$$

For the arc OP, x varies from 0 to x.

Length of the arc OP,
$$s = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$
$$= \int_0^x \sqrt{1+\frac{(1-x)^2}{4x}}\, dx$$
$$= \int_0^x \sqrt{\frac{(1+x)^2}{4x}}\, dx$$
$$= \int_0^x \frac{1+x}{2\sqrt{x}}\, dx$$
$$= \frac{1}{2}\int_0^x (x^{-\frac{1}{2}} + x^{\frac{1}{2}})\, dx$$
$$= \frac{1}{2}\left|\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^x$$
$$= \sqrt{x}\left(1+\frac{x}{3}\right)$$
$$s^2 = x\left(1+\frac{x}{3}\right)^2$$
$$= x\left(1-\frac{x}{3}\right)^2 + \frac{4}{3}x^2$$
$$= y^2 + \frac{4}{3}x^2$$

The points of intersection of the curve $y^2 = x\left(1-\frac{1}{3}x\right)^2$ and x-axis are obtained as,
$$0 = x\left(1-\frac{1}{3}x\right)^2$$
$$x = 0,\ 3,\ 3 \text{ and } y = 0,\ 0,\ 0$$

Hence, A: (3, 0) is the point of intersection.

Length of the upper half of the loop $= \sqrt{3}\left(1+\frac{3}{3}\right) = 2\sqrt{3}$

Length of the complete loop $= 4\sqrt{3}$

Example 8

Show that the length of the loop of the curve $9ay^2 = (x-2a)(x-5a)^2$ *is* $4\sqrt{3}a$.

Solution

The points of intersection of the curve $9ay^2 = (x-2a)(x-5a)^2$ and x-axis are obtained as,

$$0 = (x-2a)(x-5a)^2$$

$$x = 2a,\ 5a \text{ and } y = 0,\ 0$$

Hence, A: $(2a, 0)$ and B: $(5a, 0)$ are the points of intersection.

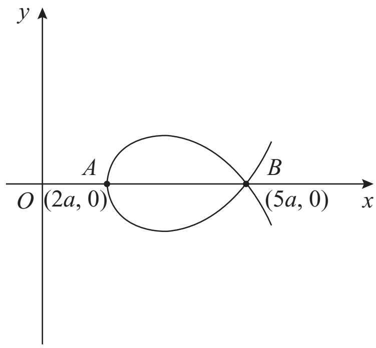


Fig. 4.10

$$9ay^2 = (x-2a)(x-5a)^2$$

$$18ay\frac{\mathrm{d}y}{\mathrm{d}x} = (x-2a)\ 2(x-5a) + (x-5a)^2$$

$$= (x-5a)(3x-9a)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-5a)(x-3a)}{6ay}$$

For the upper half of the loop, x varies from $2a$ to $5a$.

Length of the loop of the curve, $s = 2$ (Length of upper half of the loop)

$$= 2\int_{2a}^{5a}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

$$= 2\int_{2a}^{5a}\sqrt{1+\frac{(x-5a)^2(x-3a)^2}{36a^2y^2}}\,\mathrm{d}x$$

$$= 2\int_{2a}^{5a}\sqrt{1+\frac{(x-5a)^2(x-3a)^2}{4a(x-2a)(x-5a)^2}}\,\mathrm{d}x$$

$$= 2\int_{2a}^{5a}\sqrt{1+\frac{(x-3a)^2}{4a(x-2a)}}\,\mathrm{d}x$$

$$= 2\int_{2a}^{5a}\sqrt{\frac{(x-a)^2}{4a(x-2a)}}\,\mathrm{d}x$$

$$= 2\int_{2a}^{5a}\frac{x-a}{2\sqrt{a}\cdot\sqrt{x-2a}}\,\mathrm{d}x$$

$$= \frac{1}{\sqrt{a}}\int_{2a}^{5a}\frac{(x-2a)+a}{\sqrt{x-2a}}\,\mathrm{d}x$$

$$= \frac{1}{\sqrt{a}}\int_{2a}^{5a}\left[\sqrt{x-2a}+a(x-2a)^{-\frac{1}{2}}\right]\mathrm{d}x$$

$$= \frac{1}{\sqrt{a}} \left| \frac{2}{3}(x-2a)^{\frac{3}{2}} + 2a(x-2a)^{\frac{1}{2}} \right|_{2a}^{5a}$$

$$= \frac{1}{\sqrt{a}} \left[\frac{2}{3}(3a)^{\frac{3}{2}} + 2a(3a)^{\frac{1}{2}} \right]$$

$$= 2\sqrt{3}\, a + 2a\sqrt{3}$$

$$= 4\sqrt{3}\, a$$

Example 9

In the evolute $27ay^2 = 4(x-2a)^3$ *of the parabola* $y^2 = 4ax$, *show that the length of the arc from one cusp to the point where it meets the parabola is* $2a(3\sqrt{3}-1)$.

Solution

(i) The points of intersection of the parabola $y^2 = 4ax$ with its evolute $27ay^2 = 4(x-2a)^3$ are obtained as,

$$27a \cdot 4ax = 4(x-2a)^3$$

$$x^3 - 6ax^2 - 15a^2x - 8a^3 = 0$$

$$(x+a)^2(x-8a) = 0$$

$$x = -a,\ 8a$$

But $x = -a$ does not lie on the parabola.

$x = 8a$ and $y = \pm\sqrt{32}a$

Hence, B: $\left(8a, \sqrt{32}a\right)$ is the point of intersection.

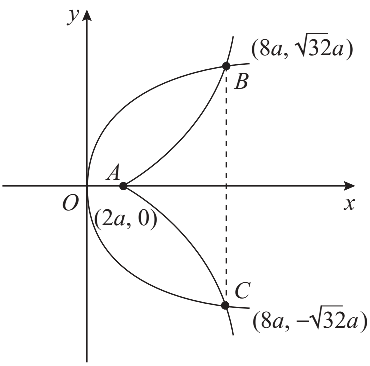


Fig. 4.11

(ii) The points of intersection of the evolute $27ay^2 = 4(x-2a)^3$ with the x-axis are obtained as,

$$x = 2a \text{ and } y = 0$$

Hence A: $(2a, 0)$ is the point of intersection.

Now, $$27ay^2 = 4(x-2a)^3$$

$$y = \frac{2}{3\sqrt{3a}}(x-2a)^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{2}{3\sqrt{3a}} \cdot \frac{3}{2}(x-2a)^{\frac{1}{2}} = \sqrt{\frac{x-2a}{3a}}$$

For the arc AB, x varies from $2a$ to $8a$.

Length of the arc AB, $s = \int_{2a}^{8a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$= \int_{2a}^{8a} \sqrt{1+\frac{x-2a}{3a}}\, dx$$

$$= \frac{1}{\sqrt{3a}} \int_{2a}^{8a} \sqrt{x+a}\, dx$$

$$= \frac{1}{\sqrt{3a}} \cdot \frac{2}{3} \left| (x+a)^{\frac{3}{2}} \right|_{2a}^{8a}$$

$$= \frac{1}{\sqrt{3a}} \cdot \frac{2}{3} \left[(9a)^{\frac{3}{2}} - (3a)^{\frac{3}{2}} \right]$$

$$= \frac{2}{3\sqrt{3a}} (3a)^{\frac{3}{2}} (3^{\frac{3}{2}} - 1)$$

$$= 2a(3\sqrt{3} - 1)$$

Example 10

Find the length of the parabola $x^2 = 4y$ which lies inside the circle $x^2 + y^2 = 6y$.

Solution

The equation of the circle is

$$x^2 + y^2 = 6y$$

$$x^2 + y^2 - 6y = 0$$

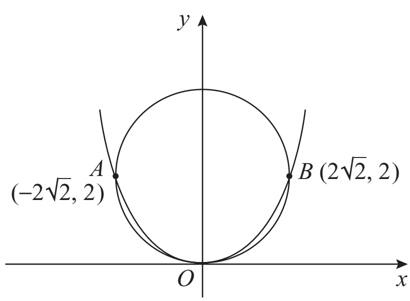


Fig. 4.12

The centre of the circle is (0, 3) and radius is 3.

The points of intersection of parabola $x^2 = 4y$ and circle $x^2 + y^2 = 6y$ are obtained as,

$$4y + y^2 = 6y$$

$$y^2 - 2y = 0$$

$$y(y-2) = 0$$

$$y = 0,\ 2$$

When $\quad y = 0, \quad x = 0$

$$y = 2, \quad x = \pm 2\sqrt{2}$$

Hence $A: \left(-2\sqrt{2}, 2\right)$ and $B: \left(2\sqrt{2}, 2\right)$ are the points of intersection.

Now, $\quad x^2 = 4y$

$$\frac{dy}{dx} = \frac{x}{2}$$

For the arc, OB, x varies from 0 to $2\sqrt{2}$.

Length of the arc OB, $s = 2(\text{Length of the arc } OB)$

$$= 2\int_0^{2\sqrt{2}} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\int_0^{2\sqrt{2}} \sqrt{1+\frac{x^2}{4}}\, dx$$

$$= \int_0^{2\sqrt{2}} \sqrt{x^2+4}\, dx$$

$$= \left|\frac{x}{2}\sqrt{x^2+4} + 2\log\left(x+\sqrt{x^2+4}\right)\right|_0^{2\sqrt{2}}$$

$$= \sqrt{2}\cdot\sqrt{12} + 2\log\left(2\sqrt{2}+\sqrt{12}\right) - 2\log 2$$

$$= 2\left[\sqrt{6} + \log\left(\sqrt{2}+\sqrt{3}\right)\right]$$

Example 11

Show that the length of the parabola $y^2 = 4ax$ from the vertex to the end of the latus rectum is $a\left[\sqrt{2}+\log\left(1+\sqrt{2}\right)\right]$. Find the length of the arc cut off by the line $3y = 8x$.

Solution

(i) The points of intersection of the parabola $y^2 = 4ax$ and its latus rectum $x = a$ are obtained as,

$$y^2 = 4a \cdot a = 4a^2$$

$$y = \pm 2a \quad \text{and } x = a$$

Hence, P: $(a, 2a)$ and $Q:(a, -2a)$ are the points of intersection.

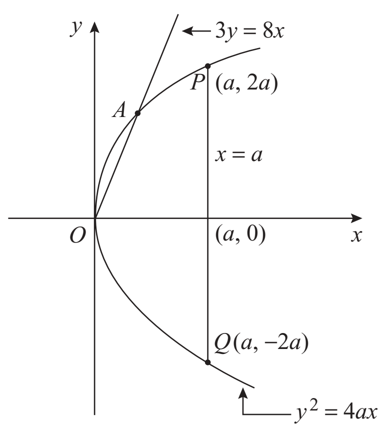


Fig. 4.13

Now, $$x = \frac{y^2}{4a}$$

$$\frac{dx}{dy} = \frac{y}{2a}$$

For the arc OP, y varies from 0 to $2a$.

Length of the arc *OP*, $s = \int_0^{2a} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

$$= \int_0^{2a} \sqrt{1 + \frac{y^2}{4a^2}}\, dy$$

$$= \frac{1}{2a}\int_0^{2a} \sqrt{y^2 + 4a^2}\, dy$$

$$= \frac{1}{2a}\left| \frac{y}{2}\sqrt{y^2 + 4a^2} + \frac{4a^2}{2}\log\left(y + \sqrt{y^2 + 4a^2}\right)\right|_0^{2a}$$

$$= \frac{1}{2a}\left[a \cdot 2a\sqrt{2} + 2a^2 \log\left(2a + 2a\sqrt{2}\right) - 2a^2 \log 2a\right]$$

$$= a\left(\sqrt{2} + \log\frac{2a + 2a\sqrt{2}}{2a}\right)$$

$$= a\left[\sqrt{2} + \log\left(1 + \sqrt{2}\right)\right]$$

(ii) The points of intersection of the parabola $y^2 = 4ax$ and the line $3y = 8x$ are obtained as,

$$y^2 = 4a\left(\frac{3y}{8}\right)$$

$$y\left(y - \frac{3a}{2}\right) = 0$$

$$y = 0, \frac{3a}{2} \text{ and } x = 0, \frac{9a}{16}$$

Hence, $A: \left(\frac{9a}{16}, \frac{3a}{2}\right)$ is the point of intersection.

For the arc *OA*, *y* varies from 0 to $\frac{3a}{2}$.

Length of the arc *OA*, $s = \frac{1}{2a}\int_0^{\frac{3a}{2}} \sqrt{y^2 + 4a^2}\, dy$

$$= \frac{1}{2a}\left| \frac{y}{2}\sqrt{y^2 + 4a^2} + \frac{4a^2}{2}\log\left(y + \sqrt{y^2 + 4a^2}\right)\right|_0^{\frac{3a}{2}}$$

$$= \frac{1}{2a}\left[\frac{3a}{4}\sqrt{\frac{9a^2}{4} + 4a^2} + 2a^2\left\{\log\left(\frac{3a}{2} + \sqrt{\frac{9a^2}{4} + 4a^2}\right) - \log 2a\right\}\right]$$

$$= \frac{1}{2a}\left[\frac{3a}{4} \cdot \frac{5a}{2} + 2a^2\left\{\log\left(\frac{3a}{2} + \frac{5a}{2}\right) - \log 2a\right\}\right]$$

$$= \frac{1}{2a}\left(\frac{15a^2}{8} + 2a^2 \log 2\right)$$
$$= a\left(\log 2 + \frac{15}{16}\right)$$

EXERCISE 4.2

1. Find the length of the arc of following curves:

   (i) $y = \log\left(\tanh\frac{x}{2}\right)$ from $x = 1$ to $x = 2$

   (ii) $24xy = x^4 + 48$ from $x = 2$ to $x = 4$

   (iii) $x = 3y^{\frac{3}{2}} - 1$ from $y = 0$ to $y = 4$

   (iv) $y = x(2 - x)$ from $x = 0$ to $x = 2$

   $\left[\textbf{Ans.: } \text{(i) } \log\left(e + \frac{1}{e}\right), \text{ (ii) } \frac{17}{6}, \text{ (iii) } \frac{8}{243}\left(82\sqrt{82} - 1\right), \text{ (iv) } \frac{1}{2}\log\left(2 + \sqrt{5}\right) + \sqrt{5}\right]$

2. Find the length of the curve $y^2 = (2x - 1)^3$ cut off by the line $x = 4$.

   $\left[\textbf{Ans.: } \frac{1022}{27}\right]$

3. Find the arc of the parabola $y^2 = 4a(a - x)$ cut off by the y-axis.

   $\left[\textbf{Ans.: } a\left[2\sqrt{2} - \log\left(3 - 2\sqrt{2}\right)\right]\right]$

4. Find the length of the arc of the parabola $y^2 = 8x$ cut off by its latus rectum. Find the length of the arc cut off by the line $3y = 8x$

   $\left[\textbf{Ans.: } 4\left[\sqrt{2} + \log(1 + \sqrt{2})\right], \quad 2\left(\log 2 + \frac{15}{16}\right)\right]$

5. Find the length of the arc of the parabola $x^2 = 4ay$ measured from the vertex to one extremity of the latus rectum.

   $\left[\textbf{Ans.: } \log\left[x + \sqrt{(1 + x^2)}\right]\right]$

6. Show that if s is the arc of the curve $9y^2 = x(3 - x)^2$ measured from the origin to the point P(x, y), then $3s^2 = 3y^2 + 4x^2$.

7. Find the length of the loop of the curve

   (i) $3ay^2 = x(x - a)^2$

(ii) $9y^2 = (x+7)(x+4)^2$

(iii) $9ay^2 = x(x-3a)^2$

(iv) $ay^2 = x^2(a-x)$

$$\left[\textbf{Ans.:}\ \text{(i)}\ \frac{4}{\sqrt{3}}a,\ \text{(ii)}\ 4\sqrt{3},\ \ \text{(iii)}\ 4\sqrt{3}a,\ \text{(iv)}\ \frac{4}{\sqrt{3}}a\right]$$

4.3.2 Length of Arc in Parametric Form

When the equation of the curve is given in parametric form $x = f_1(t)$, $y = f_2(t)$, we have, from differential calculus,

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

The length of the arc of the curve between the points $t = t_1$ and $t = t_2$ is given by,

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

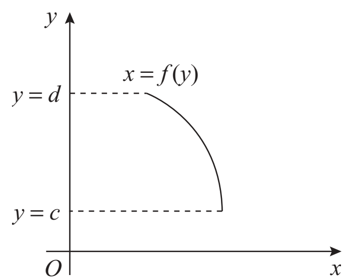


Fig. 4.14

Example 1

Find the length of the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$, *from* $\theta = 0$ *to* $\theta = 2\pi$.

Solution

$$x = a(\cos\theta + \theta\sin\theta)$$

$$\frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta) = a\theta\cos\theta$$

$$y = a(\sin\theta - \theta\cos\theta)$$

$$\frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta) = a\,\theta\sin\theta$$

For the required arc, θ varies from 0 to2π.

Length of the curve, $$s = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(a\theta\cos\theta)^2 + (a\theta\sin\theta)^2}\, d\theta$$

$$= a\int_0^{2\pi} \theta\, d\theta$$

$$= a\left|\frac{\theta^2}{2}\right|_0^{2\pi}$$

$$= 2a\pi^2$$

Example 2

Find the length of the curve $x = e^{\theta}\left(\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right)$, $y = e^{\theta}\left(\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right)$ *measured from* $\theta = 0$ *to* $\theta = \pi$.

Solution

$$x = e^{\theta}\left(\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right)$$

$$\frac{dx}{d\theta} = e^{\theta}\left(\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right) + e^{\theta}\left(\frac{1}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right) = \frac{5}{2}e^{\theta}\cos\frac{\theta}{2}$$

$$y = e^{\theta}\left(\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right)$$

$$\frac{dy}{d\theta} = e^{\theta}\left(\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right) + e^{\theta}\left(-\frac{1}{2}\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right) = -\frac{5}{2}e^{\theta}\sin\frac{\theta}{2}$$

For the required arc, θ varies from 0 to π.

Length of the curve $s = \int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

$$= \int_0^{\pi}\sqrt{\frac{25}{4}e^{2\theta}\cos^2\frac{\theta}{2} + \frac{25}{4}e^{2\theta}\sin^2\frac{\theta}{2}}\, d\theta$$

$$= \int_0^{\pi}\sqrt{\frac{25}{4}e^{2\theta}}\, d\theta$$

$$= \frac{5}{2}\int_0^{\pi} e^{\theta}\, d\theta$$

$$= \frac{5}{2}\left|e^{\theta}\right|_0^{\pi}$$

$$= \frac{5}{2}(e^{\pi} - 1)$$

Example 3

Find the length of the cycloid from one cusp to the next cusp $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.

Solution

$$x = a(\theta + \sin\theta)$$
$$\frac{dx}{d\theta} = a(1+\cos\theta)$$
$$y = a(1-\cos\theta)$$
$$\frac{dy}{d\theta} = a\sin\theta$$

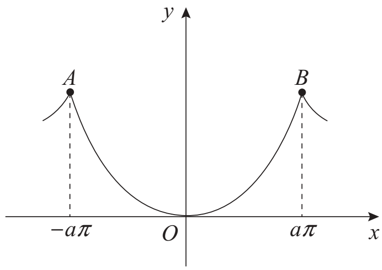


Fig. 4.15

For the arc OB, x varies from 0 to $a\pi$, hence θ varies from 0 to π.

Length of the arc AB,

$$\begin{aligned} s &= 2\ (\text{Length of arc } OB) \\ &= 2\int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\ &= 2\int_0^{\pi} \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\, d\theta \\ &= 2a\int_0^{\pi} \sqrt{2(1+\cos\theta)}\, d\theta \\ &= 4a\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta \\ &= 4a\left|2\sin\frac{\theta}{2}\right|_0^{\pi} \\ &= 8a \end{aligned}$$

Example 4

Find the length of one arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1+\cos\theta)$.

Solution

$$x = a(\theta - \sin\theta)$$
$$\frac{dx}{d\theta} = a(1-\cos\theta)$$
$$y = a(1+\cos\theta)$$
$$\frac{dy}{d\theta} = -a\sin\theta$$

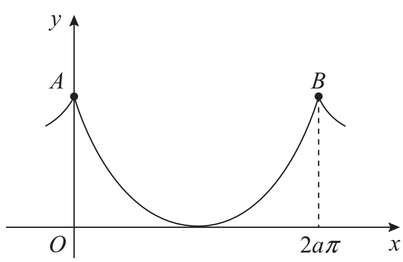


Fig. 4.16

For the arc AB, x varies from 0 to $2a\pi$, hence θ varies from 0 to 2π.

Length of the arc AB,
$$s = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$
$$= a\int_0^{2\pi} \sqrt{2-2\cos\theta}\, d\theta$$
$$= a\int_0^{2\pi} \sqrt{2\cdot 2\sin^2\frac{\theta}{2}}\, d\theta$$
$$= 2a\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta$$
$$= 2a\left|-2\cos\frac{\theta}{2}\right|_0^{2\pi}$$
$$= -4a(\cos\pi - \cos 0)$$
$$= 8a$$

Example 5

Find the length of the tractrix $x = a\left[\cos t + \log\tan\left(\frac{t}{2}\right)\right]$, $y = a\sin t$ *from* $t = \frac{\pi}{2}$ *to any point t.*

Solution

$$x = a\left[\cos t + \log\tan\left(\frac{t}{2}\right)\right]$$

$$\frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)}\sec^2\left(\frac{t}{2}\right)\cdot\frac{1}{2}\right]$$
$$= a\left(-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right)$$
$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$
$$= a\frac{(1-\sin^2 t)}{\sin t}$$
$$= a\frac{\cos^2 t}{\sin t}$$

$$y = a\sin t$$
$$\frac{dy}{dt} = a\cos t$$

For the required arc, t varies from $\frac{\pi}{2}$ to t.

Length of the curve,

$$s = \int_{\frac{\pi}{2}}^{t} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t$$

$$= \int_{\frac{\pi}{2}}^{t} \sqrt{a^2 \frac{\cos^4 t}{\sin^2 t} + a^2 \cos^2 t}\ \mathrm{d}t$$

$$= a\int_{\frac{\pi}{2}}^{t} \cos t\sqrt{\cot^2 t + 1}\ \mathrm{d}t$$

$$= a\int_{\frac{\pi}{2}}^{t} \cot\ t\ \mathrm{d}t$$

$$= a\left|\log\sin t\right|_{\frac{\pi}{2}}^{t}$$

$$= a\log\sin t$$

Example 6

For the curve $x = a(2\cos t - \cos 2t)$, $y = a(2\sin t - \sin 2t)$, *show that the length of the arc of the curve measured from* $t = 0$ *to the point where the tangent makes an angle* ψ *with the tangent, at* $t = 0$ *is given by* $s = 16a\sin^2\frac{\psi}{6}$.

Solution

$$x = a(2\cos t - \cos 2t)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(-2\sin t + 2\sin 2t) = 2a(\sin 2t - \sin t)$$

$$y = a(2\sin t - \sin 2t)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a(2\cos t - 2\cos 2t) = 2a(\cos t - \cos 2t)$$

For the required arc, t varies from 0 to t.

Length of the curve,

$$s = \int_0^t \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t$$

$$= \int_0^t \sqrt{4a^2[(\sin 2t - \sin t)^2 + (\cos t - \cos 2t)^2}\ \mathrm{d}t$$

$$= 2a\int_0^t \sqrt{2 - 2(\sin 2t\sin t + \cos \mathrm{t}\cos 2t)}\ \mathrm{d}t$$

$$= 2a\int_0^t \sqrt{2[1 - \cos(2t - t)]}\ \mathrm{d}t$$

$$= 2a\int_0^t \sqrt{2(1 - \cos t)}\ \mathrm{d}t$$

$$= 2a\int_0^t \sqrt{2 \cdot 2\sin^2 \frac{t}{2}}\, dt$$

$$= 4a\int_0^t \sin\frac{t}{2}\, dt$$

$$= 8a\left| -\cos\frac{t}{2} \right|_0^t$$

$$= 8a\left(1 - \cos\frac{t}{2}\right)$$

$$= 16a\sin^2\frac{t}{4} \qquad \ldots (1)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a(\cos t - \cos 2t)}{2a(\sin 2t - \sin t)}$$

$$= \frac{2\sin\frac{3t}{2}\sin\frac{t}{2}}{2\cos\frac{3t}{2}\sin\frac{t}{2}} = \tan\frac{3t}{2}$$

At $t = 0$, $y = 0$, $\quad \frac{dy}{dx} = 0$

Hence, the tangent is x-axis at $t = 0$.

At the point where tangent makes an angle ψ with the tangent at $t = 0$, i.e., x-axis, we get

$$\frac{dy}{dx} = \tan\psi$$

$$\tan\frac{3t}{2} = \tan\psi$$

$$\psi = \frac{3t}{2}$$

$$t = \frac{2\psi}{3}$$

Putting t in Eq. (1),

$$s = 16a\ \sin^2\frac{2\psi}{12}$$

$$= 16a\ \sin^2\frac{\psi}{6}$$

Example 7

Find the total length of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$. *Hence, deduce the total length of the curve* $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. *Also show that the line* $\theta=\frac{\pi}{3}$ *divides the length of the curve* $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ *in the first quadrant in the ratio 1:3.*

Solution

(i) The parametric equations of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$ are given by,

$$x=a\cos^3\theta, \quad y=b\sin^3\theta$$

$$\frac{dx}{d\theta}=-3a\cos^2\theta\sin\theta$$

$$\frac{dy}{d\theta}=3b\sin^2\theta\cos\theta$$

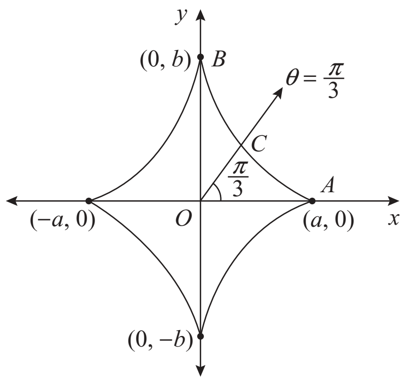


Fig. 4.17

For the arc AB, x varies from a to 0, hence θ varies from 0 to $\frac{\pi}{2}$.

Total length of the curve, $s = 4$ (Length of the arc AB)

$$=4\int_0^{\frac{\pi}{2}}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$=4\int_0^{\frac{\pi}{2}}\sqrt{9a^2\sin^2\theta\cos^4\theta+9b^2\sin^4\theta\cos^2\theta}\,d\theta$$

$$=12\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\sqrt{a^2+(b^2-a^2)\sin^2\theta}\,d\theta$$

Putting $a^2+(b^2-a^2)\sin^2\theta=t^2$,

$$2(b^2-a^2)\sin\theta\cos\theta\,d\theta=2t\,dt$$

$$\sin\theta\cos\theta\,d\theta=\frac{t}{b^2-a^2}dt$$

When $\theta=0$, $t=a$

When $\theta=\frac{\pi}{2}$, $t=b$

$$s=12\int_a^b t\cdot\frac{t}{b^2-a^2}\,dt$$

$$= \frac{12}{b^2 - a^2}\left|\frac{t^3}{3}\right|_a^b$$

$$= \frac{4(b^3 - a^3)}{b^2 - a^2}$$

$$= \frac{4(a^2 + ab + b^2)}{a + b}$$

(ii) Putting $b = a$,

Total length of the curve $(x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}) = \frac{4(a^2 + a^2 + a^2)}{2a} = 6a$

(iii) Length of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ in the first quadrant $= \frac{6a}{4} = \frac{3}{2}a$

$$\text{Length of the arc } AC = \int_0^{\frac{\pi}{3}} 3a \sin\theta \cos\theta \, d\theta$$

$$= \frac{3a}{2}\int_0^{\frac{\pi}{3}} \sin 2\theta \, d\theta$$

$$= \frac{3a}{2}\left|\frac{-\cos 2\theta}{2}\right|_0^{\frac{\pi}{3}}$$

$$= \frac{9a}{8}$$

Length of the arc BC = length of the arc AB – length of the arc AC

$$= \frac{3a}{2} - \frac{9a}{8} = \frac{3a}{8}$$

$$\frac{\text{Length of the arc } BC}{\text{Length of the arc } AC} = \frac{1}{3}$$

Example 8

Show that the length of the arc of the curve $x\sin\theta + y\cos\theta = f'(\theta)$, $x\cos\theta - y\sin\theta = f''(\theta)$ *is given by* $s = f(\theta) + f''(\theta) + C$.

Solution

$$x\sin\theta + y\cos\theta = f'(\theta) \qquad \dots (1)$$

$$x\cos\theta - y\sin\theta = f''(\theta) \qquad \dots (2)$$

Multiplying Eq. (1) by $\sin\theta$ and (2) by $\cos\theta$ and adding,

$$x = \sin\theta \, f'(\theta) + \cos\theta \, f''(\theta) \qquad \dots (3)$$

Multiplying Eq. (1) by $\cos\theta$ and (2) by $\sin\theta$ and subtracting,

$$y = \cos\theta\, f'(\theta) - \sin\theta\, f''(\theta)$$

$$\frac{dx}{d\theta} = \cos\theta\, f'(\theta) + \sin\theta\, f''(\theta) - \sin\theta\, f''(\theta) + \cos\theta\, f'''(\theta)$$

$$= \cos\theta[f'(\theta) + f'''(\theta)]$$

$$\frac{dy}{d\theta} = \cos\theta\, f''(\theta) - \sin\theta\, f'(\theta) - \cos\theta\, f''(\theta) - \sin\theta\, f'''(\theta)$$

$$= -\sin\theta[f'(\theta) + f'''(\theta)]$$

Length of the arc,

$$s = \int\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int\sqrt{(\cos^2\theta + \sin^2\theta)\,[f'(\theta) + f'''(\theta)]^2}\, d\theta$$

$$= \int[f'(\theta) + f'''(\theta)]\, d\theta$$

$$= f(\theta) + f''(\theta) + C$$

EXAMPLE 9

Show that for the curve $8a^2y^2 = x^2(a^2 - x^2)$, *arc length* $s = \dfrac{a}{2\sqrt{2}}\,(2\theta + \sin\theta\cos\theta)$ *where* $x = a\sin\theta$ *and that the perimeter of one of the loop is* $\dfrac{\pi a}{\sqrt{2}}$.

Solution

When $x = a\sin\theta$

$$8a^2y^2 = a^2\sin^2\theta(a^2 - a^2\sin^2\theta)$$

$$= a^4\sin^2\theta\cos^2\theta$$

$$y = \frac{a}{2\sqrt{2}}\sin\theta\cos\theta = \frac{a}{4\sqrt{2}}\sin 2\theta$$

$$\frac{dx}{d\theta} = a\cos\theta$$

$$\frac{dy}{d\theta} = \frac{a}{2\sqrt{2}}\cos 2\theta$$

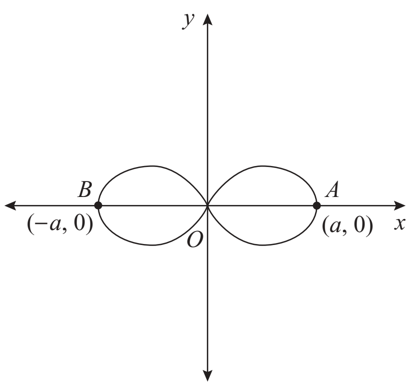


Fig. 4.18

For the upper half of the loop OA, x varies from 0 to a, hence θ varies from 0 to $\dfrac{\pi}{2}$.

Length of one loop, $s = 2(\text{Length of upper half of the loop } OA)$

$$= 2\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sqrt{a^2\cos^2\theta + \frac{a^2}{8}\cos^2 2\theta}\, d\theta$$

$$= \frac{a}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \sqrt{8\cos^2\theta + (2\cos^2\theta - 1)^2}\, d\theta$$

$$= \frac{a}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \sqrt{4\cos^4\theta + 4\cos^2\theta + 1}\, d\theta$$

$$= \frac{a}{\sqrt{2}}\int_0^{\frac{\pi}{2}} (2\cos^2\theta + 1)\, d\theta$$

$$= \frac{a}{\sqrt{2}}\int_0^{\frac{\pi}{2}} (2 + \cos 2\theta)\, d\theta$$

$$= \frac{a}{\sqrt{2}}\left|2\theta + \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{2}}$$

$$= \frac{a}{\sqrt{2}}(\pi)$$

$$= \frac{\pi a}{\sqrt{2}}$$

Example 10

Show that the length of one complete wave of the curve $y = b\cos\frac{x}{a}$ *is equal to the perimeter of the ellipse whose semi-axes are* $\sqrt{a^2+b^2}$ *and a.*

Solution

(i) $$y = b\cos\frac{x}{a}$$

$$\frac{dy}{dx} = -\frac{b}{a}\sin\frac{x}{a}$$

For one complete wave $\frac{x}{a}$ varies from 0 to 2π i.e., x varies from 0 to $2\pi a$.

Length of one complete wave, $s_1 = \int_0^{2\pi a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$= \int_0^{2\pi a} \sqrt{1 + \frac{b^2}{a^2} \sin^2 \frac{x}{a}}\, \mathrm{d}x$$

$$= \frac{1}{a} \int_0^{2\pi a} \sqrt{a^2 + b^2 \sin^2 \frac{x}{a}}\, \mathrm{d}x$$

Putting $\frac{x}{a} = t$,

$$\mathrm{d}x = a\, \mathrm{d}t$$

When $x = 0$, $t = 0$

When $x = 2\pi a$, $t = 2\pi$

$$s_1 = \frac{1}{a} \int_0^{2\pi} \sqrt{a^2 + b^2 \sin^2 t} \cdot a\, \mathrm{d}t$$

$$= \int_0^{2\pi} \sqrt{a^2 + b^2 \sin^2 t}\, \mathrm{d}t$$

$$= 2\int_0^{\pi} \sqrt{a^2 + b^2 \sin^2 t}\, \mathrm{d}t \quad \left[\because \int_0^{2a} f(t)\,\mathrm{d}t = 2\int_0^{a} f(t)\,\mathrm{d}t \text{ if } f(2a - t) = f(t)\right]$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{a^2 + b^2 \sin^2 t}\, \mathrm{d}t \quad (1)$$

(ii) Now, parametric equations of the given ellipse are $x = \sqrt{a^2 + b^2}\cos t$ and $y = a \sin t$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sqrt{a^2 + b^2}\sin t, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = a\cos t$$

For the arc AB, x varies from $\sqrt{a^2 + b^2}$ to 0, hence t varies from 0 to $\frac{\pi}{2}$.

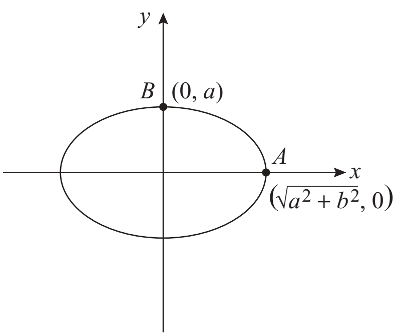


Fig. 4.19

Perimeter of the ellipse,

$$s_2 = 4 \text{ (Length of the arc } AB)$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, \mathrm{d}t$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{(a^2 + b^2)\sin^2 t + a^2 \cos^2 t}\, \mathrm{d}t$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{a^2 + b^2 \sin^2 t}\, \mathrm{d}t \quad \ldots (2)$$

From Eqs (1) and (2),

Length of one complete wave = perimeter of the ellipse.

Example 11

Show that the perimeter of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *is* $2\pi a\left[1-\frac{e^2}{2^2}-\frac{1^2\cdot 3}{2^2\cdot 4^2}e^4-\frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6^2}e^6-\ldots\right]$, *where e is the eccentricity of the ellipse.*

Solution

The parametric equations of the given ellipse are $x=a\cos\theta$ and $y=b\sin\theta$.

$$\frac{dx}{d\theta}=-a\sin\theta$$

$$\frac{dy}{d\theta}=b\cos\theta$$

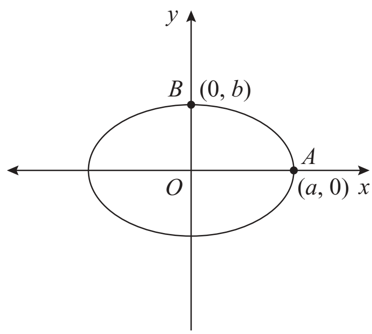


Fig. 4.20

For the arc *AB*, *x* varies from *a* to 0, hence θ varies from 0 to $\frac{\pi}{2}$.

Perimeter of the ellipse = 4(Length of the arc AB)

$$=4\int_0^{\frac{\pi}{2}}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$=4\int_0^{\frac{\pi}{2}}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\,d\theta$$

$$=4\int_0^{\frac{\pi}{2}}\sqrt{a^2\sin^2\theta+a^2(1-e^2)\cos^2\theta}\,d\theta \qquad \left[\because e=\sqrt{1-\frac{b^2}{a^2}}\right]$$

$$=4a\int_0^{\frac{\pi}{2}}(1-e^2\cos^2\theta)^{\frac{1}{2}}\,d\theta$$

$$=4a\int_0^{\frac{\pi}{2}}\left[1+\frac{1}{2}(-e^2\cos^2\theta)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-e^2\cos^2\theta)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-e^2\cos^2\theta)^3+\ldots\right]d\theta$$

$$=4a\int_0^{\frac{\pi}{2}}\left(1-\frac{1}{2}e^2\cos^2\theta-\frac{1}{2\cdot 4}e^4\cos^4\theta-\frac{1\cdot 3}{2\cdot 4\cdot 6}e^6\cos^6\theta\ldots\right)d\theta$$

$$= 4a\left[\frac{\pi}{2}-\frac{1}{2}e^2\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\frac{1}{2\cdot 4}e^4\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\frac{1\cdot 3}{2\cdot 4\cdot 6}e^6\cdot\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}-\ldots\right]$$

$$= 2\pi a\left[1-\frac{e^2}{2^2}-\frac{1^2\cdot 3}{2^2\cdot 4^2}e^4-\frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6^2}e^6\cdots\right]$$

EXERCISE 4.3

1. Find the length of the following curves:

(i) $x = a(2\cos\theta + \cos 2\theta),\ y = a(2\sin\theta + \sin 2\theta)$, from $\theta=0$ to any point θ.

(ii) $x = a(\theta - \sin\theta),\ \ y = a(1-\cos\theta)$ from $\theta = 0$ to $\theta = 2\pi$

(iii) $x = ae^{\theta}\sin\theta,\ \ y = ae^{\theta}\cos\theta$ from $\theta = 0$ to $\theta = \dfrac{\pi}{2}$

(iv) $x = \log(\sec\theta + \tan\theta) - \sin\theta,\ \ y = \cos\theta$ from $\theta = 0$ to any point θ

(v) $x = a(t - \tanh t),\ \ y = a\,\text{sech}\,t$ from $t = 0$ to any point t.

(vi) $x = (a+b)\cos\theta - b\cos\left(\dfrac{a+b}{b}\theta\right),\ y = (a+b)\sin\theta - b\sin\left(\dfrac{a+b}{b}\theta\right)$ from $\theta = \dfrac{\pi b}{a}$ to any point θ.

(vii) $x = a\sin 2\theta(1+\cos 2\theta),\ \ y = a\cos 2\theta(1-\cos 2\theta)$, from $\theta = 0$ to any point θ

Ans.: (i) $8a\sin\dfrac{\theta}{2}$, (ii) $8a$ (iii) $\sqrt{2}(e^{\frac{\pi}{2}}-1)a$, (iv) $\log\sec\theta$ (v) $\log\cosh t$ (vi) $\dfrac{4b}{a}(a+b)\cos\left(\dfrac{a\theta}{2b}\right)$ (vii) $\dfrac{4}{3}a\sin\theta$

2. Prove that the loop of the curve $x = t^2,\ \ y = t - \dfrac{1}{3}t^3$ is of length $4\sqrt{3}.$

3. Show that the length of the arc of the curve $x = a(3\sin\theta - \sin^3\theta)$, $y = a\cos^3\theta$ measured from $(0, a)$ to any point (x, y) is $\dfrac{3}{2}a(\theta + \sin\theta\cos\theta)$.

4. If 's' be the length of the arc of the curve $x = a(\theta + \sin\theta\cos\theta)$, $y = a(1 + \sin\theta)^2$, measured from the point $\theta = -\dfrac{\pi}{2}$ to a point θ, show that s^4 varies as y^3.

4.3.3 Length of Arc in Polar Form

For the curve $r = f(\theta)$, we have, from differential calculus,

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

The length of the arc of the curve $r = f(\theta)$ between the points $\theta = \theta_1$ and $\theta = \theta_2$ is given by,

$$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

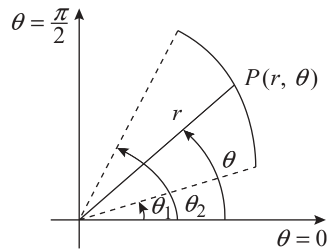


Fig. 4.21

Similarly, the length of the arc of the curve $\theta = f(r)$ between the points $r = r_1$ and $r = r_2$ is given by,

$$s = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$$

Example 1

Find the length of the spiral $r = e^{2\theta}$ from $\theta = 0$ to $\theta = 2\pi$.

Solution

$$r = e^{2\theta}$$

$$\frac{dr}{d\theta} = 2e^{2\theta}$$

For the required length of the spiral, θ varies from 0 to 2π.

Length of the spiral, $$s = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{e^{4\theta} + 4e^{4\theta}}\, d\theta$$

$$= \sqrt{5}\int_0^{2\pi} e^{2\theta} d\theta$$

$$= \sqrt{5}\left|\frac{e^{2\theta}}{2}\right|_0^{2\pi}$$

$$= \frac{\sqrt{5}}{2}(e^{4\pi} - 1)$$

Example 2

Find the length of the arc of the equiangular spiral $r = ae^{\theta\cot\alpha}$ *from the point corresponding to* $\theta = 0$ *to the point corresponding to* $\theta = \tan\alpha$.

Solution

$$r = ae^{\theta\cot\alpha}$$

$$\frac{dr}{d\theta} = a\cot\alpha\, e^{\theta\cot\alpha}$$

For the required arc, θ varies from 0 to tan α.

Length of the arc,
$$s = \int_0^{\tan\alpha}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$
$$= \int_0^{\tan\alpha}\sqrt{a^2e^{2\theta\cot\alpha} + a^2\cot^2\alpha e^{2\theta\cot\alpha}}\, d\theta$$
$$= a\sqrt{1+\cot^2\alpha}\int_0^{\tan\alpha} e^{\theta\cot\alpha}\, d\theta$$
$$= a\,\text{cosec}\,\alpha\left|\frac{e^{\theta\cot\alpha}}{\cot\alpha}\right|_0^{\tan\alpha}$$
$$= \frac{a\,\text{cosec}\,\alpha}{\cot\alpha}(e^{\tan\alpha\cot\alpha} - 1)$$
$$= a\sec\alpha(e-1)$$
$$= a(e-1)\sec\alpha$$

Example 3

Find the length of the cissoid $r = 2a\tan\theta\sin\theta$ *from* $\theta = 0$ *to* $\theta = \dfrac{\pi}{4}$.

Solution

$$r = 2a\tan\theta\sin\theta$$

$$\frac{dr}{d\theta} = 2a(\sec^2\theta\sin\theta + \tan\theta\cos\theta)$$
$$= 2a\sin\theta(\sec^2\theta + 1)$$

For the required arc length of the cissoid, θ varies from 0 to $\dfrac{\pi}{4}$.

Length of the curve,
$$s = \int_0^{\frac{\pi}{4}}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{4a^2 \tan^2\theta \sin^2\theta + 4a^2 \sin^2\theta(\sec^2\theta + 1)^2}\, \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{4a^2 \sin^2\theta(\sec^2\theta - 1 + \sec^4\theta + 2\sec^2\theta + 1)}\, \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{4a^2 \sin^2\theta \sec^2\theta(\sec^2\theta + 3)}\, \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{4a^2 \tan^2\theta\,(\tan^2\theta + 4)}\, \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{4}} 2a\tan\theta\sqrt{\tan^2\theta + 4}\, \mathrm{d}\theta$$

Putting $\tan^2\theta + 4 = t^2$,

$$2\tan\theta\sec^2\theta\, \mathrm{d}\theta = 2t\, \mathrm{d}t$$

$$\tan\theta\, \mathrm{d}\theta = \frac{t\mathrm{d}t}{\sec^2\theta} = \frac{t\,\mathrm{d}t}{1+\tan^2\theta} = \frac{t\,\mathrm{d}t}{t^2 - 3}$$

When $\theta = 0, \quad t = 2$

When $\theta = \frac{\pi}{4}, \quad t = \sqrt{5}$

$$s = \int_2^{\sqrt{5}} 2a \cdot \frac{t^2}{t^2 - 3}\, \mathrm{d}t$$

$$= \int_2^{\sqrt{5}} 2a\left(1 + \frac{3}{t^2 - 3}\right)\mathrm{d}t$$

$$= 2a\left| t + \frac{3}{2\sqrt{3}} \log\frac{t - \sqrt{3}}{t + \sqrt{3}} \right|_2^{\sqrt{5}}$$

$$= 2a\left(\sqrt{5} + \frac{\sqrt{3}}{2}\log\frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}} - 2 - \frac{\sqrt{3}}{2}\log\frac{2 - \sqrt{3}}{2 + \sqrt{3}}\right)$$

$$= 2a\left[\sqrt{5} - 2 + \frac{\sqrt{3}}{2}\log\left\{\frac{\left(\sqrt{5} - \sqrt{3}\right)^2}{5 - 3}\right\} - \log\left\{\frac{\left(2 - \sqrt{3}\right)^2}{4 - 3}\right\}\right]$$

$$= 2a\left[\sqrt{5} - 2 + \frac{\sqrt{3}}{2}\log\left(4 - \sqrt{15}\right) - \log\left(7 - 4\sqrt{3}\right)\right]$$

Note: Only positive values of t are considered since θ lies in the first quadrant.

Example 4

Find the length of the whole arc of the cardioid $r = a(1 + \cos\theta)$ *and show that the upper half is bisected by the line* $\theta = \frac{\pi}{3}$.

Solution

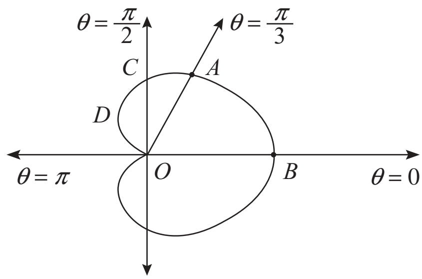


Fig. 4.22

$$r = a(1+\cos\theta)$$

$$\frac{dr}{d\theta} = -a\sin\theta$$

(i) For the arc *BACDO*, θ varies from 0 to π.

Length of the whole arc of the curve,

$$s = 2(\text{Length of arc } BACDO)$$

$$= 2\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$= 2\int_0^{\pi} \sqrt{a^2(1+\cos\theta)^2 + (-a\sin\theta)^2}\, d\theta$$

$$= 2\int_0^{\pi} a\sqrt{2+2\cos\theta}\, d\theta$$

$$= 2\int_0^{\pi} a\sqrt{2\cdot 2\cos^2\frac{\theta}{2}}\, d\theta$$

$$= 4a\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta$$

$$= 4a\left|2\sin\frac{\theta}{2}\right|_0^{\pi}$$

$$= 8a$$

Length of the upper half of the cardioid = $4a$

(ii) Let $\theta = \frac{\pi}{3}$ intersects the cardioid at point *A*.

Length of the arc, $BA = \int_0^{\frac{\pi}{3}} 2a\cos\frac{\theta}{2}\, d\theta = 2a\left|2\sin\frac{\theta}{2}\right|_0^{\frac{\pi}{3}} = 2a$

Hence, the upper half of the cardioid is bisected by the line $\theta = \frac{\pi}{3}$.

Example 5

Find the length of the cardioid $r = a(1-\cos\theta)$ *lying outside the circle* $r = a\cos\theta$.

Solution

The points of intersection of cardioid $r = a(1-\cos\theta)$ and the circle $r = a\cos\theta$ is obtained as,

$$a(1-\cos\theta) = a\cos\theta$$
$$1 = 2\cos\theta$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \pm\frac{\pi}{3}$$

Hence at A, $\theta = \frac{\pi}{3}$

$$r = a(1-\cos\theta)$$
$$\frac{dr}{d\theta} = a\sin\theta$$

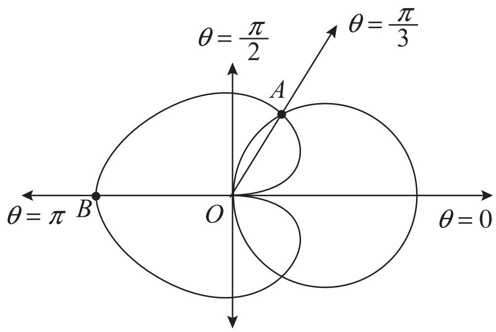


Fig. 4.23

For the arc of the cardioid lying outside the circle, θ varies from $\frac{\pi}{3}$ to π.

Length of the cardioid lying outside the circle, $s = 2$ (Length of arc AB)

$$= 2\int_{\frac{\pi}{3}}^{\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$
$$= 2\int_{\frac{\pi}{3}}^{\pi}\sqrt{a^2(1-\cos\theta)^2+(a\sin\theta)^2}\,d\theta$$
$$= 2\int_{\frac{\pi}{3}}^{\pi}a\sqrt{2-2\cos\theta}\,d\theta$$
$$= 2\int_{\frac{\pi}{3}}^{\pi}a\sqrt{2\cdot 2\sin^2\frac{\theta}{2}}\,d\theta$$
$$= 4a\int_{\frac{\pi}{3}}^{\pi}\sin\frac{\theta}{2}\,d\theta$$
$$= 4a\left|-2\cos\frac{\theta}{2}\right|_{\frac{\pi}{3}}^{\pi}$$
$$= -8a\left(-\frac{\sqrt{3}}{2}\right)$$
$$= 4a\sqrt{3}$$

Example 6

Show that the length of the arc of that part of cardioid $r = a(1+\cos\theta)$ *which lies on the side of the line* $4r = 3a\sec\theta$ *away from the pole is* $4a$.

Solution

The points of intersection of cardioid $r = a\,(1+\cos\theta)$ and the line $4r = 3a\sec\theta$ are obtained as,

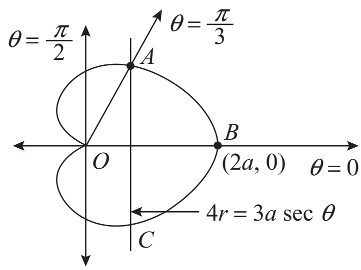


Fig. 4.24

$$a\,(1+\cos\theta) = \frac{3a}{4}\sec\theta$$

$$4\,(1+\cos\theta)\cos\theta = 3$$

$$4\cos\theta + 4\cos^2\theta - 3 = 0$$

$$(2\cos\theta+3)(2\cos\theta-1) = 0$$

$\cos\theta = \frac{1}{2}$ and $\cos\theta = \frac{-3}{2}$ (does not exist)

$$\theta = \pm\frac{\pi}{3}$$

Hence at A, $\theta = \frac{\pi}{3}$

$$r = a\,(1+\cos\theta)$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -a\sin\theta$$

For the arc BA, θ varies from 0 to $\frac{\pi}{3}$.

Length of the arc CBA, $\quad s = 2$ (length of arc BA)

$$= 2\int_0^{\frac{\pi}{3}} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

$$= 2\int_0^{\frac{\pi}{3}} \sqrt{a^2(1+\cos\theta)^2 + (-a\sin\theta)^2}\,\mathrm{d}\theta$$

$$= 2\int_0^{\frac{\pi}{3}} a\sqrt{2+2\cos\theta}\,\mathrm{d}\theta$$

$$= 2\int_0^{\frac{\pi}{3}} a\sqrt{2\cdot 2\cos^2\frac{\theta}{2}}\,\mathrm{d}\theta$$

$$= 4a\int_0^{\frac{\pi}{3}} \cos\frac{\theta}{2}\,\mathrm{d}\theta$$

$$= 4a\left|2\sin\frac{\theta}{2}\right|_0^{\frac{\pi}{3}}$$

$$= 4a$$

Example 7

Find the total length of the curve $r = a\sin^3\dfrac{\theta}{3}$.

Solution

$$r = a\sin^3\frac{\theta}{3}$$

$$\frac{dr}{d\theta} = a\cdot 3\sin^2\frac{\theta}{3}\cos\frac{\theta}{3}\cdot\frac{1}{3}$$

$$= a\sin^2\frac{\theta}{3}\cos\frac{\theta}{3}$$

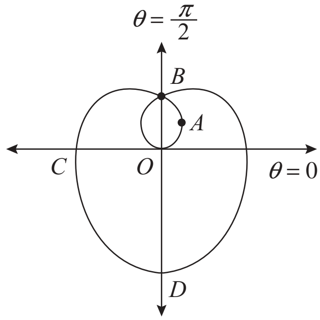


Fig. 4.25

For the arc *OABCD*, θ varies from 0 to $\dfrac{3\pi}{2}$.

$$\text{Length of the curve} = 2(\text{Length of the arc } OABCD)$$

$$= 2\int_0^{\frac{3\pi}{2}}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$= 2\int_0^{\frac{3\pi}{2}}\sqrt{a^2\sin^6\frac{\theta}{3} + a^2\sin^4\frac{\theta}{3}\cos^2\frac{\theta}{3}}\,d\theta$$

$$= 2\int_0^{\frac{3\pi}{2}} a\sin^2\frac{\theta}{3}\,d\theta$$

$$= a\int_0^{\frac{3\pi}{2}}\left(1 - \cos\frac{2\theta}{3}\right)d\theta$$

$$= a\left|\theta - \frac{3}{2}\sin\frac{2\theta}{3}\right|_0^{\frac{3\pi}{2}}$$

$$= a\cdot\frac{3\pi}{2} = \frac{3}{2}\pi a$$

Example 8

Find the perimeter of the lemniscate $r^2 = a^2\cos 2\theta$.

Solution

$$r^2 = a^2\cos 2\theta$$

$$2r\frac{dr}{d\theta} = a^2(-\sin 2\theta)\cdot 2$$

$$\frac{dr}{d\theta} = -\frac{a^2}{r}\sin 2\theta$$

For the arc *OBA*, θ varies from 0 to $\dfrac{\pi}{4}$.

Perimeter of the curve

$$= 4(\text{Length of the arc } OBA)$$

$$= 4\int_0^{\frac{\pi}{4}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \sqrt{a^2 \cos 2\theta + \frac{a^4}{r^2}\sin^2 2\theta}\, d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \sqrt{a^2 \cos 2\theta + \frac{a^4}{r^2}\sin^2 2\theta}\, d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \sqrt{a^2 \cos 2\theta + \frac{a^4 \sin^2 2\theta}{a^2 \cos 2\theta}}\, d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \sqrt{\frac{a^4}{a^2 \cos 2\theta}}\, d\theta$$

$$= 4a\int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{\cos 2\theta}}\, d\theta$$

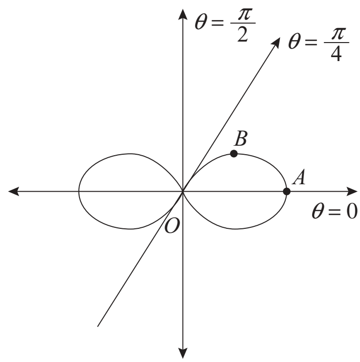


Fig. 4.26

Putting $2\theta = t,$

$$2d\theta = dt$$

When $\theta = 0, \quad t = 0$

When $\theta = \frac{\pi}{4}, \quad t = \frac{\pi}{2}$

Perimeter of the curve $= \frac{4a}{2}\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{\cos t}}$

$$= 2a\int_0^{\frac{\pi}{2}} \sin^0 t \cdot (\cos t)^{\frac{-1}{2}}\, dt$$

$$= aB\left(\frac{1}{2}, \frac{1}{4}\right)$$

$$= \frac{a\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{a\Gamma\left(\frac{1}{2}\right)\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)}$$

$$= \frac{a\Gamma\left(\frac{1}{2}\right)\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\dfrac{\pi}{\sin\dfrac{\pi}{4}}}$$

$$= \frac{a\sqrt{\pi}\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\pi\sqrt{2}} \qquad \left[\because \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{a}{\sqrt{2\pi}}\left(\Gamma\left(\frac{1}{4}\right)\right)^2$$

Example 9

Show that for the parabola $\dfrac{2a}{r} = 1 + \cos\theta$, *the arc intercepted between the vertex and the extremity of the latus rectum is* $a\left[\sqrt{2} + \log(1+\sqrt{2})\right]$.

Solution

The latus rectum is the line $\theta = \dfrac{\pi}{2}$.

$$\frac{2a}{r} = 1 + \cos\theta$$

$$r = \frac{2a}{1+\cos\theta} = \frac{2a}{2\cos^2\dfrac{\theta}{2}} = a\sec^2\frac{\theta}{2}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = a\sec^2\frac{\theta}{2}\cdot\tan\frac{\theta}{2}$$

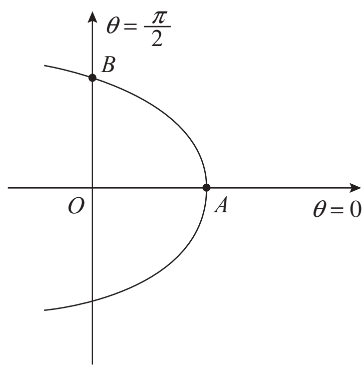


Fig. 4.27

For the arc AB, θ varies from 0 to $\dfrac{\pi}{2}$.

Length of the arc AB,

$$s = \int_0^{\frac{\pi}{2}} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{a^2\sec^4\frac{\theta}{2} + a^2\sec^4\frac{\theta}{2}\tan^2\frac{\theta}{2}}\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} a\sec^2\frac{\theta}{2}\sqrt{1+\tan^2\frac{\theta}{2}}\,\mathrm{d}\theta$$

Putting $$\tan\frac{\theta}{2} = t,$$

$$\frac{1}{2}\sec^2\frac{\theta}{2}\,\mathrm{d}\theta = \mathrm{d}t,\ \sec^2\frac{\theta}{2}\,\mathrm{d}\theta = 2\mathrm{d}t$$

When $\theta = 0,\quad t = 0$

When $\theta = \dfrac{\pi}{2},\quad t = 1$

$$\begin{aligned}s &= \int_0^1 2a\sqrt{1+t^2}\,\mathrm{d}t\\ &= 2a\left|\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\log\left(t+\sqrt{1+t^2}\right)\right|_0^1\\ &= 2a\left[\frac{1}{2}\sqrt{2} + \frac{1}{2}\log\left(1+\sqrt{2}\right)\right]\\ &= a\left[\sqrt{2} + \log\left(1+\sqrt{2}\right)\right]\end{aligned}$$

Example 10

Show that the whole length of the limacon $r = a\cos\theta + b$ ($a < b$) is equal to that of an ellipse whose semi-axes are equal in length to the maximum and minimum radii vectors of the limacon.

Solution

$$r = a\cos\theta + b$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -a\sin\theta$$

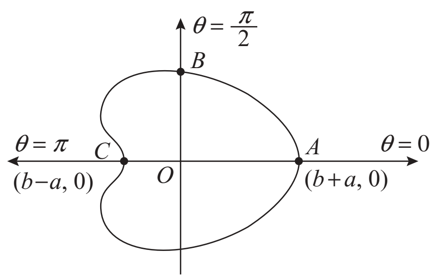


Fig. 4.28

For the arc ABC, θ varies from 0 to π.

Whole length of the limacon

$$\begin{aligned}&= 2(\text{length of the arc } ABC)\\ &= 2\int_0^\pi \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta\\ &= 2\int_0^\pi \sqrt{(a\cos\theta + b)^2 + (-a\sin\theta)^2}\,\mathrm{d}\theta\\ &= 2\int_0^\pi \sqrt{a^2 + b^2 + 2ab\cos\theta}\,\mathrm{d}\theta \qquad \ldots (1)\end{aligned}$$

Maximum radius vector of the limacon $= a(1) + b = b + a$

$[\because \text{Maximum value of } \cos\theta = 1]$

Minimum radius vector of the limacon $= a(-1) + b = b - a$

$\left[\because \text{Minimum value of } \cos\theta = -1\right]$

The parametric equations of the ellipse with above radii vectors as semi-axes are given as, $x = (b+a)\cos\theta$ and $y = (b-a)\sin\theta$

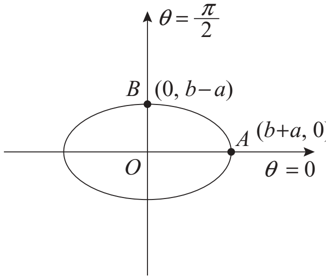


Fig. 4.29

$$\frac{dx}{d\theta} = -(b+a)\sin\theta$$

$$\frac{dy}{d\theta} = (b-a)\cos\theta$$

For the arc AB, θ varies from 0 to $\frac{\pi}{2}$.

Whole length of the ellipse = 4(length of the arc AB)

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{[-(b+a)\sin\theta]^2 + [(b-a)\cos\theta]^2}\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{(a^2 + b^2 + 2ab\sin^2\theta - 2ab\cos^2\theta)}\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{(a^2 + b^2 - 2ab\cos 2\theta)}\, d\theta$$

Putting $2\theta = t$,

$$d\theta = \frac{dt}{2}$$

When $\theta = 0, \quad t = 0$

When $\theta = \frac{\pi}{2}, \quad t = \pi$

Whole length of the ellipse $= 2\int_0^{\pi} \sqrt{a^2 + b^2 - 2ab\cos t}\, dt$

$$= 2\int_0^{\pi} \sqrt{a^2 + b^2 - 2ab\cos(\pi - t)}\, dt$$

$$= 2\int_0^{\pi} \sqrt{a^2 + b^2 + 2ab\cos t}\, dt$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right] \quad \ldots (2)$$

From Eqs. (1) and (2),

Whole length of the limacon = Whole length of the ellipse

Example 11

Find the length of the arc of the hyperbolic spiral $r\theta = a$ from the point $r = a$ to $r = 2a$.

Solution

$$r\theta = a$$

$$r\frac{d\theta}{dr} + \theta = 0$$

Length of the arc,

$$s = \int_a^{2a} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$$

$$= \int_a^{2a} \sqrt{1+\theta^2}\, dr$$

$$= \int_a^{2a} \sqrt{1+\frac{a^2}{r^2}}\, dr$$

$$= \int_a^{2a} \sqrt{\frac{r^2+a^2}{r^2}}\, dr$$

Putting $r^2 + a^2 = t^2$,

$$2r\, dr = 2t\, dt, \quad dr = \frac{t\, dt}{\sqrt{t^2 - a^2}}$$

When $r = a, \quad t = a\sqrt{2}$

When $r = 2a, \quad t = a\sqrt{5}$

$$s = \int_{a\sqrt{2}}^{a\sqrt{5}} \frac{t}{\sqrt{t^2-a^2}} \frac{t\, dt}{\sqrt{t^2-a^2}}$$

$$= \int_{a\sqrt{2}}^{a\sqrt{5}} \frac{t^2 - a^2 + a^2}{t^2 - a^2}\, dt$$

$$= \int_{a\sqrt{2}}^{a\sqrt{5}} dt + a^2 \int_{a\sqrt{2}}^{a\sqrt{5}} \frac{dt}{t^2 - a^2}$$

$$= \left| t \right|_{a\sqrt{2}}^{a\sqrt{5}} + \frac{a^2}{2a}\left|\log\frac{t-a}{t+a}\right|_{a\sqrt{2}}^{a\sqrt{5}}$$

$$= a\left(\sqrt{5} - \sqrt{2}\right) + \frac{a}{2}\left[\log\frac{a\sqrt{5}-a}{a\sqrt{5}+a} - \log\frac{a\sqrt{2}-a}{a\sqrt{2}+a}\right]$$

$$= a\left(\sqrt{5} - \sqrt{2}\right) + \frac{a}{2}\left[\log\frac{\sqrt{5}-1}{\sqrt{5}+1} - \log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right]$$

$$= a\left(\sqrt{5}-\sqrt{2}\right)+\frac{a}{2}\log\left[\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right]$$

$$= a\left(\sqrt{5}-\sqrt{2}\right)+\frac{a}{2}\log\left[\frac{5-1}{\left(\sqrt{5}+1\right)^2}\frac{\left(\sqrt{2}+1\right)^2}{2-1}\right]$$

$$= a\left(\sqrt{5}-\sqrt{2}\right)+a\log\frac{2\left(\sqrt{2}+1\right)}{\sqrt{5}+1}$$

EXERCISE 4.4

1. Find the perimeter of the following curves:

(i) $r = a\cos\theta$ (ii) $r = a(\theta^2 - 1)$

(iii) $r = a\cos^3\left(\frac{\theta}{3}\right)$ (iv) $r = ae^{m\theta}$

(v) $r = a\theta$ (vi) $r = a\sec^2\frac{\theta}{2}$

(vii) $r = 4\sin^2\theta$

[**Ans.:**

(i) πa (ii) $\frac{8a}{3}$

(iii) $\frac{3\pi a}{2}$ (iv) $(r_2 - r_1)\frac{\sqrt{1+m^2}}{m}$

(v) $\frac{a}{2}\left[\theta\sqrt{1+\theta^2}+\sinh^{-1}\theta\right]$ (vi) $2a\left[\sqrt{2}+\log(\sqrt{2}+1)\right]$

(vii) $8+\frac{4}{\sqrt{3}}\log(\sqrt{3}+2)$]

2. Find the perimeter of the cardioid $r = a(1-\cos\theta)$ and prove that the line $\theta = \frac{2\pi}{3}$ bisects the upper half of the cardioid.

[Ans.: $8a$**]**

3. Find the length of the cardioid $r = a(1+\cos\theta)$ which lies outside the circle $r + a\cos\theta = 0$.

$\left[\textbf{Ans.: } 4\sqrt{3}a\right]$

4. Prove that the length of the spiral $r = ae^{\theta\cot\alpha}$ as r increases from r_1 to r_2 is given by $(r_2 - r_1)\sec\alpha$.

5. Find the length of the cardioid $r = a(1-\cos\theta)$ lying inside the circle $r = a\cos\theta$.

$$\left[\textbf{Ans.}: 8a\left(1-\frac{\sqrt{3}}{2}\right)\right]$$

6. Find the length of the spiral $r = ae^{m\theta}$ lying inside the circle $r = a$.

$$\left[\textbf{Ans.}: \frac{a}{m}\sqrt{1+m^2}\right]$$

7. Find the length of the arc of parabola $\frac{l}{r} = 1+\cos\theta$ cut off by its latus rectum.

$$\left[\textbf{Ans.}: l\left[\sqrt{2}+\log\left(1+\sqrt{2}\right)\right]\right]$$

4.4 AREA OF SURFACE OF SOLID OF REVOLUTION

Let $y = f(x)$ be a curve included between two lines $x = a$ and $x = b$. Let $P(x, y)$ be any point on the curve. When the chord PQ is revolved about the x-axis, a solid of revolution is generated. The elementary surface area δS is approximately equal to the circumference of the circle multiplied by the PQ.

$$\delta S = 2\pi y PQ = 2\pi y\,\delta s$$

The total surface area of the solid of revolution about x-axis is given by,

$$S = \int 2\pi y\,\mathrm{d}s$$

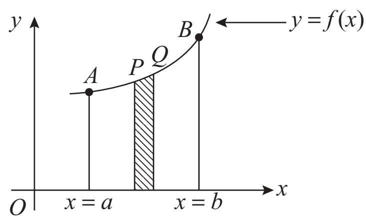


Fig. 4.30

4.4.1 Area of Surface of Solid of Revolution in Cartesian Form

Area of surface generated by revolving the arc of the curve $y = f(x)$ about the x-axis is given by,

$$\begin{aligned} S &= \int_a^b 2\pi y\,\mathrm{d}s \\ &= \int_a^b 2\pi y \frac{\mathrm{d}s}{\mathrm{d}x}\,\mathrm{d}x \\ &= \int_a^b 2\pi y \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \end{aligned}$$

Similarly, the area of the surface generated by revolving the arc of the curve $x = f(y)$ about y-axis is given by,

$$S = \int_c^d 2\pi x \, \mathrm{d}s$$

$$= \int_c^d 2\pi x \frac{\mathrm{d}s}{\mathrm{d}y} \mathrm{d}y$$

$$= \int_c^d 2\pi x \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} \mathrm{d}y$$

Example 1

Find the area of the surface of revolution generated by revolving the curve $x = y^3$ from $y = 0$ to $y = 2$.

Solution

$$x = y^3 \qquad \frac{\mathrm{d}x}{\mathrm{d}y} = 3y^2$$

$$\frac{\mathrm{d}s}{\mathrm{d}y} = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} = \sqrt{1 + 9y^4}$$

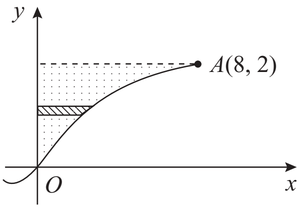


Fig. 4.31

The area of the surface is generated by revolving the region about the y-axis. For the region shown, y varies from 0 to 2.

Surface area,

$$S = \int_0^2 2\pi x \frac{\mathrm{d}s}{\mathrm{d}y} \mathrm{d}y$$

$$= \int_0^2 2\pi y^3 \sqrt{1 + 9y^4} \mathrm{d}y$$

$$= \frac{2\pi}{36} \int_0^2 (1 + 9y^4)^{\frac{1}{2}} (36y^3) \mathrm{d}y$$

$$= \frac{\pi}{18} \left| \frac{(1 + 9y^4)^{\frac{3}{2}}}{\frac{3}{2}} \right|_0^2 \qquad \left[\because \int [f(y)]^n f'(y) \, \mathrm{d}y = \frac{[f(y)]^{n+1}}{n+1} \right]$$

$$= \frac{\pi}{27} \left(145\sqrt{145} - 1\right)$$

Example 2

Find the area of the surface of revolution of the solid generated by revolving the ellipse $\frac{x^2}{16}+\frac{y^2}{4}=1$ *about the x-axis.*

Solution

$$\frac{x^2}{16}+\frac{y^2}{4}=1$$

$$\frac{2x}{16}+\frac{2y}{4}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x}{4y}$$

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$=\sqrt{1+\frac{x^2}{16y^2}}$$

$$=\frac{\sqrt{x^2+16y^2}}{4y}$$

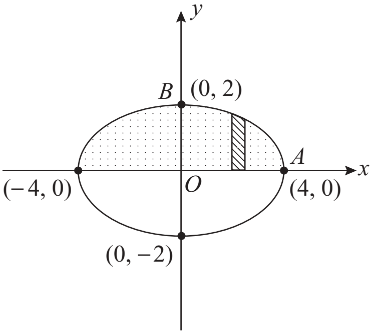


Fig. 4.32

The area of the surface of solid is generated by revolving the upper half of the ellipse about the x-axis. For the region above the x-axis, x varies from -4 to 4. Due to symmetry about the y-axis, considering the region in the first quadrant where x varies from 0 to 4,

Surface area,
$$S=2\int_0^4 2\pi y\frac{ds}{dx}\,dx$$

$$=4\pi\int_0^4 y\frac{\sqrt{x^2+16y^2}}{4y}\,dx$$

$$=\pi\int_0^4\sqrt{x^2+64-4x^2}\,dx$$

$$=\pi\sqrt{3}\int_0^4\sqrt{\left(\frac{8}{\sqrt{3}}\right)^2-x^2}\,dx$$

$$=\pi\sqrt{3}\left|\frac{x}{2}\sqrt{\frac{64}{3}-x^2}+\frac{1}{2}\cdot\frac{64}{3}\sin^{-1}\frac{x\sqrt{3}}{8}\right|_0^4$$

$$=\pi\sqrt{3}\left[2\sqrt{\frac{64}{3}-16}+\frac{32}{3}\sin^{-1}\frac{\sqrt{3}}{2}\right]$$

$$=8\pi\left(1+\frac{4\pi}{3\sqrt{3}}\right)$$

Example 3

The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the surface area of the revolution.

Solution

The points of intersection of the parabola $y^2 = 4ax$ and its latus rectum $x = a$ are obtained as,

$$y^2 = 4a \cdot a = 4a^2$$
$$y = \pm 2a \text{ and } x = a$$

Hence, A: $(a, 2a)$ and B: $(a, -2a)$ are the points of intersection.

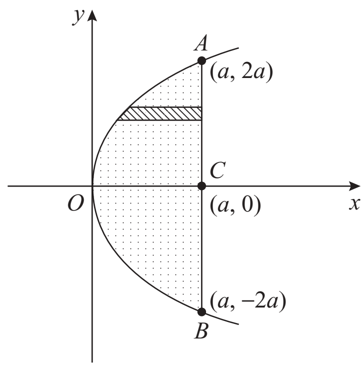


Fig. 4.33

Now,
$$x = \frac{y^2}{4a}$$
$$\frac{dx}{dy} = \frac{2y}{4a} = \frac{y}{2a}$$
$$\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$$
$$= \sqrt{1+\frac{y^2}{4a^2}}$$

The surface area is generated by revolving the region about the tangent at the vertex i.e., y-axis. For the region shown, y varies from $-2a$ to $2a$. Due to symmetry about x-axis, considering the region in the first quadrant where y varies from 0 to $2a$,

Surface area,
$$S = 2\int_0^{2a} 2\pi x \frac{ds}{dy} dy$$
$$= 2\int_0^{2a} 2\pi \cdot \frac{y^2}{4a}\sqrt{1+\frac{y^2}{4a^2}}\, dy$$

Putting
$$y = 2a \tan\theta,$$
$$dy = 2a \sec^2\theta \, d\theta$$

When $y = 0, \theta = 0$

When $y = 2a, \theta = \dfrac{\pi}{4}$

$$S = 4\pi\int_0^{\frac{\pi}{4}} \frac{4a^2\tan^2\theta}{4a}\sqrt{1+\tan^2\theta}\; 2a\sec^2\theta \, d\theta$$
$$= 8\pi a^2\int_0^{\frac{\pi}{4}} \tan^2\theta \sec^3\theta \, d\theta$$
$$= 8\pi a^2\int_0^{\frac{\pi}{4}} (\sec^5\theta - \sec^3\theta)\, d\theta$$

$$= 8\pi a^2 \left| \frac{1}{4}\tan\theta\sec^3\theta + \frac{3}{8}\tan\theta\sec\theta + \frac{3}{8}\log(\sec\theta + \tan\theta) - \frac{1}{2}\tan\theta\sec\theta - \frac{1}{2}\log(\sec\theta+\tan\theta)\right|_0^{\frac{\pi}{4}}$$

[Using reduction formula]

$$= 8\pi a^2\left[\frac{1}{4}2\sqrt{2} + \frac{3}{8}\sqrt{2} + \frac{3}{8}\log(\sqrt{2}+1) - \frac{1}{2}\sqrt{2} - \frac{1}{2}\log\left(\sqrt{2}+1\right)\right]$$

$$= \pi a^2\left[3\sqrt{2} - \log\left(\sqrt{2}+1\right)\right]$$

Example 4

Find the surface area generated by revolving the loop of the curve $9ay^2 = x(3a - x)^2$ *about the x-axis.*

Solution

The points of intersection of the curve $9ay^2 = x(3a - x)^2$ and x-axis are obtained as,

$$0 = x(3a - x)^2$$

$$x = 0, 3a, 3a \quad \text{and} \quad y = 0, 0, 0$$

Hence, A : $(3a, 0)$ is the point of intersection.

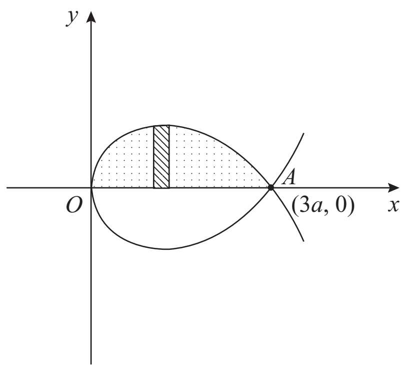


Fig. 4.34

Now, $$9ay^2 = x(3a - x)^2$$

$$18ay\frac{dy}{dx} = (3a - x)^2 - 2x(3a - x)$$

$$\frac{dy}{dx} = \frac{(3a - x)^2 - 2x(3a - x)}{18ay}$$

$$= \frac{(3a - x)(a - x)}{6ay}$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= \sqrt{1 + \frac{(3a - x)^2(a - x)^2}{36a^2y^2}}$$

$$= \sqrt{\frac{36a^2y^2 + (3a - x)^2(a - x)^2}{36a^2y^2}}$$

$$= \frac{1}{6ay}\sqrt{4ax(3a - x)^2 + (3a - x)^2(a - x)^2}$$

$$= \frac{1}{6ay}\sqrt{(3a-x)^2(a+x)^2}$$

$$= \frac{(3a-x)(a+x)}{6ay}$$

The surface area is generated by revolving the loop about the x-axis. For the loop, x varies from 0 to $3a$.

Surface area, $S = \int_0^{3a} 2\pi y \frac{ds}{dx} dx$

$$= 2\pi \int_0^{3a} y \cdot \frac{(3a-x)(a+x)}{6ay} dx$$

$$= \frac{\pi}{3a}\int_0^{3a} (3a^2 + 2ax - x^2)\, dx$$

$$= \frac{\pi}{3a}\left| 3a^2x + ax^2 - \frac{x^3}{3} \right|_0^{3a}$$

$$= 3\pi a^2$$

Example 5

Find the area of the surface of revolution of a quadrant of a circular arc as obtained by revolving it about a tangent at one of its ends.

Solution

Let $x^2 + y^2 = a^2$ be the equation of the circle and let AC be the tangent at A.

$$x^2 + y^2 = a^2$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{y^2}}$$

$$= \sqrt{\frac{x^2 + y^2}{y^2}} = \frac{a}{y}$$

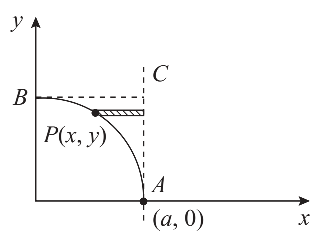


Fig. 4.35

The surface area is generated by revolving the quadrant of circular arc APB about the line AC. If $P(x, y)$ is any point on the circle, the distance of P from the tangent at $A = a - x$. For the region shown, x varies from 0 to a.

Surface area, $S = \int_0^{a} 2\pi(a-x)\frac{ds}{dx} dx$

$$= 2\pi \int_0^a (a-x)\frac{a}{y}\,dx$$

$$= 2\pi a \int_0^a \frac{a-x}{\sqrt{a^2-x^2}}\,dx$$

$$= 2\pi a \int_0^a \left(\frac{a}{\sqrt{a^2-x^2}} - \frac{x}{\sqrt{a^2-x^2}}\right)dx$$

$$= 2\pi a \int_0^a \left[\frac{a}{\sqrt{a^2-x^2}} + \frac{1}{2}(a^2-x)^{\frac{-1}{2}}(-2x)\right]dx$$

$$\left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= 2\pi a \left| a\sin^{-1}\frac{x}{a} + \sqrt{a^2-x^2}\right|_0^a$$

$$= 2\pi a\left(a\frac{\pi}{2} - a\right)$$

$$= \pi a^2(\pi - 2)$$

EXERCISE 4.5

1. Find the surface area of the solid generated by revolving the arc of the parabola $y^2 = 4ax$ bounded by its latus rectum about the x-axis.

$$\left[\textbf{Ans.: } \frac{8a^2}{3}\pi\left(2\sqrt{2}-1\right)\right]$$

2. Find the area of the curved surface generated when one loop of the curve $x^2(a^2 - x^2) = 8a^2y^2$ is revolved about the x-axis.

$$\left[\textbf{Ans.: } \frac{\pi a^2}{4}\right]$$

3. Prove that the surface area of the solid obtained by revolving the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ about the x-axis is $2\pi ab\left[\sqrt{1-e^2} + \left(\frac{1}{e}\right)\sin^{-1} e\right]$, e being the eccentricity of the ellipse.

4. Show that the surface area of the solid obtained by revolving the arc of the curve $y = \sin x$ from $x = 0$ to $x = \pi$ about the x-axis is $\pi^2\left[\sqrt{2} + \log\left(\sqrt{2}+1\right)\right]$.

5. Show that the area of the surface formed by rotating the curve $y^2 = x^3$ from $x = 0$ to $x = 4$ about the y-axis is $\frac{128\pi}{1215}\left(1+125\sqrt{10}\right)$.

6. Find the area of the curved surface of the cup formed by the revolution of the smaller part of the parabola $y^2 = 4ax$ cut off by the line $x = 3a$ about its axis.

$$\left[\textbf{Ans.: } \frac{56}{3}\pi a^2\right]$$

7. The arc of the parabola $y^2 = 4ax$ between its vertex and an extremity of its latus rectum revolves about its axis. Find the surface area traced out.

$$\left[\textbf{Ans.: } \frac{8}{3}(2\sqrt{2}-1)\pi a^2\right]$$

8. The arc of the curve $a^2y = x^3$ between $x = 0$ and $x = a$ is revolved about the x-axis. Find the area of the surface so generated.

$$\left[\textbf{Ans.: } \frac{\pi a^2}{27}\left(10\sqrt{10}-1\right)\right]$$

9. Find the surface area of the solid formed by the revolution of the loop of the curve $3ay^2 = x\,(x - a)^2$ about the x-axis.

$$\left[\textbf{Ans.: } \frac{\pi a^2}{3}\right]$$

10. Find the surface area of the solid generated by revolving the area bounded by the circle $x^2 + y^2 = a^2$ about the line $y = a$.

[**Ans.:** $4\pi^2a^2$]

4.4.2 Area of Surface of Solid of Revolution in Parametric Form

When the equation of the curve is given in parametric form $x = f_1(t)$, $y = f_2(t)$ with $t_1 \le t \le t_2$, the area of surface of solid of revolution about the x-axis is given by,

$$S = \int_{t_1}^{t_2} 2\pi y \frac{ds}{dt}\,dt$$

$$= \int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Similarly, the area of surface of solid of revolution about the y-axis is given by,

$$S = \int_{t_1}^{t_2} 2\pi x \frac{ds}{dt}\,dt$$

$$= \int_{t_1}^{t_2} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Example 1

Prove that the surface generated by the revolution of the tractrix $x = a\cos t + \frac{1}{2}a\log\tan^2\frac{t}{2}$, $y = a\sin t$ *about its asymptote is equal to the surface of the radius a.*

Solution

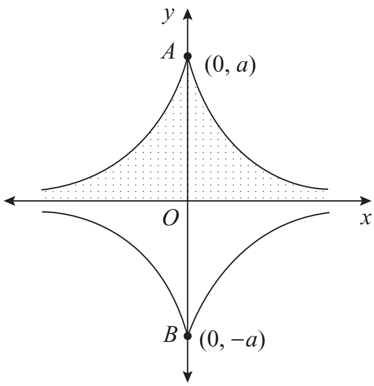


Fig. 4.36

$$x = a\cos t + \frac{1}{2}a\log\tan^2\frac{t}{2}$$

$$\frac{dx}{dt} = -a\sin t + a\cdot\frac{1}{\tan\frac{t}{2}}\cdot\sec^2\frac{t}{2}\cdot\frac{1}{2}$$

$$= -a\sin t + \frac{a}{\sin t}$$

$$= \frac{a\cos^2 t}{\sin t}$$

$$y = a\sin t$$

$$\frac{dy}{dt} = a\cos t$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{\frac{a^2\cos^4 t}{\sin^2 t} + a^2\cos^2 t}$$

$$= \frac{a\cos t}{\sin t}$$

The surface area is generated by revolving the tractrix about its asymptote, i.e., x-axis. For the region shown, x varies from $-\infty$ to ∞, hence t varies from 0 to π. Due to symmetry about the y-axis, considering the region in the second quadrant where t varies from 0 to $\frac{\pi}{2}$,

Surface area,
$$S = 2\int_0^{\frac{\pi}{2}} 2\pi y\frac{ds}{dt}\,dt$$

$$= 4\pi\int_0^{\frac{\pi}{2}} a\sin t\cdot\frac{a\cos t}{\sin t}\,dt$$

$$= 4\pi a^2\int_0^{\frac{\pi}{2}}\cos t\,dt$$

$$= 4\pi a^2\left|\sin t\right|_0^{\frac{\pi}{2}}$$

$$= 4\pi a^2$$

Example 2

Find the surface area of the solid generated by revolving the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ *about the x-axis.*

Solution

The parametric equations of the astroid are

$$x = a\cos^3\theta, \qquad y = a\sin^3\theta$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,$$

$$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$= \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}$$

$$= 3a\sin\theta\cos\theta$$

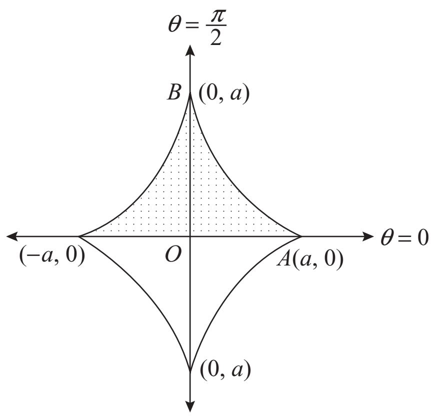


Fig. 4.37

The surface area is generated by revolving the upper half of the astroid about the x axis. For the region shown, x varies from $-a$ to a, hence θ varies from π to 0. Due to symmetry about the y-axis, considering the region in the first quadrant, where θ varies from 0 to $\frac{\pi}{2}$,

Surface area, $$S = 2\int_0^{\frac{\pi}{2}} 2\pi y \frac{ds}{d\theta}\, d\theta$$

$$= 4\pi\int_0^{\frac{\pi}{2}} a\sin^3\theta \cdot 3a\sin\theta\cos\theta\, d\theta$$

$$= 12\pi a^2\int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\, d\theta$$

$$= 12\pi a^2 \left|\frac{\sin^5\theta}{5}\right|_0^{\frac{\pi}{2}} \qquad \left[\because \int [f(\theta)^n] f'(\theta)\, d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= \frac{12}{5}\pi a^2$$

Example 3

Find the surface area of the solid formed by revolving one arch of the cycloid $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$ *about the y-axis.*

Solution

$$x = a(\theta - \sin\theta)$$

$$\frac{dx}{d\theta} = a(1-\cos\theta)$$

$$y = a(1-\cos\theta)$$

$$\frac{dy}{d\theta} = a\sin\theta$$

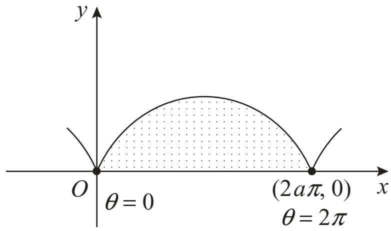


Fig. 4.38

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$= \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}$$

$$= \sqrt{2a^2(1-\cos\theta)}$$

$$= 2a\sin\frac{\theta}{2}$$

The surface area is generated by revolving one arch of the curve about the y-axis. For the region shown, θ varies from 0 to 2π.

Surface area, $S = \int_0^{2\pi} 2\pi x \frac{ds}{d\theta} d\theta$

$$= 2\pi\int_0^{2\pi} a(\theta - \sin\theta)2a\sin\frac{\theta}{2}d\theta$$

$$= 4\pi a^2\int_0^{2\pi}\left(\theta\sin\frac{\theta}{2} - 2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\right)d\theta$$

$$= 4\pi a^2\left|\left[\theta\left(-2\cos\frac{\theta}{2}\right) - 1\left(-4\sin\frac{\theta}{2}\right)\right] - \frac{4}{3}\sin^3\frac{\theta}{2}\right|_0^{2\pi}$$

$$\left[\because \int [f(\theta)]^n f'(\theta)d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= 4\pi a^2(4\pi)$$

$$= 16\pi^2 a^2$$

Example 4

A circular arc of radius a revolves round its chord. Show that the surface of the spindle generated is $4\pi a^2(\sin\alpha - \alpha\cos\alpha)$, *where* 2α *is the angle subtended by the arc at the centre. Find the surface area if the circular arc is a quadrant of circle.*

Solution

Taking the centre of the circle as origin and radius as a, the equation of the circle is $x^2 + y^2 + a^2$. The parametric equations of the circle are:

$$x = a\cos\theta, \qquad y = a\sin\theta$$

$$\frac{dx}{d\theta} = -a\sin\theta, \qquad \frac{dy}{d\theta} = a\cos\theta$$

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$= \sqrt{a^2\sin^2\theta + a^2\cos^2\theta} = a$$

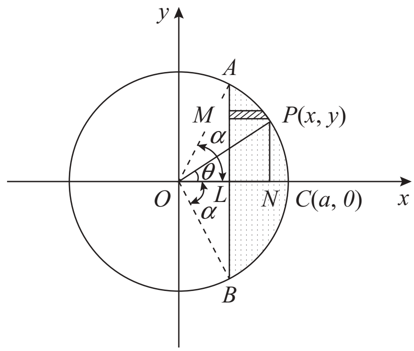


Fig. 4.39

The arc ACB is revolved about the chord AB. If $P(x, y)$ is any point on the circle and M is the foot of perpendicular from P on AB, then

$$PM = ON - OL$$
$$= x - a\cos\alpha$$

For the region shown, θ varies from $-\alpha$ to α. Due to symmetry about the x-axis, considering the region in the first quadrant where θ varies from 0 to α,

Surface area, $S = 2\int_0^{\alpha} 2\pi(PM)\frac{ds}{d\theta}d\theta$

$$= 4\pi\int_0^{\alpha}(x - a\cos\alpha)\,a\,d\theta$$

$$= 4\pi a\int_0^{\alpha}(a\cos\theta - a\cos\alpha)\,d\theta$$

$$= 4\pi a^2\left|\sin\theta - \theta\cos\alpha\right|_0^{\alpha}$$

$$= 4\pi a^2(\sin\alpha - \alpha\cos\alpha)$$

When circular arc is quadrant of a circle, $\alpha = \frac{\pi}{4}$

$$S = 4\pi a^2\left(\frac{1}{\sqrt{2}} - \frac{\pi}{4}\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi a^2}{\sqrt{2}}(4 - \pi)$$

Example 5

Show that the total surface area of the solid generated by the revolution of an ellipse about its minor axis is $2\pi a^2\left(1 + \frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right)$, *where a is the semi-major axis and e is the eccentricity.*

Solution

The parametric equations of the ellipse are,

$$x = a\cos\theta, \qquad y = b\sin\theta$$

$$\frac{dx}{d\theta} = -a\sin\theta, \qquad \frac{dy}{d\theta} = b\cos\theta$$

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$= \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$$

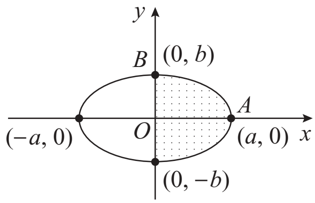


Fig. 4.40

The surface area of the solid is generated by the revolution of the ellipse about its minor axis. For the region shown, y varies from $-b$ to b, hence θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Due to symmetry about x-axis, considering the region in the first quadrant where θ varies from 0 to $\frac{\pi}{2}$,

Surface area, $$S = 2\int_0^{\frac{\pi}{2}} 2\pi x \frac{ds}{d\theta}\, d\theta$$

$$= 4\pi\int_0^{\frac{\pi}{2}} a\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$$

$$= 4a\pi\int_0^{\frac{\pi}{2}} \cos\theta\sqrt{a^2\sin^2\theta + b^2(1-\sin^2\theta)}\, d\theta$$

$$= 4a\pi\int_0^{\frac{\pi}{2}} \cos\theta\sqrt{b^2 + a^2e^2\sin^2\theta}\, d\theta \qquad \text{where } e = \sqrt{1-\frac{b^2}{a^2}}$$

Putting $\sin\theta = t$

$\cos\theta\, d\theta = dt$

When $\theta = 0, \quad t = 0$

When $\theta = \frac{\pi}{2}, \quad t = 1$

$$S = 4a\pi\int_0^1 \sqrt{b^2 + a^2e^2t^2}\, dt$$

$$= 4a\pi \cdot ae\int_0^1 \sqrt{t^2 + \left(\frac{b}{ae}\right)^2}\, dt$$

$$= 4\pi a^2 e\left|\frac{t}{2}\sqrt{t^2 + \frac{b^2}{a^2e^2}} + \frac{b^2}{2a^2e^2}\log\left(t + \sqrt{t^2 + \frac{b^2}{a^2e^2}}\right)\right|_0^1$$

$$= 4\pi a^2 e\left[\frac{1}{2ae}\sqrt{a^2e^2 + b^2} + \frac{b^2}{2a^2e^2}\log\left(1 + \frac{1}{ae}\sqrt{a^2e^2 + b^2}\right) - \frac{b^2}{2a^2e^2}\log\frac{b}{ae}\right]$$

$$= 4\pi a^2 e\left[\frac{1}{2ae}\cdot a + \frac{b^2}{2a^2e^2}\log\left(1+\frac{a}{ae}\right) - \frac{b^2}{2a^2e^2}\log\frac{b}{ae}\right] \quad \left[\because b = a\sqrt{1-e^2}\right]$$

$$= 2\pi\left[a^2 + \frac{b^2}{e}\log\frac{a(1+e)}{b}\right]$$

$$= 2\pi\left(a^2 + \frac{b^2}{2e}\log\frac{1+e}{1-e}\right) \quad \left[\because b = a\sqrt{1-e^2}\right]$$

$$= 2\pi a^2\left(1 + \frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right)$$

EXERCISE 4.6

1. Find the surface area of the reel formed by the revolution of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ about (i) the tangent at the vertex, (ii) y-axis, and (iii) base.

$$\left[\textbf{Ans.:} \text{ (i) } \frac{32}{3}\pi a^2, \text{ (ii) } 4\pi a^2\left(2\pi - \frac{4}{3}\right), \text{ (iii) } \frac{64}{3}\pi a^2\right]$$

2. Find the surface area of the solid generated by the revolution of the loop of the curve $x = t^2, y = t - \dfrac{t^3}{3}$ about x-axis.

[Ans.: 3π]

3. Show that the area of the surface of the solid generated by revolving the curve $x = a(u - \tanh u)$, $y = a\,\text{sech}\,u$, about the x-axis is equal to the area of the surface of a sphere of radius a.

4. Find the area of the surface of revolution generated by revolving the cardioid $x = 2\cos\theta - \cos 2\theta$, $y = 2\sin\theta - \sin 2\theta$, about the x-axis.

$$\left[\textbf{Ans.:} \frac{128\pi}{5}\right]$$

5. Find the area of the surface generated by revolving the curve $x = 3t(t - 2)$, $y = 8t^{\frac{3}{2}}$ with $0 \le t \le 1$ about the y-axis.

[Ans.: 39π]

6. Find the area of the surface generated by revolving the curve $x = a\cos^2 t$, $y = a\sin^2 t$ about x-axis.

$$\left[\textbf{Ans.:} \frac{12\pi}{5}a^2\right]$$

7. Show that the ratio of the areas of the surface formed by revolving the arch of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1+\cos\theta)$ between two consecutive cusps about the *x*-axis to the area enclosed by the cycloid and *x*-axis is $\frac{64}{9}$.

4.4.3 Area of Surface of Solid of Revolution in Polar Form

For the curve $r = f(\theta)$, bounded between the radii vectors at $\theta = \theta_1$ and $\theta = \theta_2$, the area of surface of the solid of revolution about the initial line $\theta = 0$ is given by,

$$S = \int_{\theta_1}^{\theta_2} 2\pi y \frac{\mathrm{d}s}{\mathrm{d}\theta}\,\mathrm{d}\theta$$

$$= \int_{\theta_1}^{\theta_2} 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

Similarly, the area of surface of solid of revolution about the line $\theta = \frac{\pi}{2}$ is given by,

$$S = \int_{\theta_1}^{\theta_2} 2\pi x \frac{\mathrm{d}s}{\mathrm{d}\theta}\,\mathrm{d}\theta$$

$$= \int_{\theta_1}^{\theta_2} 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

Example 1

The curve $r = e^{\frac{\theta}{2}}$ *is revolved about the initial line. Prove that the area of surface of revolution traced out by the part between the points* $\theta = 0$ *and* $\theta = \pi$ *is equal to* $\frac{\pi}{2}\sqrt{5}(e^{\pi} + 1)$.

Solution

$$r = e^{\frac{\theta}{2}}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{1}{2}e^{\frac{\theta}{2}}$$

$$\frac{\mathrm{d}s}{\mathrm{d}\theta} = \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2} = \sqrt{e^{\theta} + \frac{1}{4}e^{\theta}} = \frac{\sqrt{5}}{2}e^{\frac{\theta}{2}}$$

The surface area is generated by revolving the curve about the initial line. For the region shown, θ varies from 0 to π.

Surface area,
$$S = \int_0^{\pi} 2\pi y \frac{ds}{d\theta}\, d\theta$$
$$= \int_0^{\pi} 2\pi r \sin\theta \frac{ds}{d\theta}\, d\theta$$
$$= 2\pi \int_0^{\pi} e^{\frac{\theta}{2}} \sin\theta \frac{\sqrt{5}}{2} e^{\frac{\theta}{2}}\, d\theta$$
$$= \pi\sqrt{5} \int_0^{\pi} e^{\theta} \sin\theta\, d\theta$$
$$= \pi\sqrt{5} \left| \frac{e^{\theta}}{2} (\sin\theta - \cos\theta) \right|_0^{\pi}$$
$$= \frac{\pi}{2}\sqrt{5}(e^{\pi} + 1).$$

Example 2

Find the area of the surface of the solid generated by revolving upper half of the cardioid $r = a(1-\cos\theta)$ about the initial line.

Solution

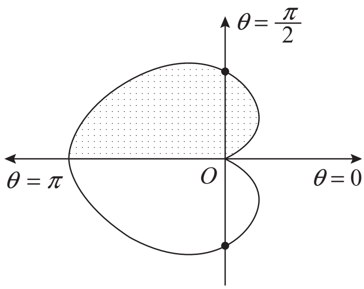


Fig. 4.41

$$r = a(1-\cos\theta)$$
$$\frac{dr}{d\theta} = a\sin\theta$$
$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$
$$= \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}$$
$$= \sqrt{a^2\left(2\sin^2\frac{\theta}{2}\right)^2 + a^2\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2}$$
$$= \sqrt{4a^2\sin^2\frac{\theta}{2}}$$
$$= 2a\sin\frac{\theta}{2}$$

The area of the surface of the solid is generated by revolving the upper half of the cardioid about the initial line $\theta = 0$. For the region shown, θ varies from 0 to π.

Surface area, $$S = \int_0^{\pi} 2\pi y \frac{ds}{d\theta}\, d\theta$$

$$= \int_0^{\pi} 2\pi r \sin\theta \frac{ds}{d\theta} d\theta$$

$$= 4\pi a^2 \int_0^{\pi} (1-\cos\theta)\sin\theta \sin\frac{\theta}{2} d\theta$$

$$= 4\pi a^2 \int_0^{\pi} \left(2\sin^2\frac{\theta}{2}\right)\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\sin\frac{\theta}{2} d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \sin^4\frac{\theta}{2}\cos\frac{\theta}{2} d\theta$$

$$= 32\pi a^2 \int_0^{\pi} \sin^4\frac{\theta}{2}\cdot\frac{1}{2}\cos\frac{\theta}{2} d\theta$$

$$= 32\pi a^2 \left| \frac{\sin^5\frac{\theta}{2}}{5} \right|_0^{\pi} \qquad \left[\because \int [f(\theta)^n f'(\theta)\, d\theta = \frac{[f(\theta)^{n+1}]}{n+1}]\right]$$

$$= \frac{32}{5}\pi a^2$$

Example 3

The arc of cardioid $r = a(1+\cos\theta)$ *included between* $\theta = -\frac{\pi}{2}$ *to* $\frac{\pi}{2}$ *is rotated about the line* $\theta = \frac{\pi}{2}$. *Show that the area of the surface generated is* $\frac{48\sqrt{2}}{5}\pi a^2$.

Solution

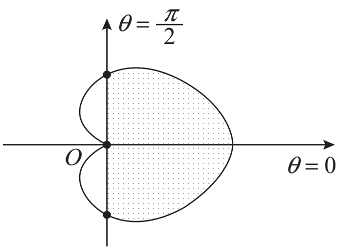


Fig. 4.42

$$r = a(1+\cos\theta)$$

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$= \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}$$

$$= \sqrt{a^2\left(2\cos^2\frac{\theta}{2}\right)^2 + a^2\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2}$$

$$= \sqrt{4a^2\cos^2\frac{\theta}{2}}$$

$$= 2a\cos\frac{\theta}{2}$$

The area of the surface is generated by revolving the cardioid about the line $\theta = \frac{\pi}{2}$. For the region shown, θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Due to symmetry about the initial line considering the region in the first quadrant where θ varies from 0 to $\frac{\pi}{2}$,

Surface area,
$$S = 2\int_0^{\frac{\pi}{2}} 2\pi y \frac{ds}{d\theta} d\theta$$
$$= 2\int_0^{\frac{\pi}{2}} 2\pi r\cos\theta \frac{ds}{d\theta} d\theta$$
$$= 8\pi a^2 \int_0^{\frac{\pi}{2}} (1+\cos\theta)\cos\theta\cos\frac{\theta}{2} d\theta$$
$$= 8\pi a^2 \int_0^{\frac{\pi}{2}} \left(2\cos^2\frac{\theta}{2}\right)\left(1-2\sin^2\frac{\theta}{2}\right)\cos\frac{\theta}{2} d\theta$$
$$= 8\pi a^2 \int_0^{\frac{\pi}{2}} 2\left(1-\sin^2\frac{\theta}{2}\right)\left(1-2\sin^2\frac{\theta}{2}\right)\cos\frac{\theta}{2} d\theta$$

Putting $\sin\frac{\theta}{2} = t,$

$$\frac{1}{2}\cos\frac{\theta}{2} d\theta = dt$$

When $\theta = 0, \quad t = 0$

When $\theta = \frac{\pi}{2}, \quad t = \frac{1}{\sqrt{2}}$

$$S = 32\pi a^2 \int_0^{\frac{1}{\sqrt{2}}} (1-t^2)(1-2t^2) dt$$
$$= 32\pi a^2 \int_0^{\frac{1}{\sqrt{2}}} (1-3t^2+2t^4) dt$$
$$= 32\pi a^2 \left| t - t^3 + \frac{2}{5}t^5 \right|_0^{\frac{1}{\sqrt{2}}}$$
$$= 32\pi a^2 \left[\frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}} + \frac{2}{5}\cdot\frac{1}{4\sqrt{2}} \right]$$
$$= \frac{48\sqrt{2}}{5}\pi a^2$$

Example 4

Find the surface area of the solid formed by the revolution of the loop about the tangent at the pole of the curve $r^2 = a^2\cos 2\theta$.

Solution

$$r^2 = a^2 \cos 2\theta$$

$$2r\frac{dr}{d\theta} = -2a^2 \sin 2\theta$$

$$\frac{dr}{d\theta} = -\frac{a^2 \sin 2\theta}{a\sqrt{\cos 2\theta}}$$

$$= -a\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$= \sqrt{a^2 \cos 2\theta + a^2 \frac{\sin^2 2\theta}{\cos 2\theta}}$$

$$= \sqrt{\frac{a^2 \cos^2 2\theta + a^2 \sin^2 2\theta}{\cos 2\theta}}$$

$$= \frac{a}{\sqrt{\cos 2\theta}}$$

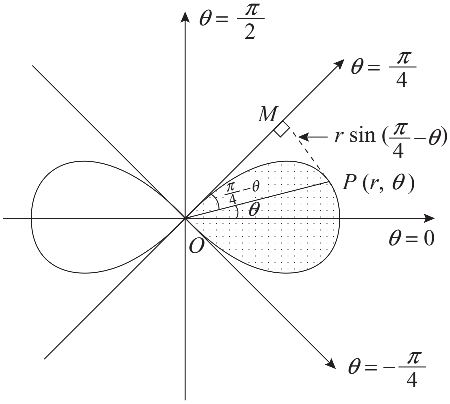


Fig. 4.43

The surface area is formed by the revolution of the loop about the tangent at the pole i.e., $\theta = \frac{\pi}{4}$. If $P(r, \theta)$ is any point on the curve, its distance from the line $\theta = \frac{\pi}{4}$ is $r\sin\left(\frac{\pi}{4} - \theta\right)$, i.e., $\frac{1}{\sqrt{2}} r(\cos\theta - \sin\theta)$. For the region shown, θ varies from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$. Due to symmetry about the initial line, considering the region in the first quadrant where θ varies from 0 to $\frac{\pi}{4}$,

Surface area,

$$S = 2\int_0^{\frac{\pi}{4}} 2\pi \frac{1}{\sqrt{2}} r(\cos\theta - \sin\theta)\frac{ds}{d\theta} d\theta$$

$$= \frac{4\pi}{\sqrt{2}}\int_0^{\frac{\pi}{4}} a\sqrt{\cos 2\theta}\,(\cos\theta - \sin\theta)\frac{a}{\sqrt{\cos 2\theta}}\, d\theta$$

$$= 2\sqrt{2}\pi a^2 \int_0^{\frac{\pi}{4}} (\cos\theta - \sin\theta)\, d\theta$$

$$= 2\sqrt{2}\pi a^2 \left|\sin\theta + \cos\theta\right|_0^{\frac{\pi}{4}}$$

$$= 2\sqrt{2}\,\pi a^2(\sqrt{2} - 1)$$

EXERCISE 4.7

1. Find the area of the surface of the solid generated by revolving the curve $r^2 = a^2\cos 2\theta$ about the initial line.

$$\left[\textbf{Ans.: } 4\pi a^2\left(1-\frac{1}{\sqrt{2}}\right)\right]$$

2. Find the area of the surface of the solid generated by revolving the curve $r = 2a\cos\theta$ about the initial line.

$$\left[\textbf{Ans.: } 4\pi a^2\right]$$

3. Find the area of the surface of the solid generated by revolving the curve $r = 4\cos\theta$ about the initial line.

$$[\textbf{Ans.: } 16\pi]$$

Points to Remember

Volume Using Cross-sections

The volume of a solid of known integrable cross-section area $A(x)$ formed by a plane perpendicular to the x-axis at any point between $x = a$ to $x = b$ is

$$V = \int_a^b A(x)\,dx$$

If the cross-section $A(y)$ is perpendicular to the y-axis at any point between $y = c$ to $y = d$, then the volume of the solid is

$$V = \int_c^d A(y)\,dy$$

Length of Plain Curves

Length of Arc

(i) Cartesian form

(a) $$s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

(b) $$s = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

(ii) Parametric form

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

(iii) Polar form

(a) $s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

(b) $s = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$

Area of Surface of Solid of Revolution

(i) Cartesian form

(a) $S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ (revolution about x-axis)

(b) $S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$ (revolution about y-axis)

(ii) Parametric form

(a) $S = \int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ (revolution about x-axis)

(b) $S = \int_{t_1}^{t_2} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ (revolution about y-axis)

(iii) Polar form

(a) $S = \int_{\theta_1}^{\theta_2} 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ (revolution about $\theta = 0$)

(b) $S = \int_{\theta_1}^{\theta_2} 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ $\left(\text{revolution about } \theta = \frac{\pi}{2}\right)$

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. The length of the arc of the curve $y = \log \sec x$ from $x = 0$ to $x = \frac{\pi}{3}$ is

(a) $\log(2 + \sqrt{3})$ (b) $\log(\sqrt{2} + 3)$

(c) $\log(\sqrt{2} + 1)$ (d) $\log(\sqrt{3} + 1)$

2. The length of the curve $y = \dfrac{(x^2+2)^{\frac{2}{3}}}{3}$ from $x = 0$ to $x = 3$ is

(a) 10 (b) 12 (c) 3π (d) 6π

3. The whole length of the curve $r = 2a\sin\theta$ is equal to

(a) πa (b) $2\pi a$ (c) $3\pi a$ (d) $4\pi a$

4. The length of the arc of the curve $6xy = x^4 + 3$ from $x = 1$ to $x = 2$ is

(a) $\dfrac{13}{12}$ (b) $\dfrac{17}{12}$ (c) $\dfrac{19}{12}$ (d) none of these

5. The arc of the sine curve $y = \sin x$ from $x = 0$ to $x = \pi$ revolved about the x-axis. The area of the surface of the solid generated is

(a) $2\pi\,[\sqrt{2} + \log(\sqrt{2}+1)]$ (b) $\dfrac{2\pi^2}{3}[\sqrt{2} + \log(\sqrt{2}+1)]$

(c) $\dfrac{\pi}{3}[\sqrt{2} + \log(\sqrt{2}+1)]$ (d) $\dfrac{\pi^2}{3}[\sqrt{2} + \log(\sqrt{2}+1)]$

6. The area of the surface of the solid generated by revolving the curve $r = 2a\cos\theta$ about the initial line is

(a) $2\pi a^2$ (b) $4\pi a^2$ (c) πa^2 (d) $8\pi a^2$

7. The area of the surface of the solid generated by revolving the curve $x = t^3 - 3t,\ y = 3t^2,\ 0 \le t \le 1$ about the x-axis is

(a) $\dfrac{40\pi}{5}$ (b) $\dfrac{24\pi}{5}$ (c) $\dfrac{36\pi}{5}$ (d) $\dfrac{48\pi}{5}$

8. The area of the surface of the solid generated by the revolution of the line segment $y = 2x$ from $x = 0$ to $x = 2$ about the x-axis is equal to

(a) $\pi\sqrt{5}$ (b) $2\pi\sqrt{5}$ (c) $4\pi\sqrt{5}$ (d) $8\pi\sqrt{5}$

Answers

1. (a) 2. (b) 3. (b) 4. (b) 5. (a) 6. (b) 7. (d) 8. (d)

UNIT-2

Chapter 5. Sequences and Series
Chapter 6. Taylor's and Maclaurin's Series

CHAPTER 5

Sequences and Series

Chapter Outline

5.1 INTRODUCTION

In this chapter, we will learn about the convergence and divergence of sequence and series. There are various methods to test the convergence and divergence of an infinite series. We will study Comparison Test, D'Alembert's ratio test, Cauchy's root test and Cauchy's integral test. We will also study alternating series, absolute and uniform convergence of the series and power series.

5.2 SEQUENCE

An ordered set of real numbers as $u_1, u_2, u_3, \ldots\ldots u_n, \ldots\ldots$ is called a sequence and is denoted by $\{u_n\}$. If the number of terms in a sequence is infinite, it is said to be an infinite sequence, otherwise it is a finite sequence and u_n is called the n^{th} term of the sequence.

5.2.1 Limit of a Sequence

A sequence $\{u_n\}$ tends to a finite number l as $n \to \infty$ if for every $\in > 0$ there exists an integer m such that, $|u_n - l| < \in$ for all $n > m$, i.e., $\lim_{n\to\infty} u_n = l$.

5.2.2 Continuous Function Theorem for Sequences

Let $\{u_n\}$ be a sequence of real numbers. If $u_n \to l$ and if f is a function that is continuous at l and defined at all u_n, then $f(u_n) \to f(l)$.

5.2.3 Convergence, Divergence and Oscillation of Finite Series

(i) If the sequence $\{u_n\}$ has a finite limit, i.e., $\lim_{n\to\infty} u_n$ is finite, the sequence is said to be convergent.

e.g.

$$\{u_n\} = \left\{\frac{1}{1+\frac{1}{n}}\right\}$$

$$\lim_{n\to\infty} u_n = 1$$

Since limit is finite, the sequence is convergent.

(ii) If the sequence $\{u_n\}$ has infinite limit, i.e., $\lim_{n\to\infty} u_n$ is infinite, the sequence is said to be divergent.

e.g.

$$\{u_n\} = \{2n+1\}$$

$$\lim_{n\to\infty} u_n = \infty$$

Since limit is infinite, the sequence is divergent.

(iii) If the limit of the sequence $\{u_n\}$ is not unique, the sequence is said to be oscillatory.

e.g.

$$\{u_n\} = (-1)^n + \frac{1}{2^n}$$

$$\lim_{n\to\infty} u_n = 1, \text{ if } n \text{ is even}$$

$$= -1, \text{ if } n \text{ is odd}$$

Since limit is not unique, the sequence is oscillatory.

5.2.4 Monotonic Sequence

A sequence is said to be monotonically increasing if $u_{n+1} \geq u_n$ for each value of n and is monotonically decreasing if $u_{n+1} \leq u_n$ for each value of n. The sequence is called alternating sequence if the terms are alternate positive and negative.

For example, (i) 1, 2, 3, 4, … is a monotonically increasing sequence.

(ii) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is a monotonically decreasing sequence.

(iii) 1, –2, 3, –4, … is an alternating sequence.

5.2.5 Bounded Sequence

A sequence $\{u_n\}$ is said to be a bounded sequence if there exists numbers m and M such that $m < u_n < M$ for all n.

***Note* 1:** Every convergent sequence is bounded but the converse is not true.

***Note* 2:** A monotonic increasing sequence converges if it is bounded above and diverges to $+\infty$ if it is not bounded above.

***Note* 3:** A monotonic decreasing sequence converges if it is bounded below and diverges to $-\infty$ if it is not bounded below.

***Note* 4:** If sequence $\{u_n\}$ and $\{v_n\}$ converges to l_1 and l_2 respectively then

(i) Sequence $\{u_n + v_n\}$ converges to $l_1 + l_2$

(ii) Sequence $\{u_n . v_n\}$ converges to $l_1 \cdot l_2$

(iii) Sequence $\left\{\frac{u_n}{v_n}\right\}$ converges to $\frac{l_1}{l_2}$ provided $l_2 \neq 0$

Example 1

Test the convergence of the sequence $\left\{\frac{n^2+n}{2n^2-n}\right\}$.

Solution

Let

$$u_n = \frac{n^2+n}{2n^2-n}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n^2+n}{2n^2-n}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2-\frac{1}{n}}$$

$$= \frac{1}{2}$$

Hence, $\{u_n\}$ is convergent.

Example 2

Test the convergence of the sequence $\{\tanh n\}$.

Solution

Let $u_n = \tanh n$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \tanh n$$
$$= \lim_{n\to\infty} \frac{\sinh n}{\cosh n}$$
$$= \lim_{n\to\infty} \frac{e^n - e^{-n}}{e^n + e^{-n}}$$
$$= \lim_{n\to\infty} \frac{e^{2n} - 1}{e^{2n} + 1}$$
$$= \lim_{n\to\infty} \frac{1 - \frac{1}{e^{2n}}}{1 + \frac{1}{e^{2n}}}$$
$$= 1$$

Hence, $\{u_n\}$ is convergent.

Example 3

Test the convergence of the sequence $\{2^n\}$.

Solution

Let $u_n = 2^n$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} 2^n$$
$$= \infty$$

Hence, $\{u_n\}$ is divergent.

Example 4

Test the convergence of the sequence $\left\{2-(-1)^n\right\}$.

Solution

Let $u_n = 2 - (-1)^n$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} 2-(-1)^n$$
$$= 2 - 1 = 1 \quad , \quad \text{if } n \text{ is even}$$
$$= 2 - (-1) = 3 \quad , \quad \text{if } n \text{ is odd}$$

Since limit is not unique, the sequence $\{u_n\}$ is oscillatory.

Example 5

Show that the sequence $\{u_n\}$ *whose* n^{th} *term is* $u_n = 1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^n}$, *is monotonic increasing and bounded. Is it convergent?*

Solution

$$u_n = 1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^n}$$

$$u_{n+1} = 1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^n}+\frac{1}{3^{n+1}}$$

$$u_{n+1}-u_n = \frac{1}{3^{n+1}} > 0$$

Hence, $\{u_n\}$ is monotonic increasing sequence.

Also,
$$u_n = 1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^n}$$

$$= \frac{1\left(1-\frac{1}{3^{n+1}}\right)}{1-\frac{1}{3}}$$

$$= \frac{3}{2}\left(1-\frac{1}{3^{n+1}}\right) < \frac{3}{2}$$

$\{u_n\}$ is bounded above by $\frac{3}{2}$.

Since $\{u_n\}$ is monotonic increasing and bounded above, it is convergent.

Example 6

Show that the sequence $\{u_n\}$ *whose* n^{th} *term is* $u_n = \frac{1}{1!}+\frac{2}{2!}+\ldots+\frac{1}{n!}$, $n \in N$, *is monotonic increasing and bounded. Is it convergent?*

Solution

$$u_n = \frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}$$

$$u_{n+1} = \frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}+\frac{1}{(n+1)!}$$

$$u_{n+1}-u_n = \frac{1}{(n+1)!} > 0$$

$$u_{n+1} > u_n$$

Hence, $\{u_n\}$ is a monotonic increasing sequence.

Also,
$$\begin{aligned} u_n &= \frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \\ &= 1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \\ &< 1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n} \\ &< \frac{1\left(1-\frac{1}{2^{n+1}}\right)}{1-\frac{1}{2}} \qquad \text{[Using sum of G.P]} \\ &< 2\left(1-\frac{1}{2^{n+1}}\right) \\ &< 2 \end{aligned}$$

$\{u_n\}$ is bounded above by 2.
Since $\{u_n\}$ is monotonic increasing and bounded above, it is convergent.

Example 7

Show that the sequence $\left\{\frac{n}{n^2+1}\right\}$ *is monotonic decreasing and bounded. Is it convergent?*

Solution

Let
$$\begin{aligned} u_n &= \frac{n}{n^2+1} \\ u_{n+1} &= \frac{n+1}{(n+1)^2+1} \\ u_{n+1}-u_n &= \frac{n+1}{(n+1)^2+1}-\frac{n}{n^2+1} \\ &= \frac{(n+1)(n^2+1)-n(n^2+2n+2)}{(n^2+2n+2)(n^2+1)} \\ &= \frac{-n^2-n+1}{(n^2+2n+2)(n^2+1)} < 0 \end{aligned}$$

Hence, $\{u_n\}$ is a monotonic decreasing sequence.

Also,
$$u_n = \frac{n}{n^2+1} > 0$$

$\{u_n\}$ is bounded below by 0.
Since $\{u_n\}$ is monotonic decreasing and bounded below, it is convergent.

5.2.6 Sandwich Theorem for Sequences

Let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be three sequences such that $u_n \le v_n \le w_n$ for all n. If $\lim_{n\to\infty} u_n = \lim_{n\to\infty} w_n = l$, then $\lim_{n\to\infty} v_n = l$

Example 1

Show that the sequence $\{u_n\}$, *where* $u_n = \dfrac{\sin n}{n}$ *converges to zero.*

Solution

We know that

$$-1 \le \sin n \le 1$$

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$$

$$\lim_{n\to\infty}\left(-\frac{1}{n}\right) = 0$$

$$\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

By sandwich theorem,

$$\lim_{n\to\infty}\frac{\sin n}{n} = 0$$

Hence, $\{u_n\}$ converges to zero.

Example 2

If $x \in R$ *with* $|x| < 1$ *then prove that* $x^n \to 0$ *as* $n \to \infty$.

Solution

For $\quad x = 0, \quad x^n = 0$

For $\quad x \ne 0, \quad \text{let } |x| = \dfrac{1}{1+y}$

$$|x|^n = \frac{1}{(1+y)^n}$$

$$= \frac{1}{1 + ny + \dfrac{n(n-1)y^2}{2!} + \cdots}$$

$$< \frac{1}{ny} \qquad \left[\because \left\{1 + ny + \frac{n(n-1)y^2}{2!} + \cdots\right\} > ny\right]$$

$$0 < |x|^n < \frac{1}{ny}$$

$$\lim_{n\to\infty} \frac{1}{ny} = 0$$

By sandwich theorem,

$$\lim_{n\to\infty} |x|^n = 0$$

Hence, $$|x|^n \to 0 \quad \text{as } n \to \infty$$

EXERCISE 5.1

1. Test the convergence of the following sequences:

(i) $\frac{2n+1}{1-3n}$ (ii) $2 + (0.1)^n$ (iii) $1 + (-1)^n$

(iv) e^n (v) $1 + (-1)^n$ (vi) $\frac{n^2}{2n-1}\sin\left(\frac{1}{n}\right)$

(vii) $\tan^{-1} n$

[Ans. : (i) convergent (ii) convergent (iii) divergent (iv) divergent (v) oscillatory (vi) convergent (vii) convergent]

2. Determine whether the following sequences are monotonically increasing/ decreasing, bounded or convergent/divergent.

(i) $1+\frac{1}{n}$ (ii) $\frac{2n-7}{3n+2}$

[Ans. : (i) decreasing, bounded, convergent (ii) increasing, bounded, convergent]

3. Show that the sequence $\{u_n\}$, where $u_n = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{(n-1)!}$; $n \ge 2$, is convergent.

4. Does the sequence $\{u_n\}$ convergent where $u_n = \left(\frac{n+1}{n-1}\right)^n$?

[Ans. : yes]

5.3 INFINITE SERIES

If $u_1, u_2, u_3, \ldots u_n, \ldots$ is an infinite sequence of real numbers, then the sum of the terms of the sequence, $u_1 + u_2 + u_3 + \ldots + u_n + \ldots \infty$ is called an infinite series.

The infinite series $u_1 + u_2 + u_3 + \ldots + u_n + \ldots \infty$ is usually denoted by $\sum_{n=1}^{\infty} u_n$ or Σu_n.

The sum of its first n terms is denoted by S_n and is also known as n^{th} partial sum of Σu_n.

5.3.1 Convergence, Divergence and Oscillation of Finite Series

Consider the infinite series $\Sigma u_n = u_1 + u_2 + u_3 + \ldots u_n + \ldots \infty$ and let the sum of the first n terms be $S_n = u_1 + u_2 + u_3 + \ldots + u_n$. As $n \to \infty$, three possibilities arise for S_n:

(i) If S_n tends to a finite limit as $n \to \infty$, the series Σu_n is said to be convergent.

(ii) If S_n tends to $\pm\infty$ as $n \to \infty$, the series Σu_n is said to be divergent.

(iii) If S_n does not tend to a unique limit as $n \to \infty$, i.e., limit does not exist, the series Σu_n is said to be oscillatory.

5.3.2 Properties of Infinite Series

1. The convergence or divergence of an infinite series remains unaffected:

(i) by addition or removal of a finite number of terms

(ii) by multiplication of each term with a finite number

2. If two series Σu_n and Σv_n are convergent, then $\Sigma(u_n + v_n)$ is also convergent.

3. If two series Σu_n and Σv_n are divergent, then $\Sigma(u_n + v_n)$ may be convergent.

4. If each term of a series Σu_n of positive terms does not exceed the corresponding term of a convergent series Σv_n of positive terms, then Σu_n is convergent.

5. If each term of a series Σu_n of positive terms exceeds the corresponding term of a divergent series Σv_n of positive terms, then Σu_n is divergent.

5.4 THE n^{th} TERM TEST FOR DIVERGENCE

If a positive term series Σu_n is convergent, then $\lim_{n\to\infty} u_n = 0$.

The converse of this result is not true, i.e., if $\lim_{n\to\infty} u_n = 0$, it is not necessary that the series will be convergent.

For example,

$$\sum u_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \ldots \infty$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Now, $$S_n = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}} > 1+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+\ldots+\frac{1}{\sqrt{n}}$$

$$S_n > \frac{n}{\sqrt{n}}$$

$$S_n > \sqrt{n}$$

and $\lim\limits_{n\to\infty} \sqrt{n} = \infty$

Thus, the series is divergent.

Hence, $\lim\limits_{n\to\infty} u_n = 0$ is a necessary but not sufficient condition for convergence of Σu_n.

If $\lim\limits_{n\to\infty} u_n \neq 0$ or $\lim\limits_{n\to\infty} u_n$ does not exist, then Σu_n is divergent.

5.5 GEOMETRIC SERIES

Consider the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ ... (1)

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$= \frac{a(1-r^n)}{1-r}, \qquad \text{if } r < 1$$

$$= \frac{a(r^n-1)}{r-1}, \qquad \text{if } r > 1$$

(i) When $|r| < 1$, $\lim\limits_{n\to\infty} r^n = 0$

$$\lim_{n\to\infty} S_n = \frac{a}{1-r} \text{ is finite.}$$

Hence, the series is convergent.

(ii) When $r > 1$, $\lim\limits_{n\to\infty} r^n \to \infty$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(r^n-1)}{r-1} \to \infty$$

Hence, the series is divergent.

(iii) When $r = 1$,

$$S_n = a + a + a + \cdots = na$$

$$\lim_{n\to\infty} S_n \to \infty$$

Hence, the series is divergent.

(iv) When $r = -1$,

$$S_n = a - a + a - \cdots (-1)^{n-1} a$$
$$= 0, \text{ if } n \text{ is even}$$
$$= a, \text{ if } n \text{ is odd}$$

Hence, the series is oscillatory.

(v) When $r < -1$, let $r = -k$ where $k > 0$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a[1-(-k)^n]}{1+k}$$
$$= \lim_{n\to\infty} \frac{a[1-(-1)^n k^n]}{1+k}$$
$$= -\infty, \text{ if } n \text{ is even}$$
$$= +\infty, \text{ if } n \text{ is odd}$$

Hence, the series is oscillatory.

From all the above cases, we conclude that the geometric series (1) is

(i) convergent if $|r| < 1$

(ii) divergent if $r \geq 1$

(iii) oscillatory if $r \leq -1$

Example 1

Prove that $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \cdots$ *converges and find its sum.*

[Winter 2014]

Solution

The given series is geometric series with $a = 1$ and $r = \frac{2}{3}$.

$$1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1}$$
$$|r| = \frac{2}{3} < 1$$

Hence, the series is convergent.

$$S_n = \frac{a}{1-r} = \frac{1}{1-\frac{2}{3}} = 3$$

Example 2

Test the convergence of the series $5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}+\cdots$ **[Summer 2017]**

Solution

The given series is geometric series with $a = 5$ and $r = -\frac{2}{3}$.

$$5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}+\cdots=\sum_{n=1}^{\infty} ar^{n-1}$$

$$=\sum_{n=1}^{\infty} 5\left(-\frac{2}{3}\right)^{n-1}$$

$$|r|=\left|-\frac{2}{3}\right|=\frac{2}{3}<1$$

Hence, the series is convergent.

$$S_n=\frac{a}{1-r}$$

$$=\frac{5}{1-\left(-\frac{2}{3}\right)}$$

$$=\frac{5}{1+\frac{2}{3}}$$

$$=\frac{5}{\frac{5}{3}}$$

$$=3$$

Example 3

Let $S=\sum_{n=1}^{\infty} n\alpha^n$ *where* $|\alpha|<1$. *Find the value of* α *in* (0, 1) *such that* $S = 2\alpha$. **[Winter 2016]**

Solution

$$S=\alpha+2\alpha^2+3\alpha^3+\ldots \quad \ldots (1)$$

$$\alpha S=\alpha^2+2\alpha^3+3\alpha^4+\ldots \quad \ldots (2)$$

Subtracting Eq. (2) from Eq. (1),

$$S(1-\alpha) = \alpha + \alpha^2 + \alpha^3 + \dots$$

$$S(1-\alpha) = \frac{\alpha}{1-\alpha}$$

$$S = \frac{\alpha}{(1-\alpha)^2}$$

If $S = 2\alpha$,

$$2\alpha = \frac{\alpha}{(1-\alpha)^2}$$

$$(1-\alpha)^2 = \frac{1}{2}$$

$$1-\alpha = \frac{1}{\sqrt{2}}$$

$$\alpha = 0.2929$$

Example 4

A ball is dropped from 'a' meters above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds a distance rh where 0 < r < 1. Find the total distance the ball travels up and down, when $a = 6$ m *and* $r = \dfrac{2}{3}$ m. **[Winter 2016]**

Solution

Total distance $(h) = a + 2ar + 2ar^2 + 2ar^3 + \dots$

$$= a + \frac{2ar}{1-r}$$

$$= \frac{a(1+r)}{1-r}$$

Here, $a = 6 \text{ m} \quad r = \dfrac{2}{3} \text{ m}$

$$h = \frac{6\left(1+\frac{2}{3}\right)}{\left(1-\frac{2}{3}\right)} = 6 \cdot \frac{5}{2} \cdot \frac{3}{1} = 30 \text{ m}$$

Example 5

The figure 5.1 shows the first seven of a sequence of squares. The outermost square has an area of $4m^2$. Each of the other squares is obtained by joining the midpoints of the sides of the squares in the infinite sequence. Find sum of the areas of all the squares in the infinite sequence. **[Winter 2015]**

Solution

Since each square is obtained by joining the midpoints of the square before its, area of each square is half the area of previous square.

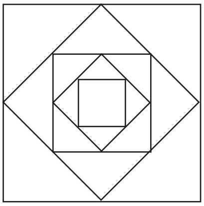


Fig. 5.1

$$\text{Area of } 2^{\text{nd}} \text{ square } = \frac{1}{2}(\text{area of outermost square})$$

$$= \frac{1}{2}(4) = 2$$

$$\text{Area of } 3^{\text{rd}} \text{ square } = \frac{1}{2}(\text{area of } 2^{\text{nd}} \text{ square})$$

$$= \frac{1}{2}(2) = 1$$

$$\text{Area of } 4^{\text{th}} \text{ square } = \frac{1}{2}(\text{area of } 3^{\text{rd}} \text{ square})$$

$$= \frac{1}{2}(1) = \frac{1}{2}$$

and so on.

Sum(s) of areas of all squares in the infinite sequence is

$$S = 4 + 2 + 1 + \frac{1}{2} + \cdots$$

$$= 4\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)$$

$$= 4\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right)$$

which is an infinite geometric series with $a = 1$, $r = \frac{1}{2}$.

$$S = 4\left(\frac{a}{1-r}\right)$$

$$= 4\left(\frac{1}{1-\frac{1}{2}}\right)$$

$$= 8$$

5.6 TELESCOPING SERIES

A telescoping series is a series in which the n^{th} partial sum S_n (sum of first n terms) can be represented in such a manner that almost each term cancels with a preceeding or following term except fixed number of terms.

Example 1

Test the convergence of the series $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots.$

[Summer 2014]

Solution

$$\begin{aligned} u_n &= \frac{1}{n(n+1)} \\ &= \frac{n+1-n}{n(n+1)} \\ &= \frac{1}{n}-\frac{1}{n+1} \end{aligned}$$

$$\begin{aligned} S_n &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right) \\ &= 1-\frac{1}{n+1} \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} S_n &= \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) \\ &= 1-0 \\ &= 1 \text{ [finite]} \end{aligned}$$

Hence, the series Σu_n is convergent.

Example 2

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$.

Solution

$$\begin{aligned}
u_n &= \frac{1}{n^2+3n+2} \\
&= \frac{1}{(n+1)(n+2)} \\
&= \frac{(n+2)-(n+1)}{(n+1)(n+2)} \\
&= \frac{1}{n+1} - \frac{1}{n+2}
\end{aligned}$$

$$\begin{aligned}
S_n &= u_1 + u_2 + u_3 + \cdots + u_n \\
&= \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{5}\right) + \cdots + \left(\frac{1}{n+1}-\frac{1}{n+2}\right) \\
&= \frac{1}{2} - \frac{1}{n+2}
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{n+2}\right) \\
&= \frac{1}{2} \quad \text{[finite]}
\end{aligned}$$

Hence, the series Σu_n is convergent.

Example 3

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3}$.

Solution

$$\begin{aligned}
u_n &= \frac{1}{n^2+4n+3} \\
&= \frac{1}{(n+1)(n+3)} \\
&= \frac{(n+3)-(n+1)}{2(n+1)(n+3)}
\end{aligned}$$

$$= \frac{1}{2}\left[\frac{1}{n+1} - \frac{1}{n+3}\right]$$

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n$$

$$= \frac{1}{2}\left[\left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right)\right.$$

$$\left. + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+2}\right) + \left(\frac{1}{n+1} - \frac{1}{n+3}\right)\right]$$

$$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$

$$= \frac{1}{2}\left(\frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}\left(\frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$

$$= \frac{5}{12} \quad \text{[finite]}$$

Hence, the series Σu_n is convergent.

Example 4

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{2n+1}{n^2 + (n+1)^2}$.

Solution

$$u_n = \frac{2n+1}{n^2(n+1)^2}$$

$$= \frac{(n+1)^2 - n^2}{n^2(n+1)^2}$$

$$= \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n$$

$$= \left(1 - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \left(\frac{1}{3^2} - \frac{1}{4^2}\right) + \cdots + \left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right]$$

$$= 1 - \frac{1}{(n+1)^2}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[1-\frac{1}{(n+1)^2}\right]$$
$$= 1 \text{ [finite]}$$

Hence, the series Σu_n is convergent.

5.7 COMBINING SERIES

If two series Σu_n and Σv_n are convergent then the basic mathematical operations between the series do not change the behaviour (convergence) of these series, i.e. combine series $\Sigma(u_n + v_n)$, $\Sigma(u_n - v_n)$, Σku_n are also convergent, where k is a constant.

Example 1

Find the sum of the series $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}}$. **[Summer 2015]**

Solution

$$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty}\left[\left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{6}\right)^{n-1}\right]$$
$$= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1} - \sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^{n-1}$$
$$= \left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right) - \left(1+\frac{1}{6}+\frac{1}{36}+\cdots\right)$$

Both the series are geometric series with $a = 1$ and $r = \frac{1}{2}$ and $r = \frac{1}{6}$ respectively.

$$S_n = \frac{a_1}{1-r_1} - \frac{a_2}{1-r_2}$$
$$= \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{6}}$$
$$= 2 - \frac{6}{5}$$
$$= \frac{4}{5}$$

Example 2

Investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{2^n + 5}{3^n}$.

[Summer 2015]

Solution

$$\sum_{n=1}^{\infty} \frac{2^n + 5}{3^n} = \sum_{n=1}^{\infty} \left[\left(\frac{2}{3}\right)^n + 5\left(\frac{1}{3^n}\right)^n \right]$$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=1}^{\infty} 5\left(\frac{1}{3}\right)^n$$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^{n-1} + \sum_{n=1}^{\infty} \left(\frac{5}{3}\right)\left(\frac{1}{3}\right)^{n-1}$$

Both the series are geometric series with $r_1 = \frac{2}{3}$ and $r_2 = \frac{1}{3}$ respectively.

$$|r_1| < 1 \quad \text{and} \quad |r_2| < 1$$

Hence, both the series are convergent.

$$S_1 = \frac{a_1}{1 - r_1} = \frac{\frac{2}{3}}{1 - \frac{2}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2$$

$$S_2 = \frac{a_2}{1 - r_2} = \frac{\frac{5}{3}}{1 - \frac{1}{3}} = \frac{\frac{5}{3}}{\frac{2}{3}} = \frac{5}{2}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{2^n + 5}{3^n} = 2 + \frac{5}{2} = \frac{9}{2}.$$

5.8 HARMONIC SERIES

The harmonic series is expressed as

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

This series is divergent in nature.

5.9 *p*-SERIES

The generalisation of harmonic series is known as *p*-series. It is represented as

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$$

This series is
(i) convergent if $p > 1$
(ii) divergent if $p \le 1$

5.10 COMPARISON TEST

If Σu_n and Σv_n are series of positive terms such that $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=l$ (finite and non-zero) then both series converge or diverge together.

Proof

$$\lim_{n\to\infty}\frac{u_n}{v_n}=l$$

By definition of limit, for a positive number $\in$, however small, there exists an integer *m* such that

$$\left|\frac{u_n}{v_n}-l\right|<\in \qquad \text{for all } n>m$$

$$-\in<\frac{u_n}{v_n}-l<\in \qquad \text{for all } n>m$$

$$l-\in<\frac{u_n}{v_n}<l+\in \qquad \text{for all } n>m$$

Neglecting the first *m* terms of Σu_n and Σv_n,

$$l-\in<\frac{u_n}{v_n}<l+\in \qquad \text{for all } n \quad \ldots (1)$$

Case I If Σv_n is convergent then $\lim\limits_{n\to\infty}(v_1+v_2+v_3+\cdots+v_n)=\text{finite}=k$, say

From Eq. (1),

$$\frac{u_n}{v_n}<l+\in$$

$$u_n<(l+\in)v_n \qquad \text{for all } n$$

$$\lim_{n\to\infty}(u_1+u_2+u_3+\cdots+u_n)<(l+\in)\lim_{n\to\infty}(v_1+v_2+v_3+\cdots+v_n)$$

$$\lim_{n\to\infty}(u_1+u_2+u_3+\cdots+u_n)<(l+\epsilon)k \text{ (finite)}$$

Hence, Σu_n is also convergent.

Case II If Σv_n is divergent then

$$\lim_{n\to\infty}(v_1+v_2+v_3+\ldots+v_n)\to\infty \qquad \text{... (2)}$$

From Eq. (1),

$$l-\epsilon<\frac{u_n}{v_n}$$

$$u_n>(l-\epsilon)v_n \qquad \text{for all } n$$

$$\lim_{n\to\infty}(u_1+u_2+u_3+\cdots+u_n)>(l-\epsilon)\lim_{n\to\infty}(v_1+v_2+v_3+\cdots+v_n)$$

$$\lim_{n\to\infty}(u_1+u_2+u_3+\cdots+u_n)\to\infty \qquad \text{[From Eq. (2)]}$$

Hence, Σu_n is also divergent.

Note

The following standard limits can be used to solve the problems:

(i) $\lim_{n\to\infty}\frac{\log n}{n}=0$

(ii) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$

(iii) $\lim_{n\to\infty}(n)^{\frac{1}{n}}=1$

(iv) $\lim_{n\to\infty}(n!)^{\frac{1}{n}}=\infty$

(v) $\lim_{n\to\infty}\left(\frac{n!}{n}\right)^{\frac{1}{n}}=\frac{1}{e}$

(vi) $\lim_{n\to\infty}x^n=0$ if $x<1$

(vii) $\lim_{n\to\infty}x^n=\infty$ if $x>1$

(viii) $\lim_{n\to\infty}\frac{x^n}{n!}=0$ for all x

(ix) $\lim_{n\to 0}\left(\frac{a^n-1}{n}\right)=\log a$

(x) $\lim_{n\to\infty}\frac{a^{\frac{1}{n}}-1}{\frac{1}{n}}=\log a$

Example 1

Test the convergence of the series $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$.

Solution

Let $$u_n = \frac{\sqrt{n}}{n^2+1} = \frac{1}{n^{\frac{3}{2}}\left(1+\frac{1}{n^2}\right)}$$

Let $$v_n = \frac{1}{n^{\frac{3}{2}}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n^2}}$$

$$= 1 \qquad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^{\frac{3}{2}}}$ is convergent as $p = \frac{3}{2} > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 2

Test the convergence of the series $\sum_{n=1}^{\infty}\frac{2n-1}{n(n+1)(n+2)}$.

Solution

Let $$u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{\left(2-\frac{1}{n}\right)}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

Let $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$= 2 \text{ [finite and non-zero]}$$

and the series $\Sigma v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 3

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{1+2^2+3^2+\cdots+n^2}$.

[Winter 2015]

Solution

Let
$$u_n = \frac{1}{1+2^2+3^2+\cdots+n^2}$$
$$= \frac{6}{n(n+1)(2n+1)}$$
$$= \frac{6}{n^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$

Let
$$v_n = \frac{1}{n^3}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{6}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$
$$= 6 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^3}$ is convergent as $p = 3 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 4

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n+2}{(n+1)\sqrt{n}}$.

Solution

Let
$$u_n = \frac{n+2}{(n+1)\sqrt{n}}$$

$$= \frac{1+\frac{2}{n}}{n^{\frac{1}{2}}\left(1+\frac{1}{n}\right)}$$

Let $$v_n = \frac{1}{n^{\frac{1}{2}}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)}$$

$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^{\frac{1}{2}}}$ is divergent as $p = \frac{1}{2} < 1$.

Hence, by comparison test, Σu_n is also divergent.

Example 5

Is the series $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$ *convergent or divergent?* **[Summer 2015]**

Solution

Let $$u_n = \frac{2n+1}{(n+1)^2}$$

$$= \frac{n\left(2+\frac{1}{n}\right)}{n^2\left(1+\frac{1}{n}\right)^2}$$

$$= \frac{1}{n}\frac{\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^2}$$

Let $$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^2}$$

$$=\frac{2}{1}=2 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n=\sum\frac{1}{n}$ is divergent as $p=1$.

Hence, by comparison test, Σu_n is also divergent.

Example 6

Test the convergence of the series $\sum_{n=1}^{\infty}\frac{(2n^2-1)^{\frac{1}{3}}}{(3n^3+2n+5)^{\frac{1}{4}}}$. **[Winter 2013]**

Solution

Let
$$u_n=\frac{(2n^2-1)^{\frac{1}{3}}}{(3n^3+2n+5)^{\frac{1}{4}}}$$

$$=\frac{n^{\frac{2}{3}}\left(2-\frac{1}{n^2}\right)^{\frac{1}{3}}}{n^{\frac{3}{4}}\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{\frac{1}{4}}}$$

$$=\frac{\left(2-\frac{1}{n^2}\right)^{\frac{1}{3}}}{n^{\frac{1}{12}}\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{\frac{1}{4}}}$$

Let
$$v_n=\frac{1}{n^{\frac{1}{12}}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\left(2-\frac{1}{n^2}\right)^{\frac{1}{3}}}{\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{\frac{1}{4}}}$$

$$=\frac{(2)^{\frac{1}{3}}}{(3)^{\frac{1}{4}}} \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^{\frac{1}{12}}}$ is divergent as $p = \frac{1}{12} < 1.$

Hence, by comparison test, Σu_n is also divergent.

Example 7

Test the convergence of the series $\frac{1}{1\cdot 3} + \frac{2}{3\cdot 5} + \frac{3}{5\cdot 7} + \frac{4}{7\cdot 9} + \cdots.$

Solution

Let
$$u_n = \frac{n}{(2n-1)(2n+1)}$$
$$= \frac{1}{n\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$

Let
$$v_n = \frac{1}{n}$$
$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$
$$= \frac{1}{4} \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

Hence, by comparison test, Σu_n is also divergent.

Example 8

Test the convergence of the series $\frac{1}{3^p} + \frac{1}{5^p} + \frac{1}{7^p} + \cdots.$

Solution

Let
$$u_n = \frac{1}{(2n+1)^p}$$
$$= \frac{1}{n^p\left(2+\frac{1}{n}\right)^p}$$

Let
$$v_n = \frac{1}{n^p}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{\left(2+\frac{1}{n}\right)^p}$$

$$=\frac{1}{2^p}\qquad \text{[finite and non-zero]}$$

and $\Sigma v_n=\sum\frac{1}{n^p}$ is convergent if $p>1$ and divergent if $p\le 1$.

Hence, by comparison test, Σu_n is also convergent if $p>1$ and divergent if $p\le 1$.

Example 9

Test the convergence of the series $\frac{2}{1}+\frac{3}{8}+\frac{4}{27}+\frac{5}{64}+\cdots+\frac{n+1}{n^3}+\cdots$.

Solution

Let $$u_n=\frac{n+1}{n^3}$$

$$=\frac{1}{n^2}\left(1+\frac{1}{n}\right)$$

Let $$v_n=\frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

$$=1\qquad \text{[finite and non-zero]}$$

and $\Sigma v_n=\sum\frac{1}{n^2}$ is convergent as $p=2>1$.

Hence, by comparison test, Σu_n is also convergent.

Example 10

Test the convergence of the series $\frac{1}{1+\sqrt{2}}+\frac{2}{1+2\sqrt{3}}+\frac{3}{1+3\sqrt{4}}+\cdots$.

Solution

Let $$u_n=\frac{n}{1+n\sqrt{n+1}}$$

$$=\frac{n}{n^{\frac{3}{2}}\left(\frac{1}{n^{\frac{3}{2}}}+\sqrt{1+\frac{1}{n}}\right)}$$

$$= \frac{1}{n^{\frac{1}{2}}\left(\frac{1}{n^{\frac{3}{2}}} + \sqrt{1+\frac{1}{n}}\right)}$$

Let $$v_n = \frac{1}{n^{\frac{1}{2}}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(\frac{1}{n^{\frac{3}{2}}} + \sqrt{1+\frac{1}{n}}\right)}$$

$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \frac{1}{n^{\frac{1}{2}}}$ is divergent as $p = \frac{1}{2} < 1$.

Hence, by comparison test, Σu_n is also divergent.

Example 11

Test the convergence of the series $\sqrt{\frac{1}{2^3}} + \sqrt{\frac{2}{3^3}} + \sqrt{\frac{3}{4^3}} + \cdots$.

Solution

Let $$u_n = \sqrt{\frac{n}{(n+1)^3}}$$

$$= \frac{n^{\frac{1}{2}}}{(n+1)^{\frac{3}{2}}}$$

$$= \frac{n^{\frac{1}{2}}}{n^{\frac{3}{2}}\left(1+\frac{1}{n}\right)^{\frac{3}{2}}}$$

$$= \frac{1}{n\left(1+\frac{1}{n}\right)^{\frac{3}{2}}}$$

Let $$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{\frac{3}{2}}}$$

$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

Hence, by comparison test, Σu_n is also divergent.

Example 12

Test the convergence of the series $\frac{14}{1^3}+\frac{24}{2^3}+\frac{34}{3^3}+\cdots$.

Solution

n^{th} term of the numerator $= a + (n-1)d = 14 + (n-1)10 = 10n + 4$
n^{th} term of the denominator $= n^3$

$$u_n = \frac{10n+4}{n^3}$$

$$= \frac{1}{n^2}\left(10+\frac{4}{n}\right)$$

Let

$$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left(10+\frac{4}{n}\right)$$

$$= 10 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 13

Test the convergence of the series $\frac{1}{a\cdot 1^2+b}+\frac{2}{a\cdot 2^2+b}+\frac{3}{a\cdot 3^2+b}+\cdots$.

Solution

Let

$$u_n = \frac{n}{a\cdot n^2+b}$$

$$= \frac{1}{n\left(a+\frac{b}{n^2}\right)}$$

Let
$$v_n = \frac{1}{n}$$
$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(a + \frac{b}{n^2}\right)}$$
$$= \frac{1}{a} \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

Hence, by comparison test, Σu_n is also divergent.

Example 14

Test the convergence of the series $\frac{1}{1^2+m} + \frac{2}{2^2+m} + \frac{3}{3^2+m} + \cdots$.

Solution

Let
$$u_n = \frac{n}{n^2+m}$$
$$= \frac{1}{n\left(1 + \frac{m}{n^2}\right)}$$

Let
$$v_n = \frac{1}{n}$$
$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(1 + \frac{m}{n^2}\right)}$$
$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

Hence, by comparison test, Σu_n is also divergent.

Example 15

Test the convergence of the series $\frac{2\cdot 1^3+5}{4\cdot 1^5+1} + \frac{2\cdot 2^3+5}{4\cdot 2^5+1} + \cdots + \frac{2\cdot n^3+5}{4\cdot n^5+1} + \cdots$.

Solution

Let
$$u_n = \frac{2n^3+5}{4n^5+1}$$

$$= \frac{\left(2+\frac{5}{n^3}\right)}{n^2\left(4+\frac{1}{n^5}\right)}$$

Let $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\left(2+\frac{5}{n^3}\right)}{\left(4+\frac{1}{n^5}\right)}$$

$$= \frac{2}{4} \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 16

Test the convergence of the series $\frac{1\cdot 2}{3^2\cdot 4^2} + \frac{3\cdot 4}{5^2\cdot 6^2} + \frac{5\cdot 6}{7^2\cdot 8^2} + \cdots$.

[Winter 2013]

Solution

Let $$u_n = \frac{(2n-1)\cdot 2n}{(2n+1)^2(2n+2)^2} \quad \text{[Using A.P.]}$$

$$= \frac{\left(2-\frac{1}{n}\right)2}{n^2\left(2+\frac{1}{n}\right)^2\left(2+\frac{2}{n}\right)^2}$$

Let $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2\left(2-\frac{1}{n}\right)}{\left(2+\frac{1}{n}\right)^2\left(2+\frac{2}{n}\right)^2}$$

$$= \frac{1}{4} \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 17

Test the convergence of the series $\frac{1}{1\cdot 2\cdot 3}+\frac{3}{2\cdot 3\cdot 4}+\frac{5}{3\cdot 4\cdot 5}+\frac{7}{4\cdot 5\cdot 6}+\cdots$.

[Winter 2014]

Solution

Let
$$u_n = \frac{(2n+1)}{n(n+1)(n+2)} \qquad \text{[Using A.P.]}$$
$$= \frac{\left(2+\frac{1}{n}\right)}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

Let
$$v_n = \frac{1}{n^2}$$
$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$
$$= 2 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 18

Test the convergence of the series $\sum_{n=1}^{\infty}\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$.

[Summer 2016]

Solution

Let
$$u_n = \sqrt{n^4+1}-\sqrt{n^4-1}$$
$$= \frac{\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)}{\left(\sqrt{n^4+1}+\sqrt{n^4-1}\right)}\left(\sqrt{n^4+1}+\sqrt{n^4-1}\right)$$

$$= \frac{(n^4+1)-(n^4-1)}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \frac{1}{n^2} \cdot \frac{2}{\left(\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\right)}$$

Let $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2}{\left(\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\right)}$$

$$= 2 \quad \text{[finite and non-zero]}$$

and $\sum v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 19

Check for convergence of the series $\sum_{n=1}^{\infty} \frac{5n^3-3n}{n^2(n-2)(n^2+5)}$.

[Winter 2016]

Solution

$$u_n = \frac{5n^3-3n}{n^2(n-2)(n^2+5)}$$

$$= \frac{n^3\left(5-\frac{3}{n^2}\right)}{n^5\left(1-\frac{2}{n}\right)\left(1+\frac{5}{n^2}\right)}$$

$$= \frac{\left(5-\frac{3}{n^2}\right)}{n^2\left(1-\frac{2}{n}\right)\left(1+\frac{5}{n^2}\right)}$$

Let $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\left(5-\frac{3}{n^2}\right)}{\left(1-\frac{2}{n}\right)\left(1+\frac{5}{n^2}\right)}$$

$$= 5 \qquad \text{[finite and non zero]}$$

and $\sum v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison text, $\sum u_n$ is also convergent.

Example 20

Test the convergence of the series $\sum_{n=1}^{\infty}\left[\frac{1}{n} - \log\left(\frac{n+1}{n}\right)\right]$.

Solution

Let $$u_n = \frac{1}{n} - \log\left(\frac{n+1}{n}\right)$$

$$= \frac{1}{n} - \log\left(1+\frac{1}{n}\right)$$

$$= \frac{1}{n} - \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \cdots\right)$$

$$= \frac{1}{2n^2} - \frac{1}{3n^3} + \frac{1}{4n^4} - \cdots$$

$$= \frac{1}{n^2}\left(\frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} - \cdots\right)$$

Let $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} - \cdots\right)$$

$$= \frac{1}{2} \qquad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 21

Test the convergence of the series $\sum_{n=1}^{\infty} \sin\frac{1}{n}$.

Solution

Let
$$u_n = \sin\frac{1}{n}$$
$$= \frac{1}{n} - \frac{1}{3!n^3} + \frac{1}{5!n^5} - \cdots$$
$$= \frac{1}{n}\left(1 - \frac{1}{3!n^2} + \frac{1}{5!n^4} - \cdots\right)$$

Let
$$v_n = \frac{1}{n}$$
$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(1 - \frac{1}{3!n^2} + \frac{1}{5!n^4} - \cdots\right)$$
$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum\frac{1}{n}$ is divergent as $p = 1$.

Hence, by comparison test, Σu_n is also divergent.

Example 22

Test the convergence of the series $\sum_{n=1}^{\infty}\left[\left(n^3+1\right)^{\frac{1}{3}} - n\right]$. **[Summer 2017]**

Solution

Let
$$u_n = \left[(n^3+1)^{\frac{1}{3}} - n\right]$$
$$= n\left(1+\frac{1}{n^3}\right)^{\frac{1}{3}} - n$$
$$= n\left[1 + \frac{1}{3}\cdot\frac{1}{n^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{n^3}\right)^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}\left(\frac{1}{n^3}\right)^3 + \cdots\right] - n$$
$$= \frac{1}{3}\cdot\frac{1}{n^2} - \frac{1}{3^2}\cdot\frac{1}{n^5} + \frac{5}{3^4}\cdot\frac{1}{n^8} - \cdots$$

$$= \frac{1}{n^2}\left(\frac{1}{3} - \frac{1}{3^2}\cdot\frac{1}{n^3} + \frac{5}{3^4}\cdot\frac{1}{n^6} - \cdots\right)$$

Let $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{1}{3} - \frac{1}{3^2}\cdot\frac{1}{n^3} + \frac{5}{3^4}\cdot\frac{1}{n^6} - \cdots\right)$$

$$= \frac{1}{3} \quad \text{[finite and non-zero]}$$

and the series $\Sigma v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 23

Test the convergence of the series $\sum_{n=1}^{\infty}\left[(n+1)^{\frac{1}{3}} - n^{\frac{1}{3}}\right]$.

Solution

Let $$u_n = (n+1)^{\frac{1}{3}} - n^{\frac{1}{3}}$$

$$= n^{\frac{1}{3}}\left[\left(1+\frac{1}{n}\right)^{\frac{1}{3}} - 1\right]$$

$$= n^{\frac{1}{3}}\left[\left\{1 + \frac{1}{3n} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\cdot\frac{1}{n^2} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}\cdot\frac{1}{n^3} + \cdots\right\} - 1\right]$$

$$= \frac{1}{3n^{\frac{2}{3}}} - \frac{1}{9n^{\frac{5}{3}}} + \frac{5}{81n^{\frac{8}{3}}} - \cdots$$

$$= \frac{1}{n^{\frac{2}{3}}}\left(\frac{1}{3} - \frac{1}{9n} + \frac{5}{81n^2} - \cdots\right)$$

Let $$v_n = \frac{1}{n^{\frac{2}{3}}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{9n}+\frac{5}{81n^2}-\cdots\right)$$

$$=\frac{1}{3} \quad \text{[finite and non-zero]}$$

and $\Sigma v_n=\sum\frac{1}{n^{\frac{2}{3}}}$ is divergent as $p=\frac{2}{3}<1$.

Hence, by comparison test, Σu_n is also divergent.

Example 24

Test the convergence of the series $\sum_{n=1}^{\infty}\frac{\sqrt{n^2+n+1}-\sqrt{n^2-n+1}}{n}$.

Solution

Let

$$u_n=\frac{\sqrt{n^2+n+1}-\sqrt{n^2-n+1}}{n}$$

$$=\left[1+\left(\frac{1}{n}+\frac{1}{n^2}\right)\right]^{\frac{1}{2}}-\left[1+\left(-\frac{1}{n}+\frac{1}{n^2}\right)\right]^{\frac{1}{2}}$$

Expanding using binomial expansion,

$$u_n=\left[1+\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n^2}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{1}{n}+\frac{1}{n^2}\right)^2+\cdots\right]$$

$$-\left[1+\frac{1}{2}\left(-\frac{1}{n}+\frac{1}{n^2}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{n}+\frac{1}{n^2}\right)^2+\cdots\right]$$

$$=\left[1+\frac{1}{2n}+\frac{1}{2n^2}-\frac{1}{8}\left(\frac{1}{n^2}+\frac{2}{n^3}+\frac{1}{n^4}\right)+\cdots\right]$$

$$-\left[1-\frac{1}{2n}+\frac{1}{2n^2}-\frac{1}{8}\left(\frac{1}{n^2}-\frac{2}{n^3}+\frac{1}{n^4}\right)+\cdots\right]$$

$$= \frac{1}{n} - \frac{1}{2n^3} + \cdots$$

$$= \frac{1}{n}\left(1 - \frac{1}{2n^2} + \cdots\right)$$

Let $$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(1 - \frac{1}{2n^2} + \cdots\right)$$

$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

Hence, by comparison test, Σu_n is also divergent.

Example 25

Test the convergence of the series $\sum\left(\frac{\sqrt{n^2+1}-n}{n^p}\right)$.

Solution

Let $$u_n = \frac{\sqrt{n^2+1}-n}{n^p}$$

$$= \frac{n\left[\left(1+\frac{1}{n^2}\right)^{\frac{1}{2}} - 1\right]}{n^p}$$

$$= \frac{n}{n^p}\left[\left\{1 + \frac{1}{2n^2} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\cdot\frac{1}{n^4} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\cdot\frac{1}{n^6} + \cdots\right\} - 1\right]$$

$$= \frac{n}{n^p}\left(\frac{1}{2n^2} - \frac{1}{8n^4} + \frac{1}{16n^6} - \cdots\right)$$

$$= \frac{1}{n^{p+1}}\left(\frac{1}{2} - \frac{1}{8n^2} + \frac{1}{16n^4} - \cdots\right)$$

Let $$v_n = \frac{1}{n^{p+1}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{8n^2}+\frac{1}{16n^4}-\cdots\right)$$

$$= \frac{1}{2} \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^{p+1}}$ is convergent if $p + 1 > 1$, i.e., $p > 0$ and divergent if $p + 1 \le 1$, i.e., $p \le 0$.

Hence, by comparison test, Σu_n is also convergent if $p > 0$ and divergent if $p \le 0$.

Example 26

Test the convergence of the series $\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x+1}+\frac{1}{x-2}+\frac{1}{x+2}+\cdots$, *where x is a positive fraction.*

Solution

Since it is an infinite series, by ignoring the first term, the series can be rewritten as

$$\sum u_n = \left(\frac{1}{x-1}+\frac{1}{x+1}\right)+\left(\frac{1}{x-2}+\frac{1}{x+2}\right)+\cdots$$

$$= \frac{2x}{x^2-1^2}+\frac{2x}{x^2-2^2}+\cdots$$

$$= \sum \frac{2x}{x^2-n^2}$$

$$u_n = \frac{2x}{x^2-n^2}$$

$$= \frac{2x}{n^2\left(\frac{x^2}{n^2}-1\right)}$$

Let $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2x}{\left(\frac{x^2}{n^2}-1\right)}$$

$$= -2x \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

EXERCISE 5.2

1. Test the convergence of the following series:

(i) $\sum \frac{1}{n^2+1}$ (ii) $\sum\left(\sqrt{n+1}-\sqrt{n}\right)$

(iii) $\sum\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$ (iv) $\sum\left(\frac{n^p}{\sqrt{n+1}+\sqrt{n}}\right)$

(v) $\sum \frac{n^p}{(n+1)^q}$ (vi) $\sum \frac{1}{\sqrt{n}}\tan\left(\frac{1}{n}\right)$

(vii) $\sum \tan^{-1}\left(\frac{1}{n}\right)$ (viii) $\sum \frac{1}{n^{\left(a+\frac{b}{n}\right)}}$

[**Ans.:**
(i) Convergent (ii) Divergent (iii) Convergent
(iv) Convergent if $p < -\frac{1}{2}$ Divergent if $p \geq -\frac{1}{2}$
(v) Convergent if $p - q + 1 < 0$, Divergent if $p - q + 1 \geq 0$
(vi) Convergent (vii) Divergent
(viii) Convergent if $a > 1$, Divergent if $a \leq 1$]

2. Test the convergence of the series

$$\frac{(1+a)(1+b)}{1\cdot 2\cdot 3}+\frac{(2+a)(2+b)}{2\cdot 3\cdot 4}+\frac{(3+a)(3+b)}{3\cdot 4\cdot 5}+\dots$$

[**Ans.:** Divergent]

5.11 D'ALEMBERT'S RATIO TEST

If Σu_n is a positive-term series and $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = l$ then

(i) Σu_n is convergent if $l < 1$

(ii) Σu_n is divergent if $l > 1$

Proof

Case I If $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = l < 1.$

Consider a number $l < r < 1$ such that $\frac{u_{n+1}}{u_n} < r$ for all $n > m$ ... (1)

Neglecting the first m terms,

$$\sum_{n=m+1}^{\infty} u_n = u_{m+1} + u_{m+2} + u_{m+3} + \dots \infty$$

$$= u_{m+1}\left(1 + \frac{u_{m+2}}{u_{m+1}} + \frac{u_{m+3}}{u_{m+1}} + \frac{u_{m+4}}{u_{m+1}} + \dots\right)$$

$$= u_{m+1}\left(1 + \frac{u_{m+2}}{u_{m+1}} + \frac{u_{m+3}}{u_{m+2}} \cdot \frac{u_{m+2}}{u_{m+1}} + \frac{u_{m+4}}{u_{m+3}} \cdot \frac{u_{m+3}}{u_{m+2}} \cdot \frac{u_{m+2}}{u_{m+1}} + \dots\right)$$

$$< u_{m+1}(1 + r + r \cdot r + r \cdot r \cdot r + \dots) \qquad \text{[Using Eq. (1)]}$$

$$= u_{m+1}(1 + r + r^2 + r^3 + \dots)$$

$$= u_{m+1} \cdot \frac{1}{1-r} \qquad (r < 1)$$

$$\therefore \qquad \sum_{n=m+1}^{\infty} u_n < \frac{u_{m+1}}{1-r} \quad \text{(finite)}$$

Thus, the series $\sum\limits_{n=m+1}^{\infty} u_n$ is convergent.

The nature of a series remains unchanged if we neglect a finite number of terms in the beginning. Hence, the series $\sum\limits_{n=1}^{\infty} u_n$ is convergent.

Case II If $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = l > 1.$

$$\frac{u_{n+1}}{u_n} > 1 \text{ for all } n > m \qquad \text{... (2)}$$

Neglecting the first m terms,

$$\sum_{n=m+1}^{\infty} u_n = u_{m+1} + u_{m+2} + u_{m+3} + u_{m+4} + \ldots \infty$$

$$= u_{m+1}\left(1 + \frac{u_{m+2}}{u_{m+1}} + \frac{u_{m+3}}{u_{m+1}} + \frac{u_{m+4}}{u_{m+1}} + \ldots\right)$$

$$= u_{m+1}\left(1 + \frac{u_{m+2}}{u_{m+1}} + \frac{u_{m+3}}{u_{m+2}} \cdot \frac{u_{m+2}}{u_{m+1}} + \frac{u_{m+4}}{u_{m+3}} \cdot \frac{u_{m+3}}{u_{m+2}} \cdot \frac{u_{m+2}}{u_{m+1}} + \ldots\right)$$

$$> u_{m+1}(1+1+1+1+\ldots)$$

$$\therefore \quad (u_{m+1} + u_{m+2} + \ldots \text{to } n \text{ terms}) > u_{m+1}(1+1+1\ldots \text{ to } n \text{ terms})$$

$$S_n > u_{m+1} \cdot n$$

$$\lim_{n\to\infty} S_n > \lim_{n\to\infty} n u_{m+1} \to \infty \qquad [\because u_{m+1} \text{ is positive}]$$

Thus, the series $\sum_{n=m+1}^{\infty} u_n$ is divergent.

The nature of a series remains unchanged if we neglect a finite number of terms in the beginning. Hence, the series $\sum_{n=1}^{\infty} u_n$ is divergent.

Note 1: If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$ the ratio test fails, i.e. no conclusion can be drawn about the convergence or divergence of the series.

Note 2: It is convenient to use D'Alembert's ratio test in the following form:

If Σu_n is a positive term series and $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l$, then

(i) Σu_n is convergent if $l > 1$

(ii) Σu_n is divergent if $l < 1$

(iii) The ratio test fails if $l = 1$

Example 1

Test the convergence of the series $\sum_{n=0}^{\infty} \frac{3^{2n}}{2^{3n}}$. **[Summer 2014]**

Solution

Let $$u_n = \frac{3^{2n}}{2^{3n}}$$

$$u_{n+1} = \frac{3^{2(n+1)}}{2^{3(n+1)}} = \frac{3^{2n+2}}{2^{3n+3}}$$

$$\frac{u_n}{u_{n+1}} = \frac{3^{2n}}{2^{3n}} \cdot \frac{2^{3n+3}}{3^{2n+2}}$$

$$= \frac{2^3}{3^2} = \frac{8}{9}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{8}{9} = \frac{8}{9} < 1$$

Hence, by D'Alembert's ratio test, the series is divergent.

Example 2

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{5^{n-1}}{n!}$.

Solution

Let

$$u_n = \frac{5^{n-1}}{n!}$$

$$u_{n+1} = \frac{5^n}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{5^{n-1}}{n!} \cdot \frac{(n+1)!}{5^n}$$

$$= \frac{n+1}{5}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n+1}{5} \to \infty > 1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 3

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{2^n}{n^3+1}$. **[Winter 2016]**

Solution

Let

$$u_n = \frac{2^n}{n^3+1}$$

$$u_{n+1} = \frac{2^{n+1}}{(n+1)^3+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{2^n}{n^3+1} \cdot \frac{(n+1)^3+1}{2^{n+1}}$$

$$= \frac{\left(1+\frac{1}{n}\right)^3 + \frac{1}{n^3}}{1+\frac{1}{n^3}} \cdot \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^3 + \frac{1}{n^3}}{1+\frac{1}{n^3}} \cdot \frac{1}{2}$$

$$= \frac{1}{2} < 1$$

Hence, by D'Alembert's ratio test, the series is divergent.

Example 4

Test the convergence of the series $\sum \frac{n!}{n^n}$.

Solution

Let

$$u_n = \frac{n!}{n^n}$$

$$u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{\frac{n!}{n^n}}{\frac{(n+1)!}{(n+1)^{n+1}}}$$

$$= \frac{(n+1)(n+1)^n}{(n+1)n^n}$$

$$= \left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$$= e > 1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 5

Test the convergence of the series $\sum \frac{n!(2)^n}{n^n}$.

Solution

Let

$$u_n = \frac{n!(2)^n}{n^n}$$

$$u_{n+1} = \frac{(n+1)!(2)^{n+1}}{(n+1)^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{n!2^n}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!2^{n+1}}$$

$$= \frac{1}{2}\left(\frac{n+1}{n}\right)^n$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= \frac{e}{2} > 1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 6

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^3}{(n-1)!}$.

Solution

Let

$$u_n = \frac{n^3}{(n-1)!}$$

$$u_{n+1} = \frac{(n+1)^3}{n!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^3}{(n-1)!} \cdot \frac{n!}{(n+1)^3}$$

$$= \frac{n^3}{(n-1)!} \cdot \frac{n(n-1)!}{(n+1)^3}$$

$$= \frac{n}{\left(1+\frac{1}{n}\right)^3}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n}{\left(1+\frac{1}{n}\right)^2} \to \infty > 1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 7

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(n+1)^n}{n!}$.

Solution

Let

$$u_n = \frac{(n+1)^n}{n!}$$

$$u_{n+1} = \frac{(n+2)^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)^n}{n!} \cdot \frac{(n+1)!}{(n+2)^{n+1}}$$

$$= \frac{(n+1)^n}{n!} \cdot \frac{(n+1)(n!)}{[(n+1)+1]^{n+1}}$$

$$= \frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}$$

$$= \frac{1}{e} < 1$$

Hence, by D'Alembert's ratio test, the series is divergent.

Example 8

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n+1}$.

Solution

Let

$$u_n = \frac{2^n+1}{3^n+1}$$

$$u_{n+1} = \frac{2^{n+1}+1}{3^{n+1}+1}$$

$$\frac{u_n}{u_{n+1}} = \left(\frac{2^n+1}{3^n+1}\right)\left(\frac{3^{n+1}+1}{2^{n+1}+1}\right)$$

$$= \frac{\left(1+\frac{1}{2^n}\right)\left(3+\frac{1}{3^n}\right)}{\left(1+\frac{1}{3^n}\right)\left(2+\frac{1}{2^n}\right)}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{2^n}\right)\left(3+\frac{1}{3^n}\right)}{\left(1+\frac{1}{3^n}\right)\left(2+\frac{1}{2^n}\right)}$$

$$= \frac{3}{2} > 1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 9

Test the convergence of the series $\sum_{n=1}^{\infty}\frac{5^n+a}{3^n+b}, a>0, b>0.$

Solution

Let

$$u_n = \frac{5^n+a}{3^n+b}$$

$$u_{n+1} = \frac{5^{n+1}+a}{3^{n+1}+b}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{5^n+a}{3^n+b}\cdot\frac{3^{n+1}+b}{5^{n+1}+a}$$

$$= \lim_{n\to\infty}\frac{1+\frac{a}{5^n}}{1+\frac{b}{3^n}}\cdot\frac{3+\frac{b}{3^n}}{5+\frac{a}{5^n}}$$

$$= \frac{3}{5} < 1$$

Hence, by D'Alembert's ratio test, the series is divergent.

Example 10

Test the convergence of the series $\frac{2!}{3}+\frac{3!}{3^2}+\frac{4!}{3^3}+\cdots.$

Solution

Let

$$u_n = \frac{(n+1)!}{3^n}$$

$$u_{n+1} = \frac{(n+2)!}{3^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(n+1)!}{3^n} \cdot \frac{3^{n+1}}{(n+2)!}$$

$$= \lim_{n\to\infty} \left(\frac{3}{n+2}\right)$$

$$= 0 < 1$$

Hence, by D'Alembert's ratio test, the series is divergent.

Example 11

Test the convergence of the series $1 + \frac{4}{2!} + \frac{4^2}{3!} + \frac{4^3}{4!} + \frac{4^4}{5!} + \cdots$.

Solution

Let

$$u_n = \frac{4^{n-1}}{n!}$$

$$u_{n+1} = \frac{4^n}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{4^{n-1}}{n!} \cdot \frac{(n+1)!}{4^n}$$

$$= \frac{(n+1)}{4}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(n+1)}{4} \to \infty > 1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 12

Test the convergence of the series $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \cdots$.

Solution

Let

$$u_n = \frac{n^2}{n!}$$

$$u_{n+1} = \frac{(n+1)^2}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^2}{n!} \cdot \frac{(n+1)!}{(n+1)^2}$$

$$= \frac{n^2(n+1)\cdot n!}{n!(n+1)^2}$$

$$= \frac{n^2}{n+1}$$

$$= \frac{n}{1+\frac{1}{n}}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{n}{1+\frac{1}{n}} \to \infty > 1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 13

Test the convergence of $1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\ldots,(p>0).$

Solution

Let
$$u_n = \frac{n^p}{n!}$$

$$u_{n+1} = \frac{(n+1)^p}{(n+1)!}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\frac{n^p}{n!}}{\frac{(n+1)^p}{(n+1)!}}$$

$$= \lim_{n\to\infty}\frac{n^p}{(n+1)^p}\cdot\frac{(n+1)!}{n!}$$

$$= \lim_{n\to\infty}\frac{(n+1)}{\left(1+\frac{1}{n}\right)^p} \to \infty > 1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 14

Test the convergence of the series $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\frac{4}{1+2^4}+\cdots.$

Solution

Let
$$u_n = \frac{n}{1+2^n}$$
$$u_{n+1} = \frac{n+1}{1+2^{n+1}}$$
$$\frac{u_n}{u_{n+1}} = \frac{n}{1+2^n} \cdot \frac{1+2^{n+1}}{n+1}$$
$$= \frac{\frac{1}{2^n}+2}{\left(\frac{1}{2^n}+1\right)\left(1+\frac{1}{n}\right)}$$
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\frac{1}{2^n}+2}{\left(\frac{1}{2^n}+1\right)\left(1+\frac{1}{n}\right)}$$
$$= 2 > 1 \qquad \left[\because \lim_{n\to\infty} 2^n \to \infty\right]$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 15

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6 \ldots 2n}{5\cdot 8\cdot 11 \ldots (3n+2)}$.

Solution

Let
$$u_n = \frac{2\cdot 4\cdot 6 \ldots 2n}{5\cdot 8\cdot 11 \ldots (3n+2)} \qquad \text{[Using A.P.]}$$
$$u_{n+1} = \frac{2\cdot 4\cdot 6 \ldots 2n(2n+2)}{5\cdot 8\cdot 11 \ldots (3n+2)(3n+5)}$$
$$\frac{u_n}{u_{n+1}} = \frac{2\cdot 4\cdot 6 \ldots 2n}{5\cdot 8\cdot 11 \ldots (3n+2)} \cdot \frac{5\cdot 8\cdot 11 \ldots (3n+2)(3n+5)}{2\cdot 4\cdot 6 \ldots 2n(2n+2)}$$
$$= \frac{3n+5}{2n+2}$$
$$= \frac{3+\frac{5}{n}}{2+\frac{2}{n}}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{3+\frac{5}{n}}{2+\frac{2}{n}}$$

$$= \frac{3}{2} > 1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 16

Test the convergence of the series $\frac{2}{1}+\frac{2\cdot 5}{1\cdot 5}+\frac{2\cdot 5\cdot 8}{1\cdot 5\cdot 9}+\frac{2\cdot 5\cdot 8\cdot 11}{1\cdot 5\cdot 9\cdot 13}+\cdots,$

Solution

Let $u_n = \frac{2\cdot 5\cdot 8\cdot 11\ldots(3n-1)}{1\cdot 5\cdot 9\cdot 13\ldots(4n-3)}$ [Using A.P.]

$$u_{n+1} = \frac{2\cdot 5\cdot 8\cdot 11\ldots(3n-1)(3n+2)}{1\cdot 5\cdot 9\cdot 13\ldots(4n-3)(4n+1)}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\frac{2\cdot 5\cdot 8\cdot 11\ldots(3n-1)}{1\cdot 5\cdot 9\cdot 13\ldots(4n-3)}}{\frac{2\cdot 5\cdot 8\cdot 11\ldots(3n-1)(3n+2)}{1\cdot 5\cdot 9\cdot 13\ldots(4n-3)(4n+1)}}$$

$$= \lim_{n\to\infty}\frac{4n+1}{3n+2}$$

$$= \lim_{n\to\infty}\frac{4+\frac{1}{n}}{3+\frac{2}{n}}$$

$$= \frac{4}{3} > 1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 17

Test the convergence of the series $\left(\frac{1}{3}\right)^2+\left(\frac{1\cdot 2}{3\cdot 5}\right)^2+\left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2+\ldots\infty.$

Solution

Let $u_n = \left[\frac{1\cdot 2\cdot 3\ldots n}{3\cdot 5\cdot 7\ldots(2n+1)}\right]^2$ [Using A.P.]

$$u_{n+1} = \left[\frac{1\cdot 2\cdot 3\ldots n(n+1)}{3\cdot 5\cdot 7\ldots(2n+1)(2n+3)}\right]^2$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{\left[\dfrac{1\cdot 2\cdot 3\ldots n}{3\cdot 5\cdot 7\ldots(2n+1)}\right]^2}{\left[\dfrac{1\cdot 2\cdot 3\ldots n(n+1)}{3\cdot 5\cdot 7\ldots(2n+1)(2n+3)}\right]^2}$$

$$=\lim_{n\to\infty}\left[\frac{2n+3}{n+1}\right]^2$$

$$=\lim_{n\to\infty}\left[\frac{2+\frac{3}{n}}{1+\frac{1}{n}}\right]^2$$

$$=4>1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 18

Test the convergence of the series $2+\frac{3}{2}x+\frac{4}{3}x^2+\frac{5}{4}x^3+\cdots$.

[Summer 2014]

Solution

Let
$$u_n=\frac{n+1}{n}x^{n-1}$$

$$u_{n+1}=\frac{n+2}{n+1}x^n$$

$$\frac{u_n}{u_{n+1}}=\frac{(n+1)x^{n-1}}{n}\cdot\frac{n+1}{(n+2)x^n}$$

$$=\frac{1+\frac{1}{n}}{1}\cdot\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\cdot\frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{1}\cdot\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\cdot\frac{1}{x}$$

$$=\frac{1}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x} > 1$ or $x < 1$

(ii) divergent if $\frac{1}{x} < 1$ or $x > 1$

The test fails if $\frac{1}{x} = 1$, i.e., $x = 1$.

Then
$$u_n = \frac{n+1}{n} = 1 + \frac{1}{n}$$
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$
$$= 1 \quad \text{[finite and non-zero]}$$

Σu_n is divergent for $x = 1$.

Hence, the series is convergent for $x < 1$ and is divergent for $x \geq 1$.

Example 19

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(n+1)}{n^2} x^n$.

Solution

Let
$$u_n = \frac{n+1}{n^2} x^n$$
$$u_{n+1} = \frac{n+2}{(n+1)^2} x^{n+1}$$
$$\frac{u_n}{u_{n+1}} = \frac{(n+1)x^n}{n^2} \cdot \frac{(n+1)^2}{(n+2)x^{n+1}}$$
$$= \frac{(n+1)^3}{n^2(n+2)x}$$
$$= \frac{\left(1+\frac{1}{n}\right)^3}{\left(1+\frac{2}{n}\right)} \cdot \frac{1}{x}$$
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^3}{\left(1+\frac{2}{n}\right)} \cdot \frac{1}{x}$$
$$= \frac{1}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x} > 1$ or $x < 1$

(ii) divergent if $\frac{1}{x} < 1$ or $x > 1$

The test fails if $\frac{1}{x} = 1$, i.e., $x = 1$.

Then
$$u_n = \frac{n+1}{n^2}$$
$$= \frac{1}{n}\left(1+\frac{1}{n}\right)$$

Let
$$v_n = \frac{1}{n}$$
$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$
$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

By comparison test, Σu_n is also divergent for $x = 1$.
Hence, the series is convergent for $x < 1$ and is divergent for $x \geq 1$.

Example 20

Test the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{x^n}{n^2+1}\right)$, *for* $x > 0$.

Solution

Let
$$u_n = \frac{x^n}{n^2+1}$$
$$u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$$
$$\frac{u_n}{u_{n+1}} = \frac{x^n}{n^2+1}\cdot\frac{(n+1)^2+1}{x^{n+1}}$$
$$= \frac{\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}}{1+\frac{1}{n^2}}\cdot\frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}}{1+\frac{1}{n^2}}\cdot\frac{1}{x}$$
$$=\frac{1}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x}>1$ or $x<1$

(ii) divergent if $\frac{1}{x}<1$ or $x>1$

The test fails if $x = 1$.

Then
$$u_n=\frac{1}{n^2+1}$$
$$=\frac{1}{n^2\left(1+\frac{1}{n^2}\right)}$$

Let
$$v_n=\frac{1}{n^2}$$
$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n^2}\right)}$$
$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n=\sum\frac{1}{n^2}$ is convergent as $p = 2 > 1$.

By comparison test, Σu_n is also convergent if $x = 1$.

Hence, the series is convergent for $x \le 1$ and is divergent for $x > 1$.

Example 21

Test the convergence of the series $\sum\frac{2^n}{n^4+1}x^n$, $x > 0$.

Solution

Let
$$u_n=\frac{2^n}{n^4+1}x^n$$

$$u_{n+1} = \frac{2^{n+1}}{(n+1)^4+1}x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{2^n x^n}{n^4+1}\cdot\frac{(n+1)^4+1}{2^{n+1}x^{n+1}}$$

$$= \frac{\left(1+\frac{1}{n}\right)^4+\frac{1}{n^4}}{\left(1+\frac{1}{n^4}\right)2x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^4+\frac{1}{n^4}}{\left(1+\frac{1}{n^4}\right)2x}$$

$$= \frac{1}{2x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{2x} > 1$ or $x < \frac{1}{2}$

(ii) divergent if $\frac{1}{2x} < 1$ or $x > \frac{1}{2}$

The test fails if $\frac{1}{2x} = 1$ or $x = \frac{1}{2}$.

Then
$$u_n = \frac{2^n}{n^4+1}\cdot\frac{1}{2^n}$$

$$= \frac{1}{n^4+1}$$

$$= \frac{1}{n^4\left(1+\frac{1}{n^4}\right)}$$

Let
$$v_n = \frac{1}{n^4}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n^4}\right)}$$

$= 1$ [finite and non-zero]

and $\Sigma v_n = \sum\frac{1}{n^4}$ is convergent as $p = 4 > 1$.

By comparison test, Σu_n is also convergent if $x = \frac{1}{2}$.

Hence, the series is convergent for $x \le \frac{1}{2}$ and is divergent for $x > \frac{1}{2}$.

Example 22

Test the convergence of the series $\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n.$

Solution

Let
$$u_n = \sqrt{\frac{n}{n^2+1}} \cdot x^n$$
$$u_{n+1} = \sqrt{\frac{(n+1)}{(n+1)^2+1}} \cdot x^{n+1}$$
$$\frac{u_n}{u_{n+1}} = \sqrt{\frac{n}{n^2+1}} \cdot x^n \sqrt{\frac{(n+1)^2+1}{n+1}} \cdot \frac{1}{x^{n+1}}$$
$$= \sqrt{\frac{n}{(n+1)} \cdot \frac{(n^2+2n+2)}{(n^2+1)}} \cdot \frac{1}{x}$$
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \sqrt{\frac{\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n^2}\right)}} \cdot \frac{1}{x}$$
$$= \frac{1}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x} > 1$, or $x < 1$

(ii) divergent if $\frac{1}{x} < 1$, or $x > 1$

The test fails for $x = 1$.

Then
$$u_n = \sqrt{\frac{n}{n^2+1}}$$
$$= \frac{n^{\frac{1}{2}}}{n\sqrt{1+\frac{1}{n^2}}}$$

$$= \frac{1}{n^{\frac{1}{2}}\sqrt{1+\frac{1}{n^2}}}$$

Let

$$v_n = \frac{1}{n^{\frac{1}{2}}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^{\frac{1}{2}}}$ is divergent for $p = \frac{1}{2} < 1$

By comparison test, Σu_n is also divergent for $x = 1$.
Hence, the series is convergent for $x < 1$ and is divergent for $x \geq 1$.

Example 23

Test the convergence of the series $\frac{x}{x+1} + \frac{x^2}{x+2} + \frac{x^3}{x+3} + \frac{x^4}{x+4} + \cdots$.

Solution

Let

$$u_n = \frac{x^n}{x+n}$$

$$u_{n+1} = \frac{x^{n+1}}{x+n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^n}{x+n}\cdot\frac{x+n+1}{x^{n+1}}$$

$$= \frac{\frac{x}{n}+1+\frac{1}{n}}{\frac{x}{n}+1}\cdot\frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\frac{x}{n}+1+\frac{1}{n}}{\frac{x}{n}+1}\cdot\frac{1}{x}$$

$$= \frac{1}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x} > 1$ or $x < 1$

(ii) divergent if $\frac{1}{x} < 1$ or $x > 1$

The test fails if $\frac{1}{x} = 1$ or $x = 1$.

Then
$$u_n = \frac{1}{1+n}$$
$$= \frac{1}{n\left(\frac{1}{n}+1\right)}$$

Let
$$v_n = \frac{1}{n}$$
$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\left(\frac{1}{n}+1\right)}$$
$$= 1 \text{ [finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

By comparison test, Σu_n is also divergent for $x = 1$.
Hence, the series is convergent for $x < 1$ and is divergent for $x \geq 1$.

Example 24

Test the convergence of the series $\frac{x}{1\cdot 2}+\frac{x^2}{3\cdot 4}+\frac{x^3}{5\cdot 6}+\frac{x^4}{7\cdot 8}+\cdots$.

Solution

Let
$$u_n = \frac{x^n}{(2n-1)2n}$$
$$u_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$$
$$\frac{u_n}{u_{n+1}} = \frac{x^n}{(2n-1)2n}\cdot\frac{(2n+1)(2n+2)}{x^{n+1}}$$
$$= \frac{\left(2+\frac{1}{n}\right)}{\left(2-\frac{1}{n}\right)}\left(1+\frac{1}{n}\right)\cdot\frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{\left(2+\frac{1}{n}\right)}{\left(2-\frac{1}{n}\right)}\left(1+\frac{1}{n}\right)\cdot\frac{1}{x}$$

$$=\frac{1}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x}>1$ or $x<1$

(ii) divergent if $\frac{1}{x}<1$ or $x>1$

The test fails if $\frac{1}{x}=1$ or $x=1$.

Then
$$u_n=\frac{1}{(2n-1)2n}$$

$$=\frac{1}{2n^2\left(2-\frac{1}{n}\right)}$$

Let
$$v_n=\frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{2\left(2-\frac{1}{n}\right)}$$

$$=\frac{1}{4}\quad\text{[finite and non-zero]}$$

and $\Sigma v_n=\sum\frac{1}{n^2}$ is convergent as $p=2>1$.

By comparison test, Σu_n is also convergent.

Hence, the series is convergent for $x\le 1$ and is divergent for $x>1$.

Example 25

Test the convergence of the series $\dfrac{1}{1\cdot 2\cdot 3}+\dfrac{x}{4\cdot 5\cdot 6}+\dfrac{x^2}{7\cdot 8\cdot 9}+\cdots$.

[Summer 2017]

Solution

Let $$u_n = \frac{x^{n-1}}{(3n-2)(3n-1)\,3n}$$

$$u_{n+1} = \frac{x^n}{(3n+1)\,(3n+2)(3n+3)}$$

$$\frac{u_n}{u_n+1} = \frac{x^{n-1}}{(3n-2)\,(3n-1)\,(3n)} \cdot \frac{(3n+1)\,(3n+2)\,(3n+3)}{x^n}$$

$$= \frac{\left(3+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{3}{n}\right)}{\left(3-\frac{2}{n}\right)\left(3-\frac{1}{n}\right)3} \cdot \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\left(3+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{3}{n}\right)}{\left(3-\frac{2}{n}\right)\left(3-\frac{1}{n}\right)3} \cdot \frac{1}{x}$$

$$= \frac{1}{x}$$

By D′ Alembert's ratio text, the series is

(i) convergent if $\frac{1}{x} > 1$ or $x < 1$

(ii) divergent if $\frac{1}{x} < 1$ or $x > 1$

The test fails if $\frac{1}{x} = 1$ or $x = 1$.

Then $$u_n = \frac{1}{(3n-2)(3n-1)(3n)}$$

$$= \frac{1}{n^3\left(3-\frac{2}{n}\right)\left(3-\frac{1}{n}\right)3}$$

Let $$v_n = \frac{1}{n^3}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(3-\frac{2}{n}\right)\left(3-\frac{1}{n}\right)3}$$

$$= \frac{1}{27} \qquad \text{[finite and non-zero]}$$

and $\sum v_n = \sum \frac{1}{n^3}$ is convergent as $p = 3 > 1$.

By comparison test, $\sum u_n$ is also convergent for $x = 1$.

Hence, the series is convergent for $x \le 1$ and is divergent for $x > 1$.

Example 26

Test the convergence of the series $2x + \frac{3}{8}x^2 + \frac{4}{27}x^3 + \cdots + \frac{(n+1)}{n^3}x^n + \cdots$.

Solution

Let

$$u_n = \frac{n+1}{n^3} \cdot x^n$$

$$u_{n+1} = \frac{n+2}{(n+1)^3} \cdot x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)x^n}{n^3} \cdot \frac{(n+1)^3}{(n+2)x^{n+1}}$$

$$= \frac{\left(1+\frac{1}{n}\right)^4}{\left(1+\frac{2}{n}\right)} \cdot \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^4}{\left(1+\frac{2}{n}\right)} \cdot \frac{1}{x}$$

$$= \frac{1}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x} > 1$ or $x < 1$

(ii) divergent if $\frac{1}{x} < 1$ or $x > 1$

The test fails if $\frac{1}{x} = 1$ or $x = 1$.

Then

$$u_n = \frac{n+1}{n^3}$$

$$= \frac{1}{n^2}\left(1+\frac{1}{n}\right)$$

Let
$$v_n = \frac{1}{n^2}$$
$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$
$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

By comparison test, Σu_n is also convergent for $x = 1$.

Hence, the series is convergent for $x \le 1$ and is divergent for $x > 1$.

Example 27

Test the convergence of the series $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots$.

[Winter 2013]

Solution

Let
$$u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$$
$$u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$
$$\frac{u_n}{u_{n+1}} = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \cdot \frac{(n+2)\sqrt{n+1}}{x^{2n}}$$
$$= \frac{(n+2)}{(n+1)}\sqrt{\frac{n+1}{n}} \cdot \frac{1}{x^2}$$
$$= \frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)}\sqrt{1+\frac{1}{n}} \cdot \frac{1}{x^2}$$
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)}\sqrt{1+\frac{1}{n}} \cdot \frac{1}{x^2}$$
$$= \frac{1}{x^2}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x^2} > 1$ or $x^2 < 1$

(ii) divergent if $\frac{1}{x^2} < 1$ or $x^2 > 1$

The test fails if $\frac{1}{x^2} = 1$ or $x^2 = 1$.

Then
$$u_n = \frac{1}{(n+1)\sqrt{n}}$$
$$= \frac{1}{n^{\frac{3}{2}}\left(1+\frac{1}{n}\right)}$$

Let
$$v_n = \frac{1}{n^{\frac{3}{2}}}$$
$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)}$$
$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \frac{1}{n^{\frac{3}{2}}}$ is convergent as $p = \frac{3}{2} > 1$.

By comparison test, Σu_n is also convergent for $x^2 = 1$.

Hence, the series is convergent for $x^2 \le 1$ and is divergent for $x^2 > 1$.

Example 28

Test the convergence of the series $1 + \frac{3}{2}x + \frac{5}{9}x^2 + \frac{7}{28}x^3 + \frac{9}{65}x^4 + \cdots$.

Solution

Let
$$u_n = \frac{2n+1}{n^3+1}x^n \quad \text{[Neglecting the first term]}$$
$$u_{n+1} = \frac{2n+3}{(n+1)^3+1}x^{n+1}$$
$$\frac{u_n}{u_{n+1}} = \frac{(2n+1)x^n}{n^3+1} \cdot \frac{[(n+1)^3+1]}{(2n+3)x^{n+1}}$$
$$= \frac{\left(2+\frac{1}{n}\right)\left[\left(1+\frac{1}{n}\right)^3+\frac{1}{n^3}\right]}{\left(2+\frac{3}{n}\right)\left(1+\frac{1}{n^3}\right)x}$$
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{x\to\infty} \frac{\left(2+\frac{1}{n}\right)\left[\left(1+\frac{1}{n}\right)^3+\frac{1}{n^3}\right]}{\left(2+\frac{3}{n}\right)\left(1+\frac{1}{n^3}\right)x}$$

$$= \frac{1}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x} > 1$ or $x < 1$

(ii) divergent if $\frac{1}{x} < 1$ or $x > 1$

The test fails if $x = 1$.

Then
$$u_n = \frac{2n+1}{n^3+1}$$
$$= \frac{2+\frac{1}{n}}{n^2\left(1+\frac{1}{n^3}\right)}$$

Let
$$v_n = \frac{1}{n^2}$$
$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2+\frac{1}{n}}{\left(1+\frac{1}{n^3}\right)}$$
$$= 2 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

By comparison test, Σu_n is also convergent if $x = 1$.

Hence, the series is convergent for $x \le 1$ and is divergent for $x > 1$.

EXERCISE 5.3

Test the convergence of the following series:

1. $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\ldots\infty$

[Ans.: Convergent]

2. $\sum_{n=1}^{\infty}\frac{4\cdot 7\cdot 10\ldots(3n+1)}{1\cdot 2\cdot 3\cdot 4\ldots n}$

[Ans.: Convergent]

3. $\frac{1}{1+5}+\frac{2}{1+5^2}+\frac{3}{1+5^3}+\ldots\infty$

[Ans.: Convergent]

4. $1+\frac{2!}{2^2}+\frac{3!}{3^3}+\frac{4!}{4^4}+\ldots\infty$ **[Ans.:** Convergent]

5. $\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\cdots$ [**Ans.:** Convergent]

6. $1+\frac{3}{2!}+\frac{3^2}{3!}+\frac{3^3}{4!}+\frac{3^4}{5!}+\cdots$ [**Ans.:** Convergent]

7. $\sum_{n=1}^{\infty}\frac{n^2}{3^n}$ [**Ans.:** Convergent]

8. $\sum_{n=1}^{\infty}\frac{2^{n-1}}{3^n+1}$ [**Ans.:** Convergent]

9. $\sum_{n=1}^{\infty}\frac{1}{n!}$ [**Ans.:** Convergent]

10. $\sum_{n=1}^{\infty}\frac{n^2(n+1)^2}{n!}$ [**Ans.:** Convergent]

11. $\sum_{n=1}^{\infty}\frac{3^n+4^n}{4^n+5^n}$ [**Ans.:** Divergent]

12. $\sum_{n=1}^{\infty}\frac{x^n}{3^n\cdot n^2},\ x>0$

[**Ans.:** Convergent for $x<3$, divergent for $x>3$]

13. $\sum_{n=1}^{\infty}\frac{3^n-2}{3^n+1}\cdot x^{n-1},\ x>0$

[**Ans.:** Convergent for $x<1$, divergent for $x>3$]

14. $\sum_{n=1}^{\infty}\frac{x^n}{(2^n)!}$ [**Ans.:** Convergent]

15. $\sum_{n=1}^{\infty}\sqrt{\frac{n+1}{n^2+1}}\cdot x^n,\ x>0$ [**Ans.:** Convergent for $x<1$, divergent for $x>1$]

16. $x+2x^2+3x^3+4x^4+\ldots\infty$

[**Ans.:** Convergent for $x<1$, divergent for $x>1$]

17. $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\ldots+\frac{x^n}{n^2+1}+\ldots\infty$

[**Ans.:** Convergent for $x<1$, divergent for $x>1$]

18. $\frac{x}{1\cdot 3}+\frac{x^2}{3\cdot 5}+\frac{x^3}{5\cdot 7}+\ldots\infty$

[**Ans.:** Convergent for $x<1$, divergent for $x>1$]

19. $x+\frac{3}{5}x^2+\frac{8}{10}x^3+\frac{15}{17}x^4+\ldots+\frac{n^2-1}{n^2+1}x^n+\ldots\infty$

[**Ans.:** Convergent for $x < 1$, divergent for $x > 1$]

20. $\frac{x}{2\sqrt{3}}+\frac{x^2}{3\sqrt{4}}+\frac{x^3}{4\sqrt{5}}+\ldots\infty$

[**Ans.:** Convergent for $x < 1$, divergent for $x > 1$]

5.12 RAABE'S TEST

If Σu_n is a positive term series and $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=l$, then

(i) Σu_n is convergent if $l > 1$

(ii) Σu_n is divergent if $l < 1$

(iii) Test fails if $l = 1$

Proof:

(i) Consider a number p such that $p > 1$. The series $\Sigma u_n = \Sigma\frac{1}{n^p}$ is convergent if $p > 1$. By comparison test, Σu_n will be convergent if from and after some term

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} = \frac{(n+1)^p}{n^p} = \left(1+\frac{1}{n}\right)^p$$

$$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)^p = 1+\frac{p}{n}+\frac{p(p-1)}{2!n^2}+\ldots$$

$$n\left(\frac{u_n}{u_{n+1}}-1\right) > n\left[\frac{p}{n}+\frac{p(p-1)}{2n^2}+\ldots\right]$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) > \lim_{n\to\infty}\left[p+\frac{p(p-1)}{2n}+\ldots\right]$$

$$l > p > 1$$

Hence, Σu_n is convergent if $l > 1$.

(ii) Consider a number p such that $p < 1$. The series $\Sigma v_n = \Sigma\frac{1}{n^p}$ is divergent if $p < 1$.

By comparison test, Σu_n will be divergent if from and after some term

$$\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$$

Proceeding as in case (i), we get

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) < \lim_{n\to\infty}\left[p+\frac{p(p-1)}{2n}+\ldots\right]$$

$$l < p < 1$$

Hence, Σu_n is divergent if $l < 1$.

(iii) Raabe's test fails if $l = 1$ and other tests are required to check the nature of the series.

Note: When Raabe's test fails, logarithmic test can be applied.

Example 1

Test the convergence of the series $\frac{2}{7}+\frac{2\cdot 5}{7\cdot 10}+\frac{2\cdot 5\cdot 8}{7\cdot 10\cdot 13}+\ldots$.

Solution

$$u_n = \frac{2\cdot 5\cdot 8\ldots\ldots(3n-1)}{7\cdot 10\cdot 13\ldots\ldots(3n+4)}$$

$$u_{n+1} = \frac{2\cdot 5\cdot 8\ldots(3n-1)(3n+2)}{7\cdot 10\cdot 13\ldots(3n+4)(3n+7)}$$

$$\frac{u_n}{u_{n+1}} = \frac{2\cdot 5\cdot 8\ldots(3n-1)}{7\cdot 10\cdot 13\ldots(3n+4)}\cdot\frac{7\cdot 10\cdot 13\ldots(3n+4)(3n+7)}{2\cdot 5\cdot 8\ldots(3n-1)(3n+2)}$$

$$= \frac{3n+7}{3n+2}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{3n+7}{3n+2}-1\right)$$

$$= \lim_{n\to\infty}\frac{5n}{3n+2}$$

$$= \lim_{n\to\infty}\frac{5}{3+\frac{2}{n}} = \frac{5}{3} > 1$$

Hence, by Raabe's test, the series is convergent.

Example 2

Test the convergence of the series $\sum\frac{4\cdot 7\ldots\ldots(3n+1)x^n}{n!}$.

Solution

$$u_n = \frac{4 \cdot 7 \ldots (3n+1)x^n}{n!}$$

$$u_{n+1} = \frac{4 \cdot 7 \ldots (3n+1)(3n+4)x^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{4 \cdot 7 \ldots (3n+1)x^n}{n!} \cdot \frac{(n+1)!}{4 \cdot 7 \ldots (3n+1)(3n+4)x^{n+1}}$$

$$= \frac{n+1}{(3n+4)x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{\left(3+\frac{4}{n}\right)x}$$

$$= \frac{1}{3x}$$

By ratio test, the series is

(i) Convergent if $\frac{1}{3x} > 1$ or $x < \frac{1}{3}$

(ii) Divergent if $\frac{1}{3x} < 1$ or $x > \frac{1}{3}$

(iii) Test fails if $x = \frac{1}{3}$

Then
$$\frac{u_n}{u_{n+1}} = \frac{n+1}{(3n+4)\ \frac{1}{3}}$$

$$= \frac{3n+3}{3n+4}$$

Applying Raabe's test,

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{3n+3}{3n+4} - 1\right)$$

$$= \lim_{n\to\infty}\left(\frac{-n}{3n+4}\right)$$

$$= \lim_{n\to\infty}\left(\frac{-1}{3+\frac{4}{n}}\right)$$

$$= -\frac{1}{3} < 1$$

By Raabe's test, the series is divergent if $x = \frac{1}{3}$.

Hence, the series is convergent if $x < \frac{1}{3}$ and divergent if $x \geq \frac{1}{3}$.

Example 3

Test the convergence of the series $\sum \frac{1\cdot 3\cdot 5\ldots\ldots(2n-1)x^{2n+1}}{2\cdot 4\cdot 6\ldots\ldots 2n(2n+1)}$.

Solution

$$u_n = \frac{1\cdot 3\cdot 5\ldots\ldots(2n-1)x^{2n+1}}{2\cdot 4\cdot 6\ldots\ldots(2n+1)}$$

$$u_{n+1} = \frac{1\cdot 3\cdot 5\ldots(2n-1)(2n+1)\,x^{2n+3}}{2\cdot 4\cdot 6\ldots 2n\,(2n+2)(2n+3)}$$

$$\frac{u_n}{u_{n+1}} = \frac{1\cdot 3\cdot 5\ldots\ldots(2n-1)x^{2n+1}}{2\cdot 4\cdot 6\ldots\ldots 2n(2n+1)} \cdot \frac{2\cdot 4\cdot 6\ldots\ldots 2n(2n+2)(2n+3)}{1\cdot 3\cdot 5\ldots\ldots(2n-1)(2n+1)x^{2n+3}}$$

$$= \frac{(2n+2)(2n+3)}{(2n+1)^2\,x^2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\left(2+\frac{2}{n}\right)\cdot\left(2+\frac{3}{n}\right)}{\left(2+\frac{1}{n}\right)^2 x^2}$$

$$= \frac{1}{x^2}$$

By ratio test, the series is

(i) Convergent if $\frac{1}{x^2} > 1$ or $x^2 < 1$

(ii) Divergent if $\frac{1}{x^2} < 1$ or $x^2 > 1$

(iii) Test fails if $x^2 = 1$

Then $$\frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2}$$

Applying Raabe's test,

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{(2n+2)(2n+3)}{(2n+1)^2}-1\right]$$

$$= \lim_{n\to\infty} \frac{n(6n+5)}{(2n+1)^2}$$

$$= \lim_{n\to\infty} \frac{\left(6+\frac{5}{n}\right)}{\left(2+\frac{1}{n}\right)^2}$$

$$= \frac{3}{2} > 1$$

By Raabe's test, the series is convergent if $x^2 = 1$.

Hence, the series is convergent if $x^2 \le 1$ and is divergent if $x^2 > 1$.

Example 4

Test the convergence of the series

$$\sum \frac{a(a+1)(a+2)\ldots\ldots(a+n-1)\cdot b(b+1)(b+2)\ldots\ldots(b+n-1)x^n}{1\cdot 2\cdot 3\ldots\ldots n\cdot c(c+1)(c+2)\ldots\ldots(c+n-1)}.$$

Solution

$$u_n = \frac{a(a+1)(a+2)\ldots\ldots(a+n-1)\cdot b(b+1)(b+2)\ldots\ldots(b+n-1)x^n}{1\cdot 2\cdot 3\ldots\ldots n\cdot c(c+1)(c+2)\ldots\ldots(c+n-1)}$$

$$\frac{u_n}{u_{n+1}} = \frac{a(a+1)\ldots\ldots(a+n-1)\cdot b(b+1)\ldots\ldots(b+n-1)x^n}{1\cdot 2\cdot 3\ldots\ldots n\cdot c(c+1)\ldots\ldots(c+n-1)} \cdot \frac{1\cdot 2\ldots\ldots n(n+1)\cdot c(c+1)\ldots(c+n-1)(c+n)}{a(a+1)\ldots\ldots(a+n-1)(a+n)\cdot b(b+1)\ldots\ldots(b+n-1)(b+n)x^{n+1}}$$

$$= \frac{(n+1)(c+n)}{(a+n)(b+n)x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\left(\frac{c}{n}+1\right)}{\left(\frac{a}{n}+1\right)\left(\frac{b}{n}+1\right)x}$$

$$= \frac{1}{x}$$

By ratio test, the series is

(i) Convergent if $\frac{1}{x} > 1$ or $x < 1$

(ii) Divergent if $\frac{1}{x} < 1$ or $x > 1$

(iii) Test fails if $x = 1$

Then $$\frac{u_n}{u_{n+1}} = \frac{(n+1)(c+n)}{(a+n)(b+n)}$$

Applying Raabe's test,

$$\begin{aligned}\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) &= \lim_{n\to\infty} n\left[\frac{(n+1)(c+n)}{(a+n)(b+n)} - 1\right] \\ &= \lim_{n\to\infty} n\left[\frac{(c-ab)+n(1+c-a-b)}{(a+n)(b+n)}\right] \\ &= \lim_{n\to\infty}\left[\frac{\frac{(c-ab)}{n}+(1+c-a-b)}{\left(\frac{a}{n}+1\right)\left(\frac{b}{n}+1\right)}\right] \\ &= 1+c-a-b\end{aligned}$$

By Raabe's test, the series is (i) convergent if $1+c-a-b>1$ or $c>a+b$, and (ii) divergent if $1+c-a-b<1$ or $c<a+b$.

Hence, the series is convergent if $x < 1$ and divergent if $x > 1$.

For $x = 1$, the series is convergent if $c > a + b$ and divergent if $c < a + b$.

EXERCISE 5.4

Test the convergence of the following series:

1. $1+\frac{1}{2}+\frac{1}{2\cdot 4}+\frac{1}{2\cdot 4\cdot 6}+\ldots.$

[Ans.: Divergent]

2. $\sum \frac{1^2\cdot 5^2\cdot 9^2\ldots\ldots(4n-3)^2}{4^2\cdot 8^2\cdot 12^2\ldots\ldots(4n)^2}$.

[Ans.: Convergent]

3. (i) $1+\frac{2^2}{3\cdot 4}+\frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6}+\frac{2^2\cdot 4^2\cdot 6^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}+\ldots.$

[Ans.: Convergent]

(ii) $1+\frac{(1!)^2}{2!}x+\frac{(2!)^2}{4!}x^2+\frac{(3!)^2}{6!}x^3+\ldots$

[Ans.: Convergent for $x < 4$ and divergent for $x \ge 4$]

(iii) $1+\frac{3}{7}x+\frac{3\cdot 6}{7\cdot 10}x^2+\frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3+\ldots$

[Ans.: Convergent for $x \le 1$ and divergent for $x > 1$]

4. $1+\frac{2^2}{3^2}+\frac{2^2\cdot 4^2}{3^2\cdot 5^2}+\frac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2}+\ldots$

[Ans. : Divergent]

5. $\frac{a(a+1)}{2!}+\frac{(a+1)(a+2)}{3!}+\frac{(a+2)(a+3)}{4!}+\ldots$

[Ans.: Convergent for $a \le 0$]

6. $\sum\frac{(n!)^2}{(2n)!}x^{2n}$.

[Ans.: Convergent for $x < 4$ and divergent for $x^2 \ge 4$]

5.13 CAUCHY'S ROOT TEST

If Σu_n is a positive term series and if $\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = l$ then

(i) Σu_n is convergent if $l < 1$.

(ii) Σu_n is divergent if $l > 1$.

Proof

Case I If $\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = l < 1$.

Consider a number $l < r < 1$ such that $(u_n)^{\frac{1}{n}} < r$ for all $n > m$

$$u_n < r^n \text{ for all } n > m \quad \ldots (1)$$

The geometric series, $\Sigma r^n = r + r^2 + r^3 + \ldots\infty$

$$S_n = r + r^2 + r^3 + \ldots + r^n$$
$$= \frac{r(1-r^n)}{1-r}$$
$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{r(1-r^n)}{1-r} \qquad \left[\begin{array}{l}\because r<1 \\ \therefore \lim\limits_{x\to\infty} r^n = 0\end{array}\right]$$
$$= \frac{r}{1-r}, \text{ which is finite}$$

Hence, the series Σr^n is convergent.

From Eq. (1), $u_n < r^n$ for all $n > m$

$$\Sigma u_n < \Sigma r^n$$

Since Σr^n is convergent, Σu_n is also convergent.

Case II: If $\lim\limits_{n\to\infty} (u_n)^{\frac{1}{n}} = l > 1$,

$$(u_n)^{\frac{1}{n}} > 1 \text{ for all } n > m \qquad \ldots (2)$$

Neglecting the first m terms,

$$\Sigma (u_n)^{\frac{1}{n}} = (u_{m+1})^{\frac{1}{m+1}} + (u_{m+2})^{\frac{1}{m+2}} + (u_{m+3})^{\frac{1}{m+3}} + \ldots \infty$$
$$> 1 + 1 + 1 \ldots \infty \qquad \text{[Using Eq. (2)]}$$

$$S_n = (u_{m+1})^{\frac{1}{m+1}} + (u_{m+2})^{\frac{1}{m+2}} + (u_{m+3})^{\frac{1}{m+3}} + \ldots + (u_{m+n})^{\frac{1}{m+n}}$$
$$> 1 + 1 + 1 \ldots n \text{ terms} = n$$

$$\lim_{n\to\infty} S_n > \lim_{n\to\infty} n \to \infty$$

The series $\sum\limits_{n=m+1}^{\infty} u_n$ is divergent. The nature of a series remains unchanged if we neglect a finite number of terms in the beginning. Hence, the series $\sum\limits_{n=1}^{\infty} u_n$ is divergent.

Note: If $\lim\limits_{n\to\infty} (u_n)^{\frac{1}{n}} = 1$, the root test fails, i.e., no conclusion can be drawn about the convergence or divergence of the series.

Example 1

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$.

Solution

Let

$$u_n = \frac{1}{(\log n)^n}$$

$$(u_n)^{\frac{1}{n}} = \frac{1}{\log n}$$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\log n}$$

$$= 0 < 1 \qquad [\because \log\infty \to \infty]$$

Hence, by Cauchy's root test, the series is convergent.

Example 2

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{a^{n+1}}{n^n}$.

Solution

Let

$$u_n = \frac{a^{n+1}}{n^n}$$

$$(u_n)^{\frac{1}{n}} = \frac{(a)^{1+\frac{1}{n}}}{n}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{a\, a^{\frac{1}{n}}}{n}$$

$$= 0 < 1$$

Hence, by Cauchy's root test, the series is convergent.

Example 3

Test the convergence of the series $\sum \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$.

Solution

Let
$$u_n = \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$$

$$(u_n)^{\frac{1}{n}} = \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$= \frac{1}{e} < 1$$

Hence, by Cauchy's root test, the series is convergent.

Example 4

Test the convergence of the series $\sum \frac{(n-\log n)^n}{2^n \cdot n^n}$.

Solution

Let
$$u_n = \frac{(n-\log n)^n}{2^n \cdot n^n}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{(n-\log n)}{2n} \qquad \left[\frac{\infty}{\infty}\text{ form}\right]$$

$$= \lim_{n\to\infty}\left(\frac{1}{2}-\frac{\log n}{2n}\right)$$

$$= \frac{1}{2} - \lim_{n\to\infty}\frac{\frac{1}{n}}{2} \qquad \text{[Using L'Hospital's rule]}$$

$$= \frac{1}{2} < 1$$

Hence, by Cauchy's root test, the series is convergent.

Example 5

Test the convergence of the series

$$\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+\cdots+\left(\frac{n}{2n+1}\right)^n+\cdots.$$

Solution

Let
$$u_n = \left(\frac{n}{2n+1}\right)^n$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{2n+1}$$
$$= \lim_{n\to\infty}\frac{1}{2+\frac{1}{n}}$$
$$= \frac{1}{2} < 1$$

Hence, by Cauchy's root test, the series is convergent.

Example 6

Test the convergence of the series $\frac{1^3}{3}+\frac{2^3}{3^2}+\frac{3^3}{3^3}+\frac{4^3}{3^4}+\cdots$.

Solution

Let
$$u_n = \frac{n^3}{3^n}$$

$$(u_n)^{\frac{1}{n}} = \left(\frac{n^3}{3^n}\right)^{\frac{1}{n}}$$
$$= \frac{n^{\frac{3}{n}}}{3}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n^{\frac{3}{n}}}{3}$$
$$= \frac{1}{3}\lim_{n\to\infty}\left(n^{\frac{1}{n}}\right)^3 \qquad \ldots(1)$$

Let
$$l = \lim_{n\to\infty}(n)^{\frac{1}{n}}$$

$$\log l = \lim_{n\to\infty}\frac{1}{n}\log n$$
$$= \lim_{n\to\infty}\frac{\log n}{n} \qquad \left[\frac{\infty}{\infty}\text{ form}\right]$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} \qquad \text{[Applying L'Hospital's rule]}$$

$$\log l = 0$$

$$l = e^0$$

$$= 1$$

$$\therefore \lim_{n\to\infty} (n)^{\frac{1}{n}} = 1$$

Substituting in Eq. (1),

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \frac{1}{3} < 1$$

Hence, by Cauchy's root test, the series is convergent.

Example 7

Test the convergence of the series

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots.$$

Solution

Let
$$u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-n}$$

$$(u_n)^{\frac{1}{n}} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-1}$$

$$= \left[\left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$= (e-1)^{-1}$$

$$= \frac{1}{e-1} < 1$$

Hence, by Cauchy's root test, the series is convergent.

Example 8

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n x^n}{(n+1)^n}, x > 0.$

Solution

Let
$$u_n = \frac{n^n x^n}{(n+1)^n}$$
$$(u_n)^{\frac{1}{n}} = \left[\frac{n^n x^n}{(n+1)^n}\right]^{\frac{1}{n}}$$
$$= \frac{nx}{n+1}$$
$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{x}{1+\frac{1}{n}}$$
$$= x$$

Hence, by Cauchy's root test, the series is

(i) convergent if $x < 1$

(ii) divergent if $x > 1$

The test fails if $x = 1$.
Then
$$u_n = \frac{n^n}{(n+1)^n}$$
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$
$$= \frac{1}{e} \neq 0$$

The series is divergent for $x = 1$.
Hence, the series is convergent if $x < 1$ and is divergent if $x \geq 1$.

Example 9

Test the convergence of the series $\sum \frac{(n+1)^n x^n}{n^{n+1}}$. **[Summer 2014]**

Solution

Let
$$u_n = \frac{(n+1)^n x^n}{n^{n+1}}$$
$$(u_n)^{\frac{1}{n}} = \frac{(n+1)x}{n^{\frac{n+1}{n}}}$$

$$= \frac{(n+1)x}{n.n^{\frac{1}{n}}}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{x}{n^{\frac{1}{n}}}$$

$$= x \qquad \left[\because \lim_{x\to\infty} n^{\frac{1}{n}} = 1 \text{ as solved in Ex 6}\right]$$

Hence, by Cauchy's root test, the series is convergent, if $x < 1$ and divergent if $x > 1$.

The test fails for $x = 1$.

Then

$$u_n = \frac{(n+1)^n}{n^{n+1}}$$

$$= \frac{(n+1)^n}{n \cdot n^n}$$

$$= \frac{1}{n}\left(\frac{n+1}{n}\right)^n$$

$$= \frac{1}{n}\left(1+\frac{1}{n}\right)^n$$

Let

$$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= e \qquad \text{[finite and non-zero]}$$

$\Sigma v_n = \sum\frac{1}{n}$ is divergent as $p = 1$.

By comparison test, Σu_n is also divergent for $x = 1$.

Hence, the series is convergent for $x < 1$ and divergent for $x \geq 1$.

Example 10

Test the convergence of the series $\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2 x^2+\left(\frac{4}{5}\right)^3 x^3+\ldots\infty.$

Solution

Let

$$u_n = \left(\frac{n}{n+1}\right)^{n-1} x^{n-1}$$

$$(u_n)^{\frac{1}{n}} = \left(\frac{n}{n+1}\right)^{\frac{n-1}{n}} x^{\frac{n-1}{n}}$$

$$= \left(\frac{n}{n+1}\right)^{1-\frac{1}{n}} (x)^{1-\frac{1}{n}}$$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^{1-\frac{1}{n}} (x)^{1-\frac{1}{n}}$$

$$= x$$

Hence, by Cauchy's root test, the series is convergent if $x < 1$ and divergent if $x > 1$. Root test fails for $x = 1$.

EXERCISE 5.5

Test the convergence of the following series:

1. $1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\ldots\infty$ **[Ans.:** Convergent]

2. $\sum\left(\frac{n+1}{3n}\right)^n$ **[Ans.:** Convergent]

3. $\sum\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$

[Ans.: Convergent]

4. $1+\frac{x}{2}+\frac{x^2}{3^2}+\frac{x^3}{4^3}+\ldots(x>0)$

[Ans.: Convergent]

5. $\sum\left(1+\frac{1}{n}\right)^{n^2}$

[Ans.: Divergent]

6. $\sum\frac{(1+nx)^n}{n^n}$

[Ans.: Convergent if $x < 1$ and divergent if $x > 1$]

5.14 CAUCHY'S INTEGRAL TEST

If $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} f(n)$ is a positive term series where $f(n)$ decreases as n increases and let $\int_1^{\infty} f(x)\mathrm{d}x = I$ then

(i) Σu_n is convergent if I is finite

(ii) Σu_n is divergent if I is infinite

Proof Consider the area under the curve $y = f(x)$ from $x = 1$ to $x = n + 1$ represented as $\int_1^{n+1} f(x)\mathrm{d}x$. Plot the terms $f(1), f(2), f(3), \ldots, f(n), f(n + 1)$.

The area $\int_1^{n+1} f(x)\mathrm{d}x$ lies between the sum of the areas of smaller rectangles and sum of the areas of larger rectangles

$$f(2)+f(3)+\cdots+f(n+1) \le \int_1^{n+1} f(x)\mathrm{d}x \le f(1)+f(2)+f(3)+\cdots+f(n)$$

$$S_{n+1} - f(1) \le \int_1^{n+1} f(x)\mathrm{d}x \le S_n$$

As $n \to \infty$ first inequality reduces to

$$\lim_{n\to\infty} S_{n+1} \le \int_1^{\infty} f(x)\mathrm{d}x + f(1)$$

This shows that if $\int_1^{\infty} f(x)\mathrm{d}x$ is finite, $\Sigma f(n) = \Sigma u_n$ is convergent.

As $n \to \infty$ second inequality reduces to

$$\int_1^{\infty} f(x)\mathrm{d}x \le \lim_{n\to\infty} S_n$$

or

$$\lim_{n\to\infty} S_n \ge \int_1^{\infty} f(x)\mathrm{d}x$$

This shows that if $\int_1^{\infty} f(x)\mathrm{d}x$ is infinite, $\Sigma f(n) = \Sigma u_n$ is divergent.

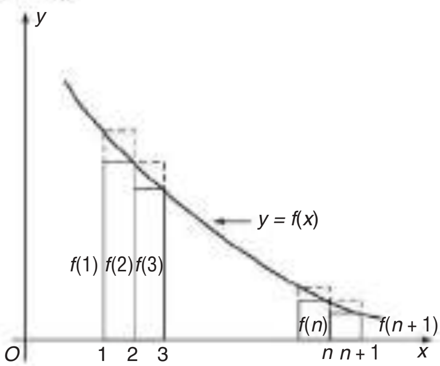


Fig. 5.2

Example 1

Test the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$.

Solution

Let
$$u_n = \frac{1}{n\log n} = f(n)$$
$$f(x) = \frac{1}{x\log x}$$
$$\int_2^\infty f(x)\mathrm{d}x = \int_2^\infty \frac{1}{x\log x}\mathrm{d}x$$
$$= \lim_{m\to\infty}\int_2^m \frac{1}{x\log x}\mathrm{d}x$$
$$= \lim_{m\to\infty}\left|\log\log x\right|_2^m$$
$$= \lim_{m\to\infty}(\log\log m - \log\log 2) \to \infty \qquad \left[\because \int \frac{f'(x)}{f(x)}\mathrm{d}x = \log f(x)\right]$$

Hence, by Cauchy's integral test, the series is divergent.

Example 2

Test the convergence of the series $\sum_{n=1}^{\infty} n^2 e^{-n^3}$.

Solution

Let
$$u_n = n^2 e^{-n^3} = f(n)$$
$$f(x) = x^2 e^{-x^3}$$
$$\int_1^\infty f(x)\mathrm{d}x = \int_1^\infty x^2 e^{-x^3}\mathrm{d}x$$
$$= \lim_{m\to\infty}\left[-\frac{1}{3}\int_1^m e^{-x^3}(-3x^2)\mathrm{d}x\right]$$
$$= \lim_{m\to\infty}\left[-\frac{1}{3}\left|e^{-x^3}\right|_1^m\right] \qquad \left[\because \int e^{f(x)}f'(x)\mathrm{d}x = e^{f(x)}\right]$$
$$= \lim_{m\to\infty}\left[-\frac{1}{3}\left(e^{-m^3} - e^{-1}\right)\right]$$
$$= -\frac{1}{3}(e^{-\infty} - e^{-1})$$
$$= -\frac{1}{3}\left(0 - \frac{1}{e}\right)$$
$$= \frac{1}{3e} \quad \text{[finite]}$$

Hence, by Cauchy's integral test, the series is convergent.

Example 3

Test the convergence of the series $\sum_{n=3}^{\infty} \frac{1}{n \log n \sqrt{\log^2 n - 1}}$.

[Winter 2015]

Solution

Let
$$u_n = \frac{1}{n \log n \sqrt{\log^2 n - 1}}$$
$$f(x) = \frac{1}{x \log x \sqrt{\log^2 x - 1}}$$
$$\int_3^{\infty} f(x)\,\mathrm{d}x = \int_3^{\infty} \frac{1}{x \log x \sqrt{\log^2 x - 1}}\,\mathrm{d}x$$

Putting $\log x = t$, $\frac{1}{x}\mathrm{d}x = \mathrm{d}t$

When $x = 3$, $t = \log 3$

When $x \to \infty$, $t = \log \infty = \infty$

$$\begin{aligned}
\int_3^{\infty} f(x)\mathrm{d}x &= \int_{\log 3}^{\infty} \frac{\mathrm{d}t}{t\sqrt{t^2 - 1}} \\
&= \lim_{m\to\infty} \int_{\log 3}^{m} \frac{\mathrm{d}t}{t\sqrt{t^2 - 1}} \\
&= \lim_{m\to\infty} \left| \sec^{-1} t \right|_{\log 3}^{m} \\
&= \lim_{m\to\infty} \left[\sec^{-1} m - \sec^{-1}(\log 3) \right] \\
&= \sec^{-1} \infty - \sec^{-1}(\log 3) \\
&= \frac{\pi}{2} - \sec^{-1}(\log 3) \\
&= \operatorname{cosec}^{-1}(\log 3) \qquad [\text{finite}]
\end{aligned}$$

Hence, by Cauchy's integral test, the series is convergent.

Example 4

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{2\tan^{-1} n}{1+n^2}$. **[Winter 2014]**

Solution

Let
$$u_n = \frac{2\tan^{-1} n}{1+n^2} = f(n)$$
$$f(x) = \frac{2\tan^{-1} x}{1+x^2}$$
$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{2\tan^{-1} x}{1+x^2}\,dx$$
$$= \lim_{m\to\infty} \int_1^{m} \frac{2\tan^{-1} x}{1+x^2}\,dx$$

Putting $\tan^{-1}x = t$, $\quad \frac{dx}{1+x^2} = dt$

When $x = 1$, $\quad t = \tan^{-1}(1) = \frac{\pi}{4}$

When $x = m$, $\quad t = \tan^{-1} m$

$$\int_1^{\infty} f(x)dx = \lim_{m\to\infty} \int_{\frac{\pi}{4}}^{\tan^{-1} m} 2t\,dt$$
$$= \lim_{m\to\infty} \left|\frac{2t^2}{2}\right|_{\frac{\pi}{4}}^{\tan^{-1} m}$$
$$= \lim_{m\to\infty} \left[\left(\tan^{-1} m\right)^2 - \frac{\pi^2}{16}\right]$$
$$= \left(\tan^{-1} \infty\right)^2 - \frac{\pi^2}{16}$$
$$= \frac{\pi^2}{4} - \frac{\pi^2}{16}$$
$$= \frac{3\pi^2}{16} \qquad \text{[finite]}$$

Hence, by Cauchy's integral test, the series is convergent.

Example 5

Show that the harmonic series of order p,

$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\infty$ *is convergent if* $p > 1$ *and is divergent if* $p \le 1$.

[Summer 2015]

Solution

Let

$$u_n = \frac{1}{n^p} = f(n)$$

$$f(x) = \frac{1}{x^p}$$

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^p}\,dx$$

$$= \lim_{m\to\infty} \left| \frac{x^{-p+1}}{-p+1} \right|_1^m$$

$$= \lim_{m\to\infty} \left(\frac{m^{1-p}}{1-p} - \frac{1}{1-p} \right)$$

$$= -\frac{1}{1-p}, \qquad p > 1$$

$$= \infty, \qquad p < 1$$

If $p = 1$,

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x}\,dx$$

$$= \lim_{m\to\infty} \int_1^m \frac{1}{x}\,dx$$

$$= \lim_{m\to\infty} \left| \log x \right|_1^m$$

$$= \lim_{m\to\infty} (\log m - \log 1)$$

$$= \log \infty \to \infty$$

The integral $\int_1^\infty f(x)\,dx$ is finite if $p > 1$ and is infinite if $p \le 1$.

Hence, by Cauchy's integral test, the series is convergent if $p > 1$ and is divergent if $p \le 1$.

EXERCISE 5.5

Test the convergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ **[Ans.:** Divergent]
2. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ **[Ans.:** Convergent]
3. $\sum_{n=1}^{\infty} ne^{-n^2}$ **[Ans.:** Convergent]
4. $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^2}$ **[Ans.:** Convergent]

5.15 ALTERNATING SERIES

An infinite series with alternate positive and negative terms is called an alternating series.

Leibnitz's Test for Alternating Series

An alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}v_n = \sum_{n=1}^{\infty}u_n$ is convergent if

(i) each term is numerically less than its preceding term, i.e. $|u_{n+1}| < |u_n|$ or $|u_n| > |u_{n+1}|$

(ii) $\lim_{n\to\infty}|u_n| = 0$

Example 1

Test the convergence of the series $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\dots$.

Solution

Let $u_n = (-1)^{n-1}\frac{1}{\sqrt{n}}$

$$|u_n| = \frac{1}{\sqrt{n}}$$

The given series is an alternating series.

(i) $|u_n|-|u_{n+1}| = \frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}$

$$= \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}\sqrt{n+1}} > 0 \qquad \text{for all } n \in N$$

$$\therefore \ |u_n| > |u_{n+1}|$$

(ii) $\lim_{n\to\infty}|u_n| = \lim_{n\to\infty}\left|\frac{1}{\sqrt{n}}\right|$

$$= 0$$

Hence, by Leibnitz's test, the series is convergent.

Example 2

Test the convergence of the series $1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+\cdots$.

Solution

Let $u_n = \frac{(-1)^{n-1}}{n\sqrt{n}}$

$$|u_n| = \frac{1}{n\sqrt{n}}$$

The given series is an alternating series.

(i) $$|u_n| - |u_{n+1}| = \frac{1}{n\sqrt{n}} - \frac{1}{(n+1)\sqrt{n+1}}$$

$$= \frac{(n+1)\sqrt{n+1} - n\sqrt{n}}{(n\sqrt{n})\left[(n+1)\sqrt{(n+1)}\right]} > 0 \quad \text{for all } n \in N$$

$$\therefore |u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{n\sqrt{n}}$$

$$= 0$$

Hence, by Leibnitz's test, the series is convergent.

Example 3

Test the convergence of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$.

Solution

Let $$u_n = (-1)^{n-1} \cdot \frac{1}{n^2}$$

$$|u_n| = \frac{1}{n^2}$$

The given series is an alternating series.

(i) $$|u_n| - |u_{n+1}| = \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

$$= \frac{2n+1}{n^2(n+1)^2} > 0 \quad \text{for all } n \in N$$

$$\therefore \quad |u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{n^2}$$

$$= 0$$

Hence, by Leibnitz's test, the series is convergent.

Example 4

Test the convergence of the series $\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots$.

Solution

Let $$u_n = (-1)^n \frac{1}{n^p}$$

$$|u_n| = \frac{1}{n^p}$$

The given series is an alternating series.

Case I: If $p > 0$,

(i) $$|u_n| - |u_{n+1}| = \frac{1}{n^p} - \frac{1}{(n+1)^p}$$

$$= \frac{(n+1)^p - n^p}{n^p (n+1)^p} > 0 \qquad [\because p > 0]$$

$$\therefore |u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{n^p}$$

$$= 0 \qquad [\because p > 0]$$

Hence, by Leibnitz's test, the series is convergent if $p > 0$.

Case II: If $p < 0$

In this case the conditions (i) and (ii) of the Leibnitz's test are not satisfied. Hence, the given series is not convergent if $p < 0$.

Example 5

Test the convergence of the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$.

Solution

Let $$u_n = (-1)^{n-1} \frac{1}{2^{n-1}}$$

$$|u_n| = \frac{1}{2^{n-1}}$$

The given series is an alternating series.

(i) $$|u_n| - |u_{n+1}| = \frac{1}{2^{n-1}} - \frac{1}{2^n}$$

$$= \frac{1}{2^{n-1}}\left(1 - \frac{1}{2}\right)$$

$$= \frac{1}{2^{n-1}} \cdot \frac{1}{2}$$

$$= \frac{1}{2^n} > 0 \quad \text{for all } n \in N$$

$\therefore \quad |u_n| > |u_{n+1}|$

(ii) $\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{2^{n-1}}$

$= 0$

Hence, by Leibnitz's test, the series is convergent.

Example 6

Test the convergence of the series $\frac{1}{2} - \frac{2}{5} + \frac{3}{10} - \frac{4}{17} + \cdots$.

Solution

Let $$u_n = (-1)^{n-1} \frac{n}{n^2+1}$$

$$|u_n| = \frac{n}{n^2+1}$$

The given series is an alternating series.

(i) $$|u_n| - |u_{n+1}| = \frac{n}{n^2+1} - \frac{n+1}{(n+1)^2+1}$$

$$= \frac{n(n^2+2n+2) - (n+1)(n^2+1)}{(n^2+1)(n^2+2n+2)}$$

$$= \frac{n^2+n-1}{(n^2+1)(n^2+2n+2)} > 0 \quad \text{for all } n \in N$$

$\therefore \quad |u_n| > |u_{n+1}|$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{n}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{1}{n + \frac{1}{n}}$$

$$= 0$$

Hence, by Leibnitz's test, the series is convergent.

Example 7

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\log(n+1)}$. **[Summer 2014]**

Solution

Let $$u_n = \frac{(-1)^{n+1}}{\log(n+1)}$$

$$|u_n| = \frac{1}{\log(n+1)}$$

The given series is an alternating series.

(i) $$|u_n| - |u_{n+1}| = \frac{1}{\log(n+1)} - \frac{1}{\log(n+2)}$$

$$= \frac{\log(n+2) - \log(n+1)}{\log(n+1) \cdot \log(n+2)} > 0$$

$$\therefore \quad |u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{\log(n+1)}$$

$$= 0$$

Hence, by Leibnitz's test, the series is convergent.

Example 8

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2(n+1)}$.

Solution

Let $$u_n = \frac{(-1)^{n-1}}{n^2(n+1)}$$

$$|u_n| = \frac{1}{n^2(n+1)}$$

The given series is an alternating series.

(i) $$|u_n| - |u_{n+1}| = \frac{1}{n^2(n+1)} - \frac{1}{(n+1)^2(n+2)}$$

$$= \frac{(n+1)(n+2) - n^2}{n^2(n+1)^2(n+2)}$$

$$= \frac{3n+2}{n^2(n+1)^2(n+2)} > 0 \quad \text{for all } n \in N$$

$$\therefore \quad |u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{n^2(n+1)}$$

$$= 0$$

Hence, by Leibnitz's test, the series is convergent.

Example 9

Test the convergence of the series $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots(if\ x<1)$.

Solution

Let $$u_n=(-1)^{n-1}\frac{x^n}{n}$$

$$|u_n|=\frac{x^n}{n}$$

The given series is an alternating series.

(i) $$|u_n|-|u_{n+1}|=\frac{x^n}{n}-\frac{x^{n+1}}{n+1}$$

$$=\frac{x^n[(n+1)-nx]}{n(n+1)}$$

$$=\frac{x^n[1+(1-x)n]}{n(n+1)}>0 \qquad [\because n\ge 1 \text{ and } 0<x<1]$$

$$\therefore \quad |u_n|>|u_{n+1}|$$

(ii) $$\lim_{n\to\infty}|u_n|=\lim_{n\to\infty}\frac{x^n}{n} \qquad \left[\because \lim_{n\to\infty}x^n=0 \text{ if } x<1\right]$$

$$=0$$

Hence, by Leibnitz's test, the series is convergent.

Example 10

Test the convergence of the series

$$\log\left(\frac{1}{2}\right)-\log\left(\frac{2}{3}\right)+\log\left(\frac{3}{4}\right)-\log\left(\frac{4}{5}\right)+\cdots$$

Solution

$$\log\left(\frac{1}{2}\right)-\log\left(\frac{2}{3}\right)+\log\left(\frac{3}{4}\right)-\log\left(\frac{4}{5}\right)+\cdots$$

$$=-\log\left(\frac{2}{1}\right)+\log\left(\frac{3}{2}\right)-\log\left(\frac{4}{3}\right)+\log\left(\frac{5}{4}\right)-\cdots$$

Let $$u_n=(-1)^n\log\left(\frac{n+1}{n}\right)$$

$$|u_n|=\log\left(\frac{n+1}{n}\right)$$

The given series is an alternating series.

(i) $|u_n| - |u_{n+1}| = \log\frac{n+1}{n} - \log\frac{n+2}{n+1} > 0 \quad \left[\because \frac{n+1}{n} > \frac{n+2}{n+1} \text{ for all } n \in N\right]$

$$\therefore \quad |u_n| > |u_{n+1}|$$

(ii)
$$\lim_{n\to\infty}|u_n| = \lim_{n\to\infty}\log\left(\frac{n+1}{n}\right)$$
$$= \lim_{n\to\infty}\log\left(1+\frac{1}{n}\right)$$
$$= \log 1$$
$$= 0$$

Hence, by Leibnitz's test, the series is convergent.

Example 11

Test the convergence of the series $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \cdots$.

Solution

Let
$$u_n = (-1)^{n-1}\cdot\frac{n}{n+1}$$
$$|u_n| = \frac{n}{n+1}$$

The given series is an alternating series.

(i)
$$|u_n| - |u_{n+1}| = \frac{n}{n+1} - \frac{n+1}{n+2}$$
$$= \frac{n^2 + 2n - n^2 - 2n - 1}{(n+1)(n+2)}$$
$$= -\frac{1}{(n+1)(n+2)} < 0$$

Since each term of the series is not numerically less than the preceding term, Leibnitz's test cannot be applied.

The series can be written as

$$\sum_{n=1}^{\infty} u_n = \left(1-\frac{1}{2}\right) - \left(1-\frac{1}{3}\right) + \left(1-\frac{1}{4}\right) - \left(1-\frac{1}{5}\right) + \cdots$$
$$= (1-1+1-1+\cdots) + \left(-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\right)$$
$$= \sum_{n=1}^{\infty}(-1)^{n-1} + (\log 2 - 1)$$

As $n \to \infty$, the sum of this series tends to $(-1 + \log 2 - 1)$ or $(1 + \log 2 - 1)$ according as n is even or odd.
Hence, the given series is an oscillatory series.

EXERCISE 5.6

Test the convergence of the following series:

1. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\dots\dots\dots$

[Ans.: Convergent]

2. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{2n-1}$

[Ans.: Oscillatory]

3. $\frac{1}{2^3}-\frac{1}{3^3}(1+2)+\frac{1}{4^3}(1+2+3)-\frac{1}{5^3}(1+2+3+4)+\dots$

[Ans.: Convergent]

4. $1-2x+3x^2-4x^3+\dots(x<1)$

[Ans.: Convergent]

5. $\frac{x}{1+x}-\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}-\frac{x^4}{1+x^4}+\dots(0<x<1)$

[Ans.: Convergent]

5.16 ABSOLUTE AND CONDITIONAL CONVERGENT OF A SERIES

Absolute Convergence of a Series The series $\sum_{n=1}^{\infty}u_n$ with both positive and negative terms (not necessarily alternative) is called absolutely convergent if the corresponding series $\sum_{n=1}^{\infty}|u_n|$ with all positive terms is convergent.

Conditional Convergence of a Series If the series $\sum_{n=1}^{\infty}u_n$ is convergent and $\sum_{n=1}^{\infty}|u_n|$ is divergent, then the series $\sum_{n=1}^{\infty}u_n$ is called conditionally convergent.

Note 1: Every absolutely convergent series is a convergent series but converse is not true.

Note 2: Any convergent series of positive terms is also absolutely convergent.

Example 1

Test the convergence of the series $\sum_{n=1}^{\infty}\frac{(-1)^n 2^{3n}}{3^{2n}}$.

Solution

Let

$$u_n = \frac{(-1)^n 2^{3n}}{3^{2n}}$$

$$|u_n| = \frac{2^{3n}}{3^{2n}}$$

$$|u_{n+1}| = \frac{2^{3(n+1)}}{3^{2(n+1)}}$$

$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{2^{3n}}{3^{2n}} \cdot \frac{3^{2n+2}}{2^{3n+3}}$$

$$= \lim_{n\to\infty} \frac{9}{8}$$

$$= \frac{9}{8} > 1$$

By D'Alembert's ratio test, $\sum_{n=1}^{\infty} |u_n|$ is convergent. Thus, the series is absolutely convergent and hence convergent.

Example 2

Test the series for absolute or conditional convergence

$$1 - \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \cdots.$$

Solution

Let

$$u_n = (-1)^{n-1} \cdot \frac{n}{3^{n-1}}$$

$$\sum_{n=1}^{\infty} |u_n| = 1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \ldots$$

$$|u_n| = \frac{n}{3^{n-1}}$$

$$|u_{n+1}| = \frac{n+1}{3^n}$$

$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{n}{3^{n-1}} \cdot \frac{3^n}{n+1}$$

$$= \lim_{n\to\infty} \frac{3}{1+\frac{1}{n}}$$

$$= 3 > 1$$

By D'Alembert's ratio test, $\sum_{n=1}^{\infty} |u_n|$ is convergent and hence, the series is absolutely convergent.

Example 3

Test the series for absolute or conditional convergence $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$.

[Winter 2016]

Solution

Let $$u_n = \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$$

$$|u_n| = \frac{1}{\sqrt{n}+\sqrt{1+n}}$$

The given series is an alternating series.

(i) $$|u_n| - |u_{n+1}| = \frac{1}{\sqrt{n}+\sqrt{1+n}} - \frac{1}{\sqrt{n+1}+\sqrt{n+2}}$$

$$= \frac{\sqrt{n+1}+\sqrt{n+2}-\sqrt{n}-\sqrt{n+1}}{(\sqrt{n}+\sqrt{1+n})(\sqrt{1+n}+\sqrt{n+2})}$$

$$= \frac{\sqrt{n+2}-\sqrt{n}}{(\sqrt{n}+\sqrt{1+n})(\sqrt{1+n}+\sqrt{n+2})} > 0 \quad \text{for all } n \in N$$

$$\therefore \quad |u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{\sqrt{n}+\sqrt{1+n}} = 0$$

By Leibnitz's test, $\sum u_n$ is convergent.

$$|u_n| = \frac{1}{\sqrt{n}+\sqrt{1+n}}$$

$$= \frac{1}{\sqrt{n}\left(1+\sqrt{1+\frac{1}{n}}\right)}$$

Let $$v_n = \frac{1}{\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{|u_n|}{v_n} = \lim_{n\to\infty} \frac{1}{\left(1+\sqrt{1+\frac{1}{n}}\right)}$$

$$= 6 \qquad \text{[finite and non-zero]}$$

and $\sum v_n = \sum \frac{1}{n^{\frac{1}{2}}}$ is divergent as $p = \frac{1}{2}$.

By comparison test, $\sum |u_n|$ is also divergent.

The series $\sum u_n$ is convergent and $\sum |u_n|$ is divergent.
Hence, the series is conditionally convergent.

Example 4

Determine absolute or conditional convergence of the series

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n^2}{n^3+1}$$ **[Winter 2013; Summer 2017]**

Solution

Let $$u_n = (-1)^n \cdot \frac{n^2}{n^3+1}$$

$$|u_n| = \frac{n^2}{n^3+1}$$

$$= \frac{1}{n\left(1+\frac{1}{n^3}\right)}$$

Let $$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{|u_n|}{v_n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^3}}$$

$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

By comparison test, $\sum |u_n|$ is also divergent.

Hence, Σu_n is not absolutely convergent.
To check the conditional convergence, applying Leibnitz's test,

(i) $|u_n| - |u_{n+1}| = \frac{n^2}{n^3+1} - \frac{(n+1)^2}{(n+1)^3+1}$

$$= \frac{n^2(n^3+3n^2+3n+2)-(n^3+1)(n^2+2n+1)}{(n^3+1)[(n+1)^3+1]}$$

$$= \frac{n^4+2n^3+n^2-2n-1}{(n^3+1)[(n+1)^3+1]}$$

$$= \frac{n^4+n^2(2n+1)-1(2n+1)}{(n^3+1)[(n+1)^3+1]}$$

$$= \frac{n^4+(2n+1)(n^2-1)}{(n^3+1)[(n+1)^3+1]} > 0 \quad \text{for all } n \in N$$

$$|u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty}|u_n| = \lim_{n\to\infty}\frac{n^2}{n^3+1}$$

$$= \lim_{n\to\infty}\frac{1}{n\left(1+\frac{1}{n^3}\right)}$$

$$= 0$$

By Leibnitz's test, $\sum u_n$ is convergent.

The series Σu_n is convergent and $\sum|u_n|$ is divergent.

Hence, the series is conditionally convergent.

Example 5

Test the series for absolute or conditional convergence

$$\frac{2}{3}-\frac{3}{4}\cdot\frac{1}{2}+\frac{4}{5}\cdot\frac{1}{3}-\frac{5}{6}\cdot\frac{1}{4}+\dots$$

Solution

Let $$u_n = (-1)^{n-1}\left(\frac{n+1}{n+2}\cdot\frac{1}{n}\right)$$

$$\sum_{n=1}^{\infty}|u_n| = \frac{2}{3}+\frac{3}{4}\cdot\frac{1}{2}+\frac{4}{5}\cdot\frac{1}{3}+\frac{5}{6}\cdot\frac{1}{4}+\cdots.$$

$$|u_n| = \frac{n+1}{n+2}\cdot\frac{1}{n}$$

Let $$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty}\frac{|u_n|}{v_n} = \lim_{n\to\infty}\frac{n+1}{n+2}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}}$$

$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

By comparison test, $\Sigma |u_n|$ is also divergent.

Hence, the series is not absolutely convergent.

To check the conditional convergence, applying Leibnitz's test,

(i) $$|u_n| - |u_{n+1}| = \frac{n+1}{n(n+2)} - \frac{n+2}{(n+1)(n+3)}$$

$$= \frac{n^2+3n+3}{n(n+1)(n+2)(n+3)} > 0 \qquad \text{for all } n \in N$$

$$|u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{n+1}{n(n+2)}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{n\left(1+\frac{2}{n}\right)}$$

$$= 0$$

By Leibnitz's test, Σu_n is convergent.
The series Σu_n is convergent and $\Sigma |u_n|$ is divergent.
Hence, the series is conditionally convergent.

Example 6

Test the convergence of the series $x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots,\ x > 0.$ **[Summer 2016]**

Solution

Let $$u_n = (-1)^{n-1} \frac{x^{2n-1}}{2n-1}$$

$$|u_n| = \frac{x^{2n-1}}{2n-1}$$

$$|u_{n+1}| = \frac{x^{2n+1}}{2n+1}$$

$$\frac{|u_n|}{|u_{n+1}|} = \frac{x^{2n-1}}{2n-1} \cdot \frac{2n+1}{x^{2n+1}}$$

$$= \frac{2+\frac{1}{n}}{2-\frac{1}{n}} \cdot \frac{1}{x^2}$$

$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \left(\frac{2+\frac{1}{n}}{2-\frac{1}{n}} \right) \cdot \frac{1}{x^2}$$

$$= \frac{1}{x^2}$$

By D'Alembert's ratio test, $\sum |u_n|$ is convergent if $\frac{1}{x^2} > 1$ or $x^2 < 1$ or $x < 1$ $[\because x > 0]$

Thus, the given series is absolutely convergent and hence, is convergent for $x < 1$.

If $x^2 = 1$ or $x = 1$ $[\because x > 0]$

$$u_n = \frac{(-1)^{n-1}}{2n-1}$$

$$|u_n| = \frac{1}{2n-1}$$

The given series is an alternating series.

(i) $|u_n| - |u_{n+1}| = \frac{1}{2n-1} - \frac{1}{2n+1}$

$$= \frac{2}{4n^2-1} > 0 \quad \text{for all } n \in N$$

(ii) $\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{2n-1}$

$= 0$

By Leibnitz's test, the series is convergent for $x = 1$

Hence, the series is convergent for $x \le 1$.

EXERCISE 5.7

Test the following series for absolute or conditional convergence:

1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$

[**Ans.:** Conditionally convergent]

2. $1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \ldots$

[**Ans.:** Absolutely convergent]

3. $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots$

[Ans.: Conditionally convergent**]**

4. $\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}-\ldots$

[Ans.: Absolutely convergent**]**

5.17 POWER SERIES

A power series is an infinite series of the form $\sum_{n=1}^{\infty} a_n x^n = a_0 + a_1(x-c)+..$ $a_2(x-c)^2 + a_3(x-c)^3 + \cdots$, where a_n represents the coefficient of the n^{th} term, c is a constant and x varies around c. When $c = 0$, the series becomes

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

5.17.1 Interval and Radius of Convergence

A power series will converge only for certain values of x. An interval $(-R, R)$ in which a power series converges is called the interval of convergence. The number R is called the radius of convergence, e.g., if a power series converges for all the values of x, then interval of convergence will be $(-\infty, \infty)$ and the radius of convergence will be ∞.

5.17.2 Test for Convergence

Since a power series may be positive, alternating or mixed series, the concept of absolute convergence is used to test the convergence of a power series. Applying D'Alembert's ratio test,

$$u_n = a_n x^n$$

$$u_{n+1} = a_{x+1} x^{n+1}$$

$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{a_n x^n}{a_{n+1} x^{n+1}}\right|$$

$$= \left|\frac{1}{x}\right| \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

If $$\lim_{n\to\infty}\left|\frac{a_n}{a_{x+1}}\right| = l,$$

$$\lim_{n\to\infty}\left|\frac{u_n}{u_{x+1}}\right| = \left|\frac{1}{x}\right| \cdot l = \left|\frac{l}{x}\right|$$

By D' Alembert's ratio test, the series is absolutely convergent and hence is convergent

If $\left|\frac{l}{x}\right| > 1$, i.e., $|x| < l$, $-l < x < l$.

Here, interval of convergence of the series is $(-l, l)$ and the radius of convergence is l.

Example 1

Obtain the range of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{2^n}, x > 0.$

Solution

Let
$$u_n = \frac{x^n}{2^n}$$
$$u_{n+1} = \frac{x^{n+1}}{2^{n+1}}$$
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^n}{2^n} \cdot \frac{2^{n+1}}{x^{n+1}}$$
$$= \frac{2}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{2}{x} > 1$ or $x < 2$

(ii) divergent if $\frac{2}{x} < 1$ or $x > 2$

The test fails if $\frac{2}{x} = 1$, or $x = 2$.

Then
$$u_n = \frac{2^n}{2^n} = 1$$
$$\sum_{n=1}^{\infty} u_n = 1 + 1 + 1 + \cdots \infty$$

which is a divergent series.

Hence, the series is convergent for $0 < x < 2$ and the range of convergence is $0 < x < 2$.

Example 2

Determine the interval of convergence for the series $\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$ *and also, their behaviour at each end point.*

Solution

Let

$$u_n = \frac{2^n x^n}{n!}$$

$$u_{n+1} = \frac{2^{n+1} x^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{2^n x^n}{n!} \cdot \frac{(n+1)!}{2^{n+1} x^{n+1}}$$

$$= \frac{n+1}{2x}$$

$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{2x}\right|$$

$$= \infty > 1$$

Hence, By D'Alembert's ratio test, the series is convergent for all values of x i.e. $-\infty < x < \infty$ and interval of convergence is $(-\infty, \infty)$.

Example 3

Obtain the range of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{a+\sqrt{n}}, x > 0, a > 0.$

Solution

Let

$$u_n = \frac{x^n}{a+\sqrt{n}}$$

$$u_{n+1} = \frac{x^{n+1}}{a+\sqrt{n+1}}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{x^n}{a+\sqrt{n}} \cdot \frac{a+\sqrt{n+1}}{x^{n+1}}$$

$$= \lim_{n\to\infty}\frac{\frac{a}{\sqrt{n}}+\sqrt{1+\frac{1}{n}}}{\frac{a}{\sqrt{n}}+1} \cdot \frac{1}{x}$$

$$= \frac{1}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x} > 1$ or $x < 1$

(ii) divergent if $\frac{1}{x} < 1$ or $x > 1$

The test fails if $x = 1$.

Then
$$u_n = \frac{1}{a+\sqrt{n}}$$
$$= \frac{1}{\sqrt{n}}\left(\frac{1}{\frac{a}{\sqrt{n}}+1}\right)$$

Let
$$v_n = \frac{1}{\sqrt{n}}$$
$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\frac{a}{\sqrt{n}}+1}$$
$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^{\frac{1}{2}}}$ is divergent as $p = \frac{1}{2} < 1$.

By comparison test, Σu_n is also divergent for $x = 1$.

Hence, the series is convergent for $0 < x < 1$ and the range of convergence is $0 < x < 1$.

Example 4

Obtain the range of convergence of $\sum_{n=1}^{\infty}\frac{(x+1)^n}{3^n \cdot n}$.

Solution

Let
$$u_n = \frac{(x+1)^n}{3^n \cdot n}$$
$$u_{n+1} = \frac{(x+1)^{n+1}}{3^{n+1}(n+1)}$$
$$\frac{u_n}{u_{n+1}} = \frac{(x+1)^n}{3^n \cdot n}\cdot\frac{3^{n+1}(n+1)}{(x+1)^{n+1}}$$
$$= \frac{3(n+1)}{(x+1)n}$$
$$= \frac{3\left(1+\frac{1}{n}\right)}{x+1}$$
$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{3\left(1+\frac{1}{n}\right)}{x+1}\right|$$
$$= \left|\frac{3}{x+1}\right|$$

The series is convergent if

$$\left|\frac{3}{x+1}\right| > 1$$
$$3 > |x+1|$$
$$|x+1| < 3$$
$$-3 < (x+1) < 3$$
$$-4 < x < 2$$

At $x = 2$,

$$u_n = \frac{1}{n}$$

$\therefore \quad \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent as $p = 1$.

At $x = -4$,

$$u_n = \frac{(-1)^n}{n}$$
$$|u_n| = \frac{1}{n}$$

The given series is an alternating series.

(i) $|u_n| - |u_{n+1}| = \frac{1}{n} - \frac{1}{n+1}$

$= \frac{1}{n(n+1)} > 0$ for all $n \in N$

$\therefore \quad |u_n| > |u_{n+1}|$

(ii) $\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{n}$

$= 0$

By Leibnitz's test, the series is convergent at $x = -4$.
Hence, the series is convergent for $-4 \le x < 2$ and the range of convergence is $-4 \le x < 2$.

Example 5

Obtain the range of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n+\sqrt{1+n^2}}$.

Solution

Let

$$u_n = \frac{x^n}{n+\sqrt{1+n^2}}$$
$$u_{n+1} = \frac{x^{n+1}}{(n+1)+\sqrt{1+(n+1)^2}}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^n}{n+\sqrt{1+n^2}} \cdot \frac{(n+1)+\sqrt{1+(n+1)^2}}{x^{n+1}}$$

$$= \frac{\left(1+\frac{1}{n}\right)+\sqrt{\frac{1}{n^2}+\left(1+\frac{1}{n}\right)^2}}{\left(1+\sqrt{\frac{1}{n^2}+1}\right)x}$$

$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\left(1+\frac{1}{n}\right)+\sqrt{\frac{1}{n^2}+\left(1+\frac{1}{n}\right)^2}}{\left(1+\sqrt{\frac{1}{n^2}+1}\right)x}\right|$$

$$= \frac{1}{|x|}$$

The series is convergent if

$$\frac{1}{|x|} > 1$$

$$|x| < 1$$

$$-1 < x < 1$$

At $x = 1$,

$$u_n = \frac{1}{n+\sqrt{1+n^2}}$$

$$= \frac{1}{n\left(1+\sqrt{\frac{1}{n^2}+1}\right)}$$

Let

$$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\left(1+\sqrt{\frac{1}{n^2}+1}\right)}$$

$$= \frac{1}{2} \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

Thus, by comparison test, Σu_n is also divergent if $x = 1$.
At $x = -1$,

$$u_n = \frac{(-1)^n}{n+\sqrt{1+n^2}}$$

$$|u_n| = \frac{1}{n+\sqrt{1+n^2}}$$

The given series is an alternating series.

(i) $$|u_n| - |u_{n+1}| = \frac{1}{n+\sqrt{1+n^2}} - \frac{1}{(n+1)+\sqrt{1+(n+1)^2}}$$

$$= \frac{(n+1)+\sqrt{1+(n+1)^2} - n - \sqrt{1+n^2}}{\left(n+\sqrt{1+n^2}\right)\left[(n+1)+\sqrt{1+(n+1)^2}\right]}$$

$$= \frac{1+\sqrt{1+(n+1)^2} - \sqrt{1+n^2}}{\left(n+\sqrt{1+n^2}\right)\left[(n+1)+\sqrt{1+(n+1)^2}\right]} > 0 \qquad \text{for all } n \in N$$

$$\therefore \quad |u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty}|u_n| = \lim_{n\to\infty}\frac{1}{n+\sqrt{1+n^2}} = 0$$

Thus, by Leibnitz's test, the series is convergent if $x = -1$.

Hence, the series is convergent for $-1 \le x < 1$ and the range of convergence is $-1 \le x < 1$.

Example 6

For the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}$, *find the radius and interval of convergence.* **[Winter 2016]**

Solution

Let $$u_n = \frac{(-1)^{n-1} x^{2n-1}}{2n-1}$$

$$u_{n+1} = \frac{(-1)^n x^{2n+2-1}}{2n+2-1} = \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{(-1)^{n-1} x^{2n-1}}{2n-1} \cdot \frac{2n+1}{(-1)^n x^{2n+1}} = -\frac{(2n+1)}{(2n-1)} \cdot \frac{1}{x^2}$$

$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(2n+1)}{(2n-1)}\cdot\frac{1}{x^2}\right|$$

$$= \left|\frac{1}{x^2}\right|$$

By D'Alembert's ratio test, the series is convergent if $\left|\frac{1}{x^2}\right| > 1$ or $1 > |x^2|$ or $|x^2| < 1$ i.e., $-1 < x < 1$ and divergent for $x > 1$.

At $x = 1$,

$$u_n = \frac{(-1)^{n-1}\,(1)^{2n-1}}{2n-1}$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$= \frac{(-1)^{n-1}}{2n-1}$$

$$|u_n| = \frac{1}{2n-1}$$

The given series is an alternating series.

(i) $$|u_n| - |u_{n+1}| = \frac{1}{2n-1} - \frac{1}{2n+1}$$

$$= \frac{2n+1-2n+1}{(2n-1)\,(2n+1)}$$

$$= \frac{2}{(2n-1)\,(2n+1)} > 0 \quad \text{for all } n \in N$$

$$|u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{2n-1} = 0$$

By Leibnitz's test, $\sum |u_n|$ is convergent.

At $x = -1$,

$$u_n = \frac{(-1)^{n-1}\,(-1)^{2n-1}}{2n-1}$$

$$= \frac{(-1)^{3n-2}}{2n-1}$$

$$|u_n| = \frac{1}{2n-1}$$

The given series is an alternating series.

Hence, by Leibnitz's test, $\sum |u_n|$ is convergent.

Thus, for the interval $-1 \le x \le 1$, given series is convergent.

Since condition for convergence is $|x^2| < 1$, radius of convergence = 1.

Since $\sum_{n=1}^{\infty} |u_n|$ is convergent, the series is absolutely convergent for $-1 \le x \le 1$.

Example 7

Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

[Winter 2013; Summer 2016]

Solution

$$u_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$$

$$u_{n+1} = \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}}$$

$$\frac{u_n}{u_{n+1}} = \frac{(-3)^n x^n}{\sqrt{n+1}} \cdot \frac{\sqrt{n+2}}{(-3)^{n+1} x^{n+1}}$$

$$= \frac{\sqrt{n+2}}{(-3)x\sqrt{n+1}}$$

$$= \frac{\sqrt{1+\frac{2}{n}}}{(-3)x\sqrt{1+\frac{1}{n}}}$$

$$\lim_{n\to\infty} \left| \frac{u_n}{u_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{\sqrt{1+\frac{2}{n}}}{(-3)x\sqrt{1+\frac{1}{n}}} \right|$$

$$= \left| \frac{1}{-3x} \right|$$

$$= \left| \frac{1}{3x} \right|$$

By D'Alembert's ratio test, the series is convergent if $\left|\frac{1}{3x}\right| > 1$ or $|3x| < 1$ or $|x| < \frac{1}{3}$ or $-\frac{1}{3} < x < \frac{1}{3}$.

The interval of convergence is $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

At $x = -\frac{1}{3}$,

$$\begin{aligned} u_n &= \frac{(-3)^n \left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} \\ &= \frac{1}{\sqrt{n+1}} \\ &= \frac{1}{\sqrt{n}\sqrt{1+\frac{1}{n}}} \end{aligned}$$

Let $v_n = \frac{1}{\sqrt{n}}$

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{u_n}{v_n}\right| &= \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}} \\ &= 1 \quad \text{[finite and non-zero]} \end{aligned}$$

and $\Sigma v_n = \sum \frac{1}{\sqrt{n}}$ is divergent as $p = \frac{1}{2}$.

Thus, by comparison test, Σu_n is also divergent if $x = -\frac{1}{3}$.

At $x = \frac{1}{3}$,

$$\begin{aligned} u_n &= \frac{(-3)^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}} \\ &= \frac{(-1)^n}{\sqrt{n+1}} \\ |u_n| &= \frac{1}{\sqrt{n+1}} \\ \lim_{n\to\infty}|u_n| &= \lim_{n\to\infty}\frac{1}{\sqrt{n+1}} = 0 \end{aligned}$$

By Leibnitz's test, the series is convergent at $x = \frac{1}{3}$.

Hence, the series is convergent at each end point.

Example 8

Determine the interval of convergence for the series $\sum_{n=1}^{\infty} (-1)^n \frac{n(x+1)^n}{2^n}$ *and also its behaviour at each end point.*

Solution

Let

$$u_n = (-1)^n \frac{n(x+1)^n}{2^n}$$

$$u_{n+1} = (-1)^{n+1} \frac{(n+1)(x+1)^{n+1}}{2^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{(-1)^n n \cdot (x+1)^n}{2^n} \cdot \frac{2^{n+1}}{(-1)^{(n+1)}(n+1)(x+1)^{n+1}}$$

$$= \frac{n}{n+1} \cdot \frac{2}{x+1}$$

$$= \frac{1}{1+\frac{1}{n}} \cdot \frac{2}{x+1}$$

$$\lim_{n\to\infty} \left| \frac{u_n}{u_{n+1}} \right| = \lim_{n\to\infty} \left| \left(\frac{1}{1+\frac{1}{n}} \right) \left(\frac{2}{x+1} \right) \right|$$

$$= \left| \frac{2}{x+1} \right|$$

The series is convergent if

$$\left| \frac{2}{x+1} \right| > 1$$

$$2 > |x+1|$$

$$|x+1| < 2$$

$$-2 < (x+1) < 2$$

$$-3 < x < 1$$

The series is convergent in the interval (– 3, 1).

At $x = -3$,

$$u_n = (-1)^n \frac{n(-3+1)^n}{2^n}$$

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} n$$

which is a divergent series.
At $x = 1$,

$$u_n = (-1)^n \frac{n(1+1)^n}{2^n}$$

$$= (-1)^n n$$

$$|u_n| = n$$

$$\lim_{n\to\infty} |u_n| \neq 0$$

By Leibnitz's test, the series is not convergent at $x = 1$.
Hence, the series is not convergent at each end point and the interval of convergence is $(-3, 1)$.

Example 9

For the series $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}$, *find the radius and interval of convergence. For what values of x does the series converge absolutely, conditionally?* **[Winter 2015]**

Solution

Let

$$u_n = \frac{(-1)^n (x+2)^n}{n}$$

$$u_{n+1} = \frac{(-1)^{n+1} (x+2)^{n+1}}{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{(-1)^n (x+2)^n}{n} \cdot \frac{(n+1)}{(-1)^{n+1} (x+2)^{n+1}}$$

$$= -\left(1+\frac{1}{n}\right) \cdot \frac{1}{(x+2)}$$

$$\lim_{n\to\infty} \left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty} \left|\left(1+\frac{1}{n}\right) \cdot \frac{1}{(x+2)}\right|$$

$$= \left|\frac{1}{x+2}\right|$$

The series is convergent if

$$\left|\frac{1}{x+2}\right| > 1$$
$$1 > |x+2|$$
$$|x+2| < 1$$
$$-1 < (x+2) < 1$$
$$-3 < x < -1$$

At $x = -3$,

$$u_n = \frac{(-1)^n(-3+2)^n}{n} = \frac{1}{n}$$
$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

which is a divergent series.
At $x = -1$,

$$u_n = \frac{(-1)^n(-1+2)^n}{n}$$
$$= \frac{(-1)^n}{n}$$
$$|u_n| = \frac{1}{n}$$
$$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{n} = 0$$

By Leibnitz's test, the series is convergent at $x = -1$.

Hence, interval of convergence is $(-3, -1]$, i.e. $-3 < x \le -1$.

Since condition for convergence is $|x+2| < 1$, radius of convergence = 1.

Since $\sum_{n=1}^{\infty} |u_n|$ is convergent, the series is absolutely convergent for $-3 < x \le -1$.

Example 10

Obtain the range of convergence of $\sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}$.

Solution

Let

$$u_n = \frac{1}{x^n + x^{-n}}$$

$$= \frac{x^n}{x^{2n}+1}$$

$$u_{n+1} = \frac{x^{n+1}}{x^{2n+2}+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^n}{x^{2n}+1} \cdot \frac{x^{2n+2}+1}{x^{n+1}}$$

$$= \frac{x^{2n+2}+1}{x(x^{2n}+1)}$$

$$\left|\frac{u_n}{u_{n+1}}\right| = \left|\frac{x^{2n+2}+1}{x(x^{2n}+1)}\right|$$

If $|x| > 1$,

$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{x^2+\frac{1}{x^{2n}}}{x\left(1+\frac{1}{x^{2n}}\right)}\right|$$

$$= |x| > 1 \quad \left[\because \lim_{n\to\infty} x^{2n} \to \infty\right]$$

Thus, the series is convergent for $|x| > 1$, i.e. $x > 1$ and $x < -1$

At $x = 1$,

$$u_n = \frac{1}{2}$$

$$\sum_{n=1}^{\infty} u_n = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \infty$$

which is a divergent series.

At $x = -1$,

$$u_n = \frac{(-1)^n}{2}$$

$$|u_n| = \frac{1}{2}$$

$$\lim_{n\to\infty}|u_n| = \frac{1}{2} \neq 0$$

Thus, by Leibnitz's test, the series is not convergent at $x = -1$.

Hence, the series is convergent for $|x| > 1$ and range of convergence is $|x| > 1$.

EXERCISE 5.9

Obtain the range of convergence of the following series:

1. $1+x+2x^2+3x^3+\cdots+nx^n+\cdots$ **[Ans.: $-1< x < 1$]**

2. $\frac{1}{2}+\frac{x}{3}+\frac{x^2}{4}+\frac{x^3}{5}+\cdots+\frac{x^n}{n+2}+\cdots$ **[Ans.: $-1 < x < 1$]**

3. $\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{(n+1)}$ **[Ans.: $|x| \le 1$]**

4. $\sum_{n=0}^{\infty}\frac{(x+2)}{\sqrt{n+1}}$ **[Ans.: $-3 \le x \le -1$]**

5. $\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{\log(n+1)}$ **[Ans.: $|x| < 1$]**

6. $\sum_{n=0}^{\infty}(-2)^n(n+1)(x-1)^n$ $\left[\text{Ans.: } \frac{1}{2}<x<\frac{3}{2}\right]$

7. $\sum_{n=1}^{\infty}n!(x-1)^n$ **[Ans.: $x = 1$]**

8. $\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}x^n$ **[Ans.: $|x| < 4$]**

9. $\sum_{n=1}^{\infty}\frac{(-2)^n(2x+1)^n}{n^2}$ $\left[\text{Ans.: } -\frac{3}{4}\le x\le -\frac{1}{4}\right]$

10. $\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n)^{\frac{3}{2}}}$ **[Ans.: $-1 \le x \le 1$]**

Points to Remember

Sequence

A sequence $\{u_n\}$ is said to be convergent, divergent or oscillatory according as $\lim_{n\to\infty} u_n$ is finite, infinite or not unique respectively.

The infinite series Σu_n is said to be convergent, divergent or oscillatory according as $\lim_{n\to\infty} S_n$ is finite, infinite or not unique respectively.

If a positive term series Σu_n is convergent then $\lim_{n\to\infty} u_n = 0$ but converse is not true, i.e., if $\lim_{n\to\infty} u_n = 0$, the series may converge or diverge. If $\lim_{n\to\infty} u_n \neq 0$, the series is not convergent.

Comparison Test

If Σu_n and Σv_n are series of positive terms such that $\lim_{n\to\infty} \frac{u_n}{v_n} = l$ (finite and non-zero) then both series converge or diverge together.

D'Alembert's Ratio Test

If Σu_n is a positive term series and $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l$ then

(i) Σu_n is convergent if $l > 1$.

(ii) Σu_n is divergent if $l < 1$.

(iii) The test fails if $l = 1$.

Raabe's Test

If Σu_n is a positive term series and $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$, then

(i) Σu_n is convergent if $l > 1$

(ii) Σu_n is divergent if $l < 1$

(iii) Test fails if $l = 1$

Cauchy's Root Test

If Σu_n is a positive term series and if $\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = l$ then

(i) Σu_n is convergent if $l < 1$.

(ii) Σu_n is divergent if $l > 1$.

This test is preferred when u_n contains n^{th} powers of itself.

Cauchy's Integral Test

If $\Sigma u_n = \Sigma f(n)$ is a positive term series where $f(n)$ decreases as n increases and let $\int_1^\infty f(x)\,dx = I$ then

(i) Σu_n is convergent if I is finite.

(ii) Σu_n is divergent if I is infinite.

This test is preferred when evaluation of the integral of $f(x)$ is easy.

Leibnitz's Test

An alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is convergent if

(i) each term is numerically less than its preceding term, i.e, $u_{n+1} < u_n$ or $u_n > u_{n+1}$

(ii) $\lim_{n\to\infty} u_n = 0$

Absolute Convergence

The series $\sum_{n=1}^{\infty} u_n$ with both positive and negative terms (not necessarily alternative) is called absolutely convergent if the corresponding series $\sum_{n=1}^{\infty} |u_n|$ with all positive terms is convergent.

Conditional Convergence

If the series $\sum_{n=1}^{\infty} u_n$ is convergent and $\sum_{n=1}^{\infty} |u_n|$ is divergent then the series $\sum_{n=1}^{\infty} u_n$ is called conditionally convergent.

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. The value of $\lim_{x \to 0} x \sin \frac{1}{x}$ is **[Summer 2014]**

(a) 1 (b) π (c) 0 (d) ∞

2. The sum of the series $\sum_{n=1}^{\infty} \frac{1}{2n}$ is **[Winter 2015]**

(a) 0 (b) $\frac{3}{4}$ (c) 1 (d) 2

3. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is **[Winter 2015]**

(a) divergent (b) absolutely convergent
(c) conditionally convergent (d) nothing can be said

4. The series $\sum n(5^{\frac{1}{n}} - 1)$ is

(a) convergent (b) divergent
(c) oscillates finitely (d) oscillates infinitely

5. The series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)}$ is

(a) convergent (b) divergent
(c) oscillates finitely (d) oscillates infinitely

6. The series $\sum \frac{1}{n^p}$ is divergent if

(a) $p > 1$ (b) $p \le 1$ (c) $p = 1$ (d) $p = 0$

7. If $\lim_{x \to 0} \frac{u_{n+1}}{u_n} > 1$, then $\sum_{n=1}^{\infty} u_n$ is

(a) convergent (b) divergent
(c) may or may not be convergent (d) oscillatory

8. The series $a + ar + ar^2 + ar^3 + \cdots$ oscillates finitely if

(a) $|r| < 1$ (b) $r > 1$ (c) $r = 1$ (d) $r \le -1$

9. The series $\frac{3}{4} + \frac{9}{8} + \frac{27}{16} + \frac{81}{32} + \cdots$ is

(a) convergent (b) divergent
(c) oscillates finitely (d) oscillates infinitely

10. The series $\sum_{n=1}^{\infty} \frac{1}{n5^n}$ is

(a) convergent (b) divergent
(c) oscillates finitely (d) oscillates infinitely

11. The series $a + ar + ar^2 + ar^3 + \cdots$ diverges if

(a) $|r| < 1$ (b) $r \ge 1$ (c) $r \le -1$ (d) $r = 1$

12. The series $\sum \frac{1}{n^p}$ converges if

(a) $p > 1$ (b) $p < 1$ (c) $p = 0$ (d) $p = 1$

13. The series $\sum \frac{1}{n^{\frac{1}{4}}}$ is

(a) convergent (b) divergent
(c) oscillates finitely (d) oscillates infinitely

14. The series $\sum_{n=1}^{\infty} \frac{3^{2n}}{4n}$ is

(a) convergent (b) divergent
(c) oscillates finitely (d) oscillates infinitely

15. The series $\sum_{n=1}^{\infty} \sin \frac{1}{n}$ is

(a) convergent (b) divergent
(c) oscillatory (d) may or may not be convergent

16. $\lim_{n \to a} \frac{u_n}{v_n} = 1$ and $\sum v_n$ diverges then $\sum u_n$ is

(a) divergent (b) convergent
(c) oscillatory (d) oscillates infinitely

17. $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\cdots\infty$ converges if

(a) $p<2$ (b) $p>2$ (c) $p>1$ (d) $p\geq 1$

18. If Σu_n is a series of positive terms such that $\lim\limits_{n\to\infty} u_n = 0$ then Σu_n is

(a) convergent (b) divergent
(c) may or may not be convergent (d) oscillatory

19. $\sum \frac{1}{n(\log n)^p}$ is convergent

(a) for $p > 1$ (b) for $p < 1$
(c) for all real values of p (d) for no value of p

20. $\sum\limits_{n=0}^{\infty}(2x)^n$ is divergent if

(a) $-1\leq x\leq 1$ (b) $-\frac{1}{2}<x<\frac{1}{2}$ (c) $-2\leq x\leq 2$ (d) $-\frac{1}{2}\geq x\geq \frac{1}{2}$

21. The geometric series $\sum\limits_{n=0}^{\infty} ar^n$, when $r = -1 \times 2$ is

(a) convergent (b) divergent (c) oscillatory (d) none of these

22. If $u_n = \frac{n!}{n^n}$ then $\lim\limits_{x\to\infty}\frac{u_n}{u_{n+1}} =$

(a) e^2 (b) e (c) e^{-1} (d) 1

23. The series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ is

(a) convergent (b) divergent (c) oscillatory (d) none of these

24. The series $\sum\frac{2^{3n}}{3^{2n}}$ is

(a) convergent (b) divergent (c) oscillatory (d) none of these

25. The power series $\sum\limits_{n=1}^{\infty}(3x)^n$ is convergent if

(a) $x=\frac{1}{3}$ (b) $x>\frac{1}{3}$ (c) $-\frac{1}{3}<x<\frac{1}{3}$ (d) $\frac{1}{3}<x<1$

26. The series $\sum\limits_{n=1}^{\infty}\frac{n^2}{n!}$ is

(a) oscillatory (b) divergent (c) convergent (d) none of these

27. The series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$ is

(a) oscillatory (b) convergent (c) divergent (d) none of these

28. The series $\frac{x}{1\cdot 2}+\frac{x^2}{3\cdot 4}+\frac{x^2}{3\cdot 4}+\frac{x^3}{5\cdot 6}+\cdots+\frac{x^n}{(2n-1)2n}+\cdots$ is

(a) p series (b) geometric series
(c) alternating series (d) power series

29. The series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ converges only if

(a) $-1 < x < 1$ (b) $-1 \le x \le 1$ (c) $-1 < x \le 1$ (d) $-1 \le x < 1$

30. Which of the following series is divergent?

(a) $\sum\left(1+\frac{1}{n}\right)$ (b) $\sum\frac{1}{n^2}$ (c) $\sum\frac{1}{n^\pi}$ (d) $\sum\frac{1}{n^e}$

31. Which of the following series is convergent?

(a) $\sum_{n=1}^{\infty}\frac{1}{n^{0.9}}$ (b) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ (c) $\sum\frac{1}{n}$ (d) $\sum\frac{1}{n^{1.001}}$

32. The series $\sum\frac{\cos n\pi}{1+n^2}$ is

(a) absolutely convergent (b) conditionally convergent
(c) convergent (d) divergent

33. The series $\sum(-n)$ is

(a) divergent (b) convergent (c) oscillatory (d) none of these

34. The sum of the series $1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\cdots$ is **[Summer 2016]**

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) $\frac{1}{2}$ (d) none of these

35. The series $\sum_{n=1}^{\infty}\frac{1}{(\log n)^n}$ is **[Summer 2016]**

(a) oscillatory (b) divergent (c) convergent (d) none of these

36. The sequence $\sin\left(\frac{\pi}{6}+\frac{1}{n}\right)$ converges to **[Winter 2016]**

(a) 0 (b) 1 (c) –1 (d) 0.5

37. The sum of the series $\sum_{n=1}^{\infty}\left(\frac{e}{\pi}\right)^n$ is **[Winter 2016]**

(a) $\frac{\pi}{\pi-e}$ (b) $\frac{e}{\pi-e}$ (c) $\frac{\pi}{e-\pi}$ (d) $\frac{e}{\pi}$

38. $\sum_{n=1}^{\infty}\frac{2^n}{3n-1}$ is **[Winter 2016]**

(a) convergent and sum is 0 (b) convergent and sum is 1
(c) divergent (d) oscillating

39. Infinite series $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$ is **[Summer 2017]**

(a) divergent (b) convergent (c) oscillatory (d) none of these

Answers

1. (c)	2. (c)	3. (c)	4. (b)	5. (b)	6. (b)	7. (b)	8. (d)	9. (b)
10. (a)	11. (b)	12. (a)	13. (b)	14. (b)	15. (b)	16. (a)	17. (b)	18. (c)
19. (a)	20. (b)	21. (c)	22. (b)	23. (a)	24. (a)	25. (c)	26. (c)	27. (c)
28. (d)	29. (c)	30. (a)	31. (d)	32. (a)	33. (a)	34. (a)	35. (c)	36. (d)
37. (b)	38. (c)	39. (b)						

CHAPTER 6

Taylor's and Maclaurin's Series

Chapter Outline

6.1 INTRODUCTION

In this chapter, we will study Taylor's and Maclaurin's series. A Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point. The concept of a Taylor series was discovered by the Scottish mathematician James Gregory and formally introduced by the English mathematician Brook Taylor in 1715. A Maclaurin series is a Taylor series expansion of a function about zero. It is named after the Scottish mathematician Colin Maclaurin, who made extensive use of this special case of Taylor series. It is common practice to approximate a function by using a finite number of terms of its Taylor series and Maclaurin's series covering expansions by definition, by standard expansion, by differentiation and integration and by substitution.

6.2 TAYLOR'S SERIES

Statement If $f(x+h)$ is a given function of h which can be expanded into a convergent series of positive ascending integral powers of h then

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots + \frac{h^n}{n!}f^n(x) + \ldots\ldots$$

Proof Let $f(x+h)$ be a function of h which can be expanded into positive ascending integral powers of h, then

$$f(x+h) = a_0 + a_1 h + a_2 h^2 + a_3 h^3 + a_4 h^4 + \ldots\ldots\ldots\ldots \quad \ldots (1)$$

Differentiating w.r.t. h successively,

$$f'(x+h) = a_1 + a_2 \cdot 2h + a_3 \cdot 3h^2 + a_4 \cdot 4h^3 + \ldots\ldots \ldots\ldots \quad \ldots (2)$$

$$f''(x+h) = a_2 \cdot 2 + a_3 \cdot 6h + a_4 \cdot 12h^2 + \ldots\ldots \ldots\ldots \quad \ldots (3)$$

$$f'''(x+h) = a_3 \cdot 6 + a_4 \cdot 24h + \ldots\ldots \ldots\ldots \quad \ldots (4)$$

and so on.

Putting $h = 0$ in Eq. (1), (2), (3) and (4),

$$a_0 = f(x)$$

$$a_1 = f'(x)$$

$$a_2 = \frac{1}{2!} f''(x)$$

$$a_3 = \frac{1}{3!} f'''(x) \text{ and so on}$$

Substituting a_0, a_1, a_2 and a_3 in Eq. (1),

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots + \frac{h^n}{n!} f^n(x) + \ldots\ldots$$

This is known as **Taylor's series.**

Putting $x = a$ and $h = x - a$ in above series, we get Taylor's series in powers of $(x - a)$ as

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots\ldots + \frac{(x-a)^n}{n!} f^n(a) + \ldots\ldots$$

Note: To express the function in ascending powers of x, express h in terms of x.

Example 1

Prove that $f(mx) = f(x) + (m-1)x f'(x) + \frac{(m-1)^2}{2!} x^2 f''(x) + \ldots.$

Solution

$$f(mx) = f(mx - x + x) = f[x + (m-1)x]$$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots\ldots$$

Putting $h = (m-1)x$,

$$f[x + (m-1)x] = f(mx) = f(x) + (m-1)x f'(x) + \frac{(m-1)^2}{2!} x^2 f''(x) + \ldots\ldots\ldots$$

Example 2

Prove that

$$f\left(\frac{x^2}{1+x}\right) = f(x) - \frac{x}{1+x} f'(x) + \frac{x^2}{2!(1+x)^2} f''(x) - \frac{x^3}{3!(1+x)^3} f'''(x) + \ldots.$$

Solution

$$\frac{x^2}{1+x} = x - \frac{x}{1+x}$$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + \ldots$$

Putting $$h = -\frac{x}{1+x},$$

$$f\left(x - \frac{x}{1+x}\right) = f\left(\frac{x^2}{1+x}\right)$$

$$f\left(\frac{x^2}{1+x}\right) = f(x) - \frac{x}{1+x} f'(x) + \frac{x^2}{2!\,(1+x)^2} f''(x) - \frac{x^3}{3!\,(1+x)^3} f'''(x) + \ldots$$

Example 3

Express $f(x) = 2x^3 + 3x^2 - 8x + 7$ *in terms of* $(x - 2)$.

Solution

$$f(x) = 2x^3 + 3x^2 - 8x + 7$$

By Taylor's series,

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(a) + \cdots$$

Putting $a = 2$,

$$f(x) = f(2) + (x-2) f'(2) + \frac{(x-2)^2}{2!} f''(2) + \frac{(x-2)^3}{3!} f'''(2) + \cdots \qquad \ldots(1)$$

$$\begin{aligned} f(x) &= 2x^3 + 3x^2 - 8x + 7, & f(2) &= 16 + 12 - 16 + 7 = 19 \\ f'(x) &= 6x^2 + 6x - 8, & f'(2) &= 24 + 12 - 8 = 28 \\ f''(x) &= 12x + 6, & f''(2) &= 24 + 6 = 30 \\ f'''(x) &= 12, & f'''(2) &= 12 \end{aligned}$$

Substituting in Eq. (1),

$$\begin{aligned} f(x) &= 19 + (x-2)28 + \frac{(x-2)^2}{2!} \cdot 30 + \frac{(x-2)^3}{3!} \cdot 12 \\ &= 19 + 28(x-2) + 15(x-2)^2 + 2(x-2)^3 \end{aligned}$$

Example 4

Express $2x^3 + 7x^2 + x - 6$ *in ascending powers of* $(x - 2)$.

Solution

Let $f(x) = 2x^3 + 7x^2 + x - 6$

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Putting $a = 2$,

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \cdots \quad ...(1)$$

$f(x) = 2x^3 + 7x^2 + x - 6,$ $f(2) = 40$

$f'(x) = 6x^2 + 14x + 1,$ $f'(2) = 53$

$f''(x) = 12x + 14,$ $f''(2) = 38$

$f'''(x) = 12,$ $f'''(2) = 12$

Substituting in Eq.(1),

$$f(x) = 40 + (x-2)(53) + \frac{(x-2)^2}{2!}(38) + \frac{(x-2)^3}{3!}(12)$$
$$= 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$$

Example 5

Expand $x^3 + 7x^2 + x - 6$ *in powers of* $(x - 3)$.

Solution

Let $f(x) = x^3 + 7x^2 + x - 6$

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Putting $a = 3$,

$$f(x) = f(3) + (x-3)f'(3) + \frac{(x-3)^2}{2!}f''(3) + \frac{(x-3)^3}{3!}f'''(3) + \cdots \quad ...(1)$$

$f(x) = x^3 + 7x^2 + x - 6,$ $f(3) = 87$

$f'(x) = 3x^2 + 14x + 1,$ $f'(3) = 70$

$f''(x) = 6x + 14,$ $f''(3) = 32$

$f'''(x) = 6,$ $f'''(3) = 6$

Substituting in Eq. (1),

$$f(x) = 87 + (x-3)(70) + \frac{(x-3)^2}{2!}(32) + \frac{(x-3)^3}{3!}(6)$$
$$= 87 + 70(x-3) + 16(x-3)^2 + (x-3)^3$$

Example 6

Expand $x^4-3x^3+2x^2-x+1$ *in powers of* $(x-3)$.

Solution

Let $f(x)=x^4-3x^3+2x^2-x+1$

By Taylor's series,

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\frac{(x-a)^4}{4!}f^{IV}(a)+\cdots$$

Putting $a=3$,

$$f(x)=f(3)+(x-3)f'(3)+\frac{(x-3)^2}{2!}f''(3)+\frac{(x-3)^3}{3!}f'''(3)+\frac{(x-3)^4}{4!}f^{IV}(3)+\cdots \quad \text{...(1)}$$

$f(x)=x^4-3x^3+2x^2-x+1,$ $\quad f(3)=16$

$f'(x)=4x^3-9x^2+4x-1,$ $\quad f'(3)=38$

$f''(x)=12x^2-18x+4,$ $\quad f''(3)=58$

$f'''(x)=24x-18,$ $\quad f'''(3)=54$

$f^{IV}(x)=24,$ $\quad f^{IV}(x)=24$

Substituting in Eq. (1),

$$f(x)=16+(x-3)(38)+\frac{(x-3)^2}{2!}(58)+\frac{(x-3)^3}{3!}(54)+\frac{(x-3)^4}{4!}(24)$$

$$=16+38(x-3)+29(x-3)^2+9(x-3)^3+(x-3)^4$$

Example 7

Expand $49+69x+42x^2+11x^3+x^4$ *in powers of* $(x+2)$.

Solution

Let $\quad f(x)=49+69x+42x^2+11x^3+x^4$

By Taylor's series,

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\frac{(x-a)^4}{4!}f^{IV}(a)+\cdots \quad \text{...(1)}$$

Putting $a = -2$,

$$f(x) = f(-2) + (x+2)f'(-2) + \frac{(x+2)^2}{2!}f''(-2) + \frac{(x+2)^3}{3!}f'''(-2) + \frac{(x+2)^4}{4!}f^{IV}(-2) + \cdots$$

$$\begin{aligned} f(x) &= 49 + 69x + 42x^2 + 11x^3 + x^4, & f(-2) &= 7 \\ f'(x) &= 69 + 84x + 33x^2 + 4x^3, & f'(-2) &= 1 \\ f''(x) &= 84 + 66x + 12x^2, & f''(-2) &= 0 \\ f'''(x) &= 66 + 24x, & f'''(-2) &= 18 \\ f^{IV}(x) &= 24, & f^{IV}(-2) &= 24 \end{aligned}$$

Substituting in Eq. (1),

$$\begin{aligned} f(x) &= 7 + (x+2)(1) + \frac{(x+2)^2}{2!}(0) + \frac{(x+2)^3}{3!}(18) + \frac{(x+2)^4}{4!}(24) \\ &= 7 + (x+2) + 3(x+2)^3 + (x+2)^4 \end{aligned}$$

Example 8

Expand $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$ in powers of $(x - 1)$ and find $f(0.99)$.

Solution

$$f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$$

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots\ldots$$

Putting $a = 1$,

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \frac{(x-1)^4}{4!}f^{IV}(1) + \frac{(x-1)^5}{5!}f^{V}(1) + \ldots\ldots \qquad \ldots (1)$$

$$\begin{aligned} f(x) &= x^5 - x^4 + x^3 - x^2 + x - 1, & f(1) &= 0 \\ f'(x) &= 5x^4 - 4x^3 + 3x^2 - 2x + 1, & f'(1) &= 5 - 4 + 3 - 2 + 1 = 3 \\ f''(x) &= 20x^3 - 12x^2 + 6x - 2, & f''(1) &= 20 - 12 + 6 - 2 = 12 \\ f'''(x) &= 60x^2 - 24x + 6, & f'''(1) &= 60 - 24 + 6 = 42 \\ f^{IV}(x) &= 120x - 24, & f^{IV}(1) &= 120 - 24 = 96 \\ f^{V}(x) &= 120, & f^{V}(1) &= 120 \end{aligned}$$

Substituting in Eq. (1),

$$f(x) = 0 + (x-1)(3) + \frac{(x-1)^2}{2!}(12) + \frac{(x-1)^3}{3!}(42) + \frac{(x-1)^4}{4!}(96) + \frac{(x-1)^5}{5!}(120)$$
$$= 3(x-1) + 6(x-1)^2 + 7(x-1)^3 + 4(x-1)^4 + (x-1)^5$$

Putting $x = 0.99$,

$$f(0.99) = 3\,(0.99-1) + 6\,(0.99-1)^2 + 7\,(0.99-1)^3 + 4\,(0.99-1)^4 + (0.99-1)^5$$
$$= 3\,(-0.01) + 6\,(-0.01)^2 + 7\,(-0.01)^3 + 4\,(-0.01)^4 + (-0.01)^5$$
$$= -0.02939 \text{ approx.}$$

Example 9

Expand $f(x) = (x+2)^4 + 5(x+2)^3 + 6(x+2)^2 + 7(x+2) + 8$ *in ascending powers of* $(x-1)$.

Solution

$$f(x) = (x+2)^4 + 5(x+2)^3 + 6(x+2)^2 + 7(x+2) + 8$$

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{iv}(a) + \cdots$$

Putting $a = 1$,

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \frac{(x-1)^4}{4!}f^{iv}(1) + \cdots \quad ...(1)$$

$$f(x) = (x+2)^4 + 5(x+2)^3 + 6(x+2)^2 + 7(x+2) + 8, \qquad f(1) = 299$$
$$f'(x) = 4(x+2)^3 + 15(x+2)^2 + 12(x+2) + 7, \qquad f'(1) = 286$$
$$f''(x) = 12(x+2)^2 + 30(x+2) + 12, \qquad f''(1) = 210$$
$$f'''(x) = 24(x+2) + 30, \qquad f'''(1) = 102$$
$$f^{IV}(x) = 24, \qquad f^{IV}(1) = 24$$

Substituting in Eq. (1),

$$f(x) = 299 + (x-1)(286) + \frac{(x-1)^2}{2!}(210) + \frac{(x-1)^3}{3!}(102) + \frac{(x-1)^4}{4!}(24)$$
$$= 299 + 286(x-1) + 105(x-1)^2 + 17(x-1)^3 + (x-1)^4$$

Example 10

Prove that $\frac{1}{1-x} = \frac{1}{3} + \frac{(x+2)}{3^2} + \frac{(x+2)^2}{3^3} + \frac{(x+2)^3}{3^4} + \ldots.$

Solution

Let $\quad f(x) = \frac{1}{1-x}$

By Taylor's series,

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\dots\dots\dots$$

Putting $a=-2$,

$$f(x)=f(-2)+(x+2)f'(-2)+\frac{(x+2)^2}{2!}f''(-2)+\frac{(x+2)^3}{3!}f'''(-2)+\dots\dots \qquad \dots(1)$$

$$f(x)=\frac{1}{1-x}, \qquad f(-2)=\frac{1}{3}$$

$$f'(x)=\frac{1}{(1-x)^2}, \qquad f'(-2)=\frac{1}{3^2}$$

$$f''(x)=\frac{2}{(1-x)^3}, \qquad f''(-2)=\frac{2!}{3^3}$$

$$f'''(x)=\frac{2\cdot 3}{(1-x)^4}, \qquad f'''(-2)=\frac{3!}{3^4} \text{ and so on}$$

Substituting in Eq. (1),

$$f(x)=\frac{1}{3}+\frac{(x+2)}{3^2}+\frac{(x+2)^2}{3^3}+\frac{(x+2)^3}{3^4}+\dots\dots\dots$$

Example 11

Expand log x *in powers of* $(x-1)$.

Solution

Let $\qquad f(x)=\log x$

By Taylor's series,

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\cdots+\frac{(x-a)^n}{n!}f^n(a)+\cdots$$

Putting $a=1$,

$$f(x)=f(1)+(x-1)f'(1)+\frac{(x-1)^2}{2!}f''(1)+\frac{(x-1)^3}{3!}f'''(1)+\cdots \qquad \dots(1)$$

$$f(x)=\log x, \qquad f(1)=\log 1=0$$

$$f'(x)=\frac{1}{x}, \qquad f'(1)=1$$

$$f''(x) = -\frac{1}{x^2}, \qquad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3}, \qquad f'''(1) = 2 \quad \text{and so on}$$

Substituting in Eq. (1),

$$f(x) = 0 + (x-1)(1) + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \cdots$$

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$$

Example 12

Expand log sin x *in powers of* $(x - 2)$. **[Summer 2014]**

Solution

Let $f(x) = \log \sin x$

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Putting $a = 2$,

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \cdots \qquad \text{...(1)}$$

$$f(x) = \log \sin x, \qquad f(2) = \log \sin (2)$$

$$f'(x) = \frac{\cos x}{\sin x} = \cot x, \qquad f'(2) = \cot (2)$$

$$f''(x) = -\text{cosec}^2 x, \qquad f''(2) = -\text{cosec}^2(2)$$

$$f'''(x) = 2\,\text{cosec}^2 x \cot x, \qquad f'''(2) = 2\,\text{cosec}^2(2)\cot(2) \quad \text{and so on}$$

Substituting in Eq. (1),

$$f(x) = \log \sin (2) + (x-2)\cot(2) + \frac{(x-2)^2}{2!}\left[-\text{cosec}^2(2)\right] + \frac{(x-2)^3}{3!}\left[2\,\text{cosec}^2(2)\cot(2)\right] + \cdots$$

$$\log \sin x = \log \sin (2) + (x-2)\cot (2) - \frac{(x-2)^2}{2}\text{cosec}^2(2) + \frac{(x-2)^3}{3}\text{cosec}^2(2)\cot(2) + \cdots$$

Example 13

Expand log (cos x) *about* $\frac{\pi}{3}$.

Solution

Let $f(x) = \log(\cos x)$

By Taylor's series,

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots\ldots$$

Putting $a = \frac{\pi}{3}$,

$$f(x) = f\left(\frac{\pi}{3}\right) + \left(x - \frac{\pi}{3}\right) f'\left(\frac{\pi}{3}\right) + \frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2 f''\left(\frac{\pi}{3}\right) + \frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3 f'''\left(\frac{\pi}{3}\right) + \ldots\ldots \quad (1)$$

$$f(x) = \log(\cos x), \qquad f\left(\frac{\pi}{3}\right) = \log\left(\cos\frac{\pi}{3}\right) = \log\left(\frac{1}{2}\right) = -\log 2$$

$$f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x, \qquad f'\left(\frac{\pi}{3}\right) = -\tan\frac{\pi}{3} = -\sqrt{3}$$

$$f''(x) = -\sec^2 x, \qquad f''\left(\frac{\pi}{3}\right) = -\sec^2\frac{\pi}{3} = -4$$

$$f'''(x) = -2\sec^2 x \tan x, \qquad f'''\left(\frac{\pi}{3}\right) = -2\sec^2\frac{\pi}{3}\tan\frac{\pi}{3}$$

$$= -2(4)\sqrt{3}$$

$$= -8\sqrt{3} \quad \text{and so on}$$

Substituting in Eq. (1),

$$f(x) = -\log 2 + \left(x - \frac{\pi}{3}\right)(-\sqrt{3}) + \frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2 (-4)$$

$$+ \frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3 (-8\sqrt{3}) + \ldots$$

$$\log(\cos x) = -\log 2 - \sqrt{3}\left(x - \frac{\pi}{3}\right) - 2\left(x - \frac{\pi}{3}\right)^2 - \frac{4\sqrt{3}}{3}\left(x - \frac{\pi}{3}\right)^3 - \ldots$$

Example 14

Obtain $\tan^{-1} x$ *in powers of* $(x - 1)$.

Solution

Let $\quad f(x) = \tan^{-1} x$

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots\ldots$$

Putting $a = 1$,

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \ldots \qquad \ldots (1)$$

$$f(x) = \tan^{-1} x, \qquad f(1) = \tan^{-1} 1 = \frac{\pi}{4}$$

$$f'(x) = \frac{1}{1+x^2}, \qquad f'(1) = \frac{1}{2}$$

$$f''(x) = -\frac{2x}{(1+x^2)^2}, \qquad f''(1) = -\frac{2}{4} = -\frac{1}{2}$$

$$f'''(x) = -\frac{2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3}, \qquad f'''(1) = \frac{1}{2} \quad \text{and so on}$$

Substituting in Eq. (1),

$$f(x) = \frac{\pi}{4} + (x-1)\left(\frac{1}{2}\right) + \frac{(x-1)^2}{2!}\left(-\frac{1}{2}\right) + \frac{(x-1)^3}{3!}\left(\frac{1}{2}\right) + \ldots\ldots$$

$$\tan^{-1} x = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 + \cdots$$

Example 15

Express $7 + (x+2) + 3(x+2)^3 + (x+2)^4 - (x+2)^5$ *in ascending powers of* x.

Solution

Let $\quad f(x+2) = 7 + (x+2) + 3(x+2)^3 + (x+2)^4 - (x+2)^5$

$$f(x) = 7 + x + 3x^3 + x^4 - x^5$$

By Taylor's series,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{IV}(x)\frac{h^5}{5!}f^{V}(x) + \cdots$$

Putting $h = 2$,

$$\begin{aligned} f(x+2) &= (7 + x + 3x^3 + x^4 - x^5) + 2(1 + 9x^2 + 4x^3 - 5x^4) \\ &\quad + \frac{2^2}{2!}(18x + 12x^2 - 20x^3) + \frac{2^3}{3!}(18 + 24x - 60x^2) \\ &\quad + \frac{2^4}{4!}(24 - 120x) + \frac{2^5}{5!}(-120) \\ &= 17 - 11x - 38x^2 - 29x^3 - 9x^4 - x^5 \end{aligned}$$

Example 16

Express $(x-1)^4 + 2(x-1)^3 + 5(x-1) + 2$ *in ascending powers of x.*

[Summer 2016]

Solution

Let $f(x-1) = (x-1)^4 + 2(x-1)^3 + 5(x-1) + 2$

$$f(x) = 2 + 5x + 2x^3 + x^4$$

By Taylor's series,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x) + \cdots$$

Putting $h = -1$,

$$\begin{aligned} f(x-1) &= f(x) + (-1)f'(x) + \frac{(-1)^2}{2!}f''(x) + \frac{(-1)^3}{3!}f''' + \frac{(-1)^4}{4!}f^{iv}(x) + \cdots \\ &= (2 + 5x + 2x^3 + x^4) + (-1)(5 + 6x^2 + 4x^3) \\ &\qquad + \frac{1}{2!}(12x + 12x^2) + \frac{(-1)}{3!}(12 + 24x) + \frac{1}{4!}(24) + 0 \\ &= (2 + 5x + 2x^3 + x^4) + (-1)(5 + 6x^2 + 4x^3) + (6x + 6x^2) + (-1)(2 + 4x) + 1 \\ &= 2 + 5x + 2x^3 + x^4 - 5 - 6x^2 - 4x^3 + 6x + 6x^2 - 2 - 4x + 1 \\ &= x^4 - 2x^3 + 7x - 4 \end{aligned}$$

Example 17

Express $5+4(x-1)^2-3(x-1)^3+(x-1)^4$ *in ascending powers of x.*

[Winter 2013]

Solution

Let
$$f(x-1)=5+4(x-1)^2-3(x-1)^3+(x-1)^4$$
$$f(x)=5+4x^2-3x^3+x^4$$

By Taylor's series,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f^{iv}(x)+\cdots$$

Putting $h=-1$,

$$f(x-1)=f(x)+(-1)f'(x)+\frac{(-1)^2}{2!}f''(x)+\frac{(-1)^3}{3!}f'''(x)+\frac{(-1)^4}{4!}f^{iv}(x)+\cdots$$
$$=(5+4x^2-3x^3+x^4)+(-1)(8x-9x^2+4x^3)+\frac{(-1)^2}{2!}(8-18x+12x^2)$$
$$+\frac{(-1)^3}{3!}(-18+24x)+\frac{(-1)^4}{4!}(24)$$
$$=5+4x^2-3x^3+x^4-8x+9x^2-4x^3+4-9x+6x^2+3-4x+1$$
$$=x^4-7x^3+19x^2-21x+13$$

Example 18

Find the expansion of $\tan\left(x+\frac{\pi}{4}\right)$ *in ascending powers of x up to terms in* x^4 *and find the approximate value of* tan (43°).

[Winter 2013; Summer 2016]

Solution

Let
$$f\left(x+\frac{\pi}{4}\right)=\tan\left(x+\frac{\pi}{4}\right)$$
$$f(x)=\tan x$$

By Taylor's series,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f^{iv}(x)+\cdots$$

Putting $x = \frac{\pi}{4}, h = x,$

$$f\left(\frac{\pi}{4}+x\right) = f\left(\frac{\pi}{4}\right) + xf'\left(\frac{\pi}{4}\right) + \frac{x^2}{2!}f''\left(\frac{\pi}{4}\right) + \frac{x^3}{3!}f'''\left(\frac{\pi}{4}\right) + \frac{x^4}{4!}f^{iv}\left(\frac{\pi}{4}\right) + \cdots \quad \text{...(1)}$$

$$f(x) = \tan x, \qquad f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$$

$$f'(x) = \sec^2 x, \qquad f'\left(\frac{\pi}{4}\right) = \sec^2\frac{\pi}{4} = 2$$

$$\begin{aligned} f''(x) &= 2\sec x \cdot \sec x \tan x, \qquad f''\left(\frac{\pi}{4}\right) = 2\tan\frac{\pi}{4} + 2\tan^3\frac{\pi}{4} = 4 \\ &= 2\sec^2 x \tan x \\ &= 2(1+\tan^2 x)\tan x \\ &= 2\tan x + 2\tan^3 x \end{aligned}$$

$$\begin{aligned} f'''(x) &= 2\sec^2 x + 6\tan^2 x \sec^2 x, \qquad f'''\left(\frac{\pi}{4}\right) = 2 + 8\tan^2\frac{\pi}{4} + 6\tan^4\frac{\pi}{4} = 16 \\ &= 2(1+\tan^2 x) + 6\tan^2 x\left(1+\tan^2 x\right) \\ &= 2 + 8\tan^2 x + 6\tan^4 x \end{aligned}$$

$$\begin{aligned} f^{IV}(x) = 16\tan x \cdot \sec^2 x + 24\tan^3 x \cdot \sec^2 x, \quad f^{IV}\left(\frac{\pi}{4}\right) &= 16\tan\frac{\pi}{4}\cdot\sec^2\frac{\pi}{4} \\ &\quad + 24\tan^3\frac{\pi}{4}\cdot\sec^2\frac{\pi}{4} \\ &= 80 \quad \text{and so on} \end{aligned}$$

Substituting in Eq. (1),

$$\begin{aligned} f\left(\frac{\pi}{4}+x\right) &= 1 + x(2) + \frac{x^2}{2!}(4) + \frac{x^3}{3!}(16) + \frac{x^4}{4!}(80) + \cdots \\ \tan\left(\frac{\pi}{4}+x\right) &= 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \cdots \end{aligned} \quad \text{...(2)}$$

Now
$$\begin{aligned} \tan 43^\circ &= \tan(45^\circ - 2^\circ) \\ &= \tan\left(\frac{\pi}{4} - \frac{2\pi}{180}\right) \\ &= \tan\left(\frac{\pi}{4} - 0.0349\right) \end{aligned}$$

$$= 1 + 2(-0.0349) + 2(-0.0349)^2 + \frac{8}{3}(-0.0349)^3 + \frac{10}{3}(-0.0349)^4$$
$$= 0\cdot 9326 \text{ approx.}$$

Example 19

Prove that

$$\log[\sin(x+h)] = \log\sin x + h\cot x - \frac{h^2}{2}\operatorname{cosec}^2 x + \frac{h^3}{3}\frac{\cos x}{\sin^3 x} + \cdots\cdots$$

Solution

Let $f(x+h) = \log[\sin(x+h)]$

$f(x) = \log(\sin x)$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots\dots\dots\dots \qquad \text{...(1)}$$

$$f'(x) = \frac{1}{\sin x}\cos x = \cot x$$
$$f''(x) = -\operatorname{cosec}^2 x$$
$$f'''(x) = 2\operatorname{cosec}^2 x\cot x = \frac{2\cos x}{\sin^3 x} \text{ and so on}$$

Substituting in Eq. (1),

$$f(x+h) = \log\sin x + h\cot x - \frac{h^2}{2!}\operatorname{cosec}^2 x + \frac{h^3}{3!}\frac{2\cos x}{\sin^3 x} + \dots\dots\dots$$
$$\log[\sin(x+h)] = \log\sin x + h\cot x - \frac{h^2}{2}\operatorname{cosec}^2 x + \frac{h^3}{3}\frac{\cos x}{\sin^3 x} + \dots\dots\dots$$

Example 20

Expand $\log\cos\left(x+\frac{\pi}{4}\right)$ *using Taylor's theorem in ascending powers of x and hence find the value of* log (cos 48°) *correct up to three decimal places.*

Solution

Let $$f\left(x+\frac{\pi}{4}\right)=\log\cos\left(x+\frac{\pi}{4}\right)$$

$$f(x)=\log\cos x$$

By Taylor's series,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\cdots$$

Putting $x=\frac{\pi}{4}, h=x,$

$$f\left(\frac{\pi}{4}+x\right)=f\left(\frac{\pi}{4}\right)+xf'\left(\frac{\pi}{4}\right)+\frac{x^2}{2!}f''\left(\frac{\pi}{4}\right)+\cdots \qquad ...(1)$$

$$f(x)=\log\cos x, \qquad f\left(\frac{\pi}{4}\right)=\log\cos\frac{\pi}{4}=\log\left(\frac{1}{\sqrt{2}}\right)=-\log\sqrt{2}$$

$$f'(x)=\frac{1}{\cos x}(-\sin x), \quad f'\left(\frac{\pi}{4}\right)=-\tan\frac{\pi}{4}=-1$$

$$=-\tan x$$

$$f''(x)=-\sec^2 x, \qquad f''\left(\frac{\pi}{4}\right)=-\sec^2\left(\frac{\pi}{4}\right)=-2 \quad \text{and so on}$$

Substituting in Eq. (1),

$$f\left(\frac{\pi}{4}+x\right)=-\log\sqrt{2}+x(-1)+\frac{x^2}{2!}(-2)+\cdots$$

$$\log\cos\left(\frac{\pi}{4}+x\right)=-\log\sqrt{2}-x-x^2+\cdots \qquad ...(2)$$

Now,

$$\begin{aligned}\log(\cos 48^\circ)&=\log[\cos(45^\circ+3^\circ)]\\&=\log\left[\cos\left(\frac{\pi}{4}+\frac{3\pi}{180}\right)\right]\\&=\log\left[\cos\left(\frac{\pi}{4}+0.0523\right)\right]\\&=-\log\sqrt{2}-0.0523-(0.0523)^2\\&=-0.402 \quad \text{approx.}\end{aligned}$$

Example 21

Show that $\tan^{-1}(x+h) = \tan^{-1} x + (h \sin\alpha)\frac{\sin\alpha}{1} - (h\sin\alpha)^2 \frac{\sin 2\alpha}{2} + (h\sin\alpha)^3 \frac{\sin 3\alpha}{3} + \cdots$

where $\alpha = \cot^{-1} x$. **[Summer 2015]**

Solution

Let $f(x+h) = \tan^{-1}(x+h)$

$$f(x) = \tan^{-1} x$$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots \quad \ldots (1)$$

$$f'(x) = \frac{1}{1+x^2}$$

$$f''(x) = -\frac{2x}{(1+x^2)^2}$$

$$f'''(x) = -\frac{2}{(1+x^2)^2} + \frac{2x \cdot 4x}{(1+x^2)^3} = \frac{2(3x^2-1)}{(1+x^2)^3} \text{ and so on}$$

Putting $x = \cot\alpha$,

$$f'(\cot\alpha) = \frac{1}{1+\cot^2\alpha} = \frac{1}{\operatorname{cosec}^2\alpha} = \sin^2\alpha$$

$$f''(\cot\alpha) = -\frac{2\cot\alpha}{(\operatorname{cosec}^2\alpha)^2} = -2\sin^2\alpha\sin^2\alpha\frac{\cos\alpha}{\sin\alpha}$$

$$= -2\sin^2\alpha\sin\alpha\cos\alpha = -\sin^2\alpha\sin 2\alpha$$

$$f'''(\cot\alpha) = \frac{2(3\cot^2\alpha - 1)}{(1+\cot^2\alpha)^3} = \frac{2(3\cos^2\alpha - \sin^2\alpha)}{\operatorname{cosec}^6\alpha\cdot\sin^2\alpha}$$

$$= 2(3-4\sin^2\alpha)\sin^4\alpha = 2(3\sin\alpha - 4\sin^3\alpha)\sin^3\alpha$$

$$= 2\sin 3\alpha\sin^3\alpha$$

Substituting in Eq. (1),

$$f(x+h) = \tan^{-1} x + h\sin^2\alpha + \frac{h^2}{2!}(-\sin^2\alpha\sin 2\alpha) + \frac{h^3}{3!}(2\sin^3\alpha\sin 3\alpha) + \cdots$$

$$= \tan^{-1} x + h\sin\alpha\left(\frac{\sin\alpha}{1}\right) - (h\sin\alpha)^2\frac{\sin 2\alpha}{2} + (h\sin\alpha)^3\frac{\sin 3\alpha}{3} + \cdots$$

Example 22

Expand $\tan^{-1}(x + h)$ *in powers of h and hence, find the value of* $\tan^{-1}(1.003)$ *up to five places of decimal.*

Solution

Let $f(x + h) = \tan^{-1}(x + h)$

$f(x) = \tan^{-1} x$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots \quad \ldots (1)$$

$$f'(x) = \frac{1}{1+x^2}$$

$$f''(x) = -\frac{2x}{(1+x^2)^2}$$

$$f'''(x) = -\frac{2}{(1+x^2)^2} + \frac{2x \cdot 4x}{(1+x^2)^3} = \frac{2(3x^2-1)}{(1+x^2)^3} \text{ and so on}$$

Substituting in Eq. (1),

$$f(x+h) = \tan^{-1}(x+h) = \tan^{-1} x + h \cdot \frac{1}{1+x^2} + \frac{h^2}{2!}\left[-\frac{2x}{(1+x^2)^2}\right] + \frac{h^3}{3!}\left[\frac{2(3x^2-1)}{(1+x^2)^3}\right] + \ldots\ldots\ldots$$

Putting $x = 1, h = 0.003$,

$$\tan^{-1}(1+0.003) = \tan^{-1} 1 + \frac{0.003}{2} + \frac{(0.003)^2}{2!}\left(-\frac{2}{4}\right) + \frac{(0.003)^3}{3!}\left(\frac{1}{2}\right) + \ldots\ldots\ldots$$

$$\tan^{-1}(1.003) = \frac{\pi}{4} + 0.00015 - 2.25 \times 10^{-6} + 2.25 \times 10^{-12} \quad \text{[Considering first 4 terms]}$$

$= 0.78540$ approx.

Example 23

Prove that $\sqrt{1+x+2x^2} = 1 + \frac{x}{2} + \frac{7x^2}{8} - \frac{7x^3}{16} + \cdots$.

Solution

Let $f(x) = \sqrt{x}$

$f(x+h) = \sqrt{x+h}$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots$$

Putting $\quad x = 1,\ h = x + 2x^2,$

$$f(x+h) = \sqrt{x+h} = \sqrt{1+x+2x^2}$$

$$= f(1) + (x+2x^2) f'(1) + \frac{(x+2x^2)^2}{2!} f''(1) + \frac{(x+2x^2)^3}{3!} f'''(1) + \ldots\ldots\ldots \quad \ldots (1)$$

$$f(x) = \sqrt{x}, \qquad f(1) = 1$$

$$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(1) = \frac{1}{2}$$

$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{x^{\frac{3}{2}}}, \qquad f''(1) = -\frac{1}{4}$$

$$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{x^{\frac{5}{2}}}, \qquad f'''(1) = \frac{3}{8} \quad \text{and so on}$$

Substituting in Eq. (1),

$$\sqrt{1+x+2x^2} = 1 + \frac{1}{2}(x+2x^2) - \frac{1}{4}\frac{(x^2+4x^3+4x^4)}{2} + \frac{3}{8}\frac{(x^3+\ldots)}{6} + \ldots\ldots\ldots$$

$$= 1 + \frac{x}{2} + \frac{7x^2}{8} - \frac{7x^3}{16} + \ldots$$

Example 24

Expand $\sqrt{1+x+2x^2}$ *in powers of* $(x-1)$.

Solution

$$\sqrt{1+x+2x^2} = \sqrt{4+2(x-1)^2+5(x-1)} \qquad \text{[Expressing in terms of } (x-1)]$$

Let

$$f(x) = \sqrt{x}$$

$$f(x+h) = \sqrt{x+h}$$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots$$

Putting $x = 4$, $h = 2(x-1)^2 + 5(x-1)$,

$$f(x+h) = \sqrt{x+h} = \sqrt{4+2(x-1)^2+5(x-1)}$$

$$= f(4) + [2(x-1)^2 + 5(x-1)]f'(4) + \frac{[2(x-1)^2+5(x-1)]^2}{2!}f''(4) + \ldots \quad \ldots (1)$$

$$f(x) = \sqrt{x}, \qquad f(4) = 2$$

$$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(4) = \frac{1}{4}$$

$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{x^{\frac{3}{2}}}, \qquad f''(4) = -\frac{1}{32} \qquad \text{and so on}$$

Substituting in Eq. (1),

$$\sqrt{4+2(x-1)^2+5(x-1)} = 2 + [2(x-1)^2+5(x-1)]\left(\frac{1}{4}\right) + \frac{[2(x-1)^2+5(x-1)]^2}{2!}\left(-\frac{1}{32}\right) + \ldots\ldots$$

$$\sqrt{1+x+2x^2} = 2 + \frac{5}{4}(x-1) + \frac{7}{64}(x-1)^2 + \ldots\ldots\ldots\ldots$$

Example 25

Show that

$$\frac{1}{2}[f(x) - f(2a-x)] = (x-a)f'(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^5}{5!}f^{v}(a) + \cdots$$

Solution

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{iv}(a) + \frac{(x-a)^5}{5!}f^{v}(a) + \cdots \quad \ldots (1)$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x) + \frac{h^5}{5!}f^{v}(x) + \cdots \quad \ldots (2)$$

Now, $f(2a-x) = f[a+(a-x)]$

Putting $x = a$, $h = a - x$ in Eq. (2),

$$f[a+(a-x)] = f(a)+(a-x)f'(a)+\frac{(a-x)^2}{2!}f''(a)+\frac{(a-x)^3}{3!}f'''(a) + \frac{(a-x)^4}{4!}f^{iv}(a)+\frac{(a-x)^5}{5!}f^{v}(a)+\cdots$$

$$f(2a-x) = f(a)-(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)-\frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{iv}(a)-\frac{(x-a)^5}{5!}f^{v}(a)+\cdots \qquad ...(3)$$

From Eqs (1) and (3),

$$\frac{1}{2}[f(x)-f(2a-x)] = \frac{1}{2}\left[2(x-a)f'(a)+2\cdot\frac{(x-a)^3}{3!}f'''(a)+2\cdot\frac{(x-a)^5}{5!}f^{v}(a)+\cdots\right]$$

$$= (x-a)f'(a)+\frac{(x-a)^3}{3!}f'''(a)+\frac{(x-a)^5}{5!}f^{v}(a)+\cdots$$

Example 26

Using Taylor's theorem, evaluate up to four places of decimals:

(i) $\sqrt{1.02}$ (ii) $\sqrt{25.15}$ (iii) $\sqrt{9.12}$

(iv) $\sqrt{10}$ (v) $\sqrt{36.12}$ **[Winter 2014]**

Solution

Let $f(x) = \sqrt{x}$

$f(x+h) = \sqrt{x+h}$

By Taylor's series,

$$f(x+h) = f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots\ldots\ldots\ldots \qquad ... (1)$$

(i) Putting $x = 1$, $h = 0.02$,

$$f(x+h) = \sqrt{x+h} = \sqrt{1+0.02} = \sqrt{1.02}$$

$$= f(1)+(0.02)f'(1)+\frac{(0.02)^2}{2!}f''(1)+\ldots\ldots\ldots \qquad ... (2)$$

$f(x) = \sqrt{x}$, $f(1) = 1$

$f'(x) = \dfrac{1}{2\sqrt{x}}$, $f'(1) = \dfrac{1}{2}$

$f''(x) = -\dfrac{1}{4x^{\frac{3}{2}}}$, $f''(1) = -\dfrac{1}{4}$ and so on

Substituting in Eq. (2) and considering only first three terms,

$$\sqrt{1.02} = 1 + (0.02)\left(\frac{1}{2}\right) + \frac{(0.02)^2}{2!}\left(-\frac{1}{4}\right)$$

$$= 1.0099 \text{ approx.}$$

(ii) Putting $x = 25$, $h = 0.15$ in Eq. (1),

$$f(x+h) = \sqrt{x+h} = \sqrt{25+0.15}$$

$$= f(25) + (0.15) f'(25) + \frac{(0.15)^2}{2!} f''(25) + \dots \qquad \text{... (3)}$$

$$f(x) = \sqrt{x}, \qquad f(25) = 5$$

$$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(25) = \frac{1}{10} = 0.1$$

$$f''(x) = -\frac{1}{4x^{\frac{3}{2}}}, \qquad f''(25) = -\frac{1}{500} = -0.002 \qquad \text{and so on}$$

Substituting in Eq. (3) and considering only first three terms,

$$\sqrt{25.15} = 5 + (0.15)(0.1) + \frac{(0.15)^2}{2}(-0.002)$$

$$= 5.0150 \text{ approx.}$$

(iii) Putting $x = 9$, $h = 0.12$ in Eq. (1),

$$f(x+h) = \sqrt{x+h} = \sqrt{9+0.12}$$

$$= f(9) + (0.12) f'(9) + \frac{(0.12)^2}{2!} f''(9) + \dots \qquad \text{... (4)}$$

$$f(x) = \sqrt{x}, \qquad f(9) = 3$$

$$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(9) = \frac{1}{6}$$

$$f''(x) = -\frac{1}{4x^{\frac{3}{2}}}, \qquad f''(9) = -\frac{1}{108} \quad \text{and so on}$$

Substituting in Eq. (4) and considering only first three terms,

$$\sqrt{9.12} = 3 + (0.12)\left(\frac{1}{6}\right) + \frac{(0.12)^2}{2}\left(-\frac{1}{108}\right)$$

$$= 3 + 0.02 - (0.12)(0.06)(0.0093)$$

$$= 3.0199 \text{ approx.}$$

(iv) Putting $x = 9$, $h = 1$ in Eq. (1),

$$f(x+h) = \sqrt{x+h} = \sqrt{9+1} = f(9) + f'(9) + \frac{1}{2!} f''(9) + \dots \qquad \text{... (5)}$$

$$\sqrt{10} = 3 + \frac{1}{6} - \frac{1}{216}$$ [Refer (iii)]

$= 3.1620$ approx.

(v) Putting $x = 36$, $h = 0.12$ in Eq. (1),

$$f(x+h) = \sqrt{x+h} = \sqrt{36+0.12} = \sqrt{36.12}$$

$$= f(36) + (0.12) f'(36) + \frac{(0.12)^2}{2!} f''(36) + \cdots \quad \text{... (6)}$$

$$f(x) = \sqrt{x}, \qquad f(36) = \sqrt{36} = 6$$

$$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(36) = \frac{1}{2\sqrt{36}} = \frac{1}{12}$$

$$f''(x) = -\frac{1}{4x^{\frac{3}{2}}}, \qquad f''(36) = -\frac{1}{4(36)^{\frac{3}{2}}} = -\frac{1}{864}$$ and so on

Substituting in Eq. (6) and considering only first three terms,

$$\sqrt{36.12} = 6 + (0.12)\left(\frac{1}{12}\right) + \frac{(0.12)^2}{2!}\left(-\frac{1}{864}\right) + \cdots$$

$= 6.0099$ approx.

Example 27

Find cosh (1.505), *given* sinh (1.5) = 2.1293 *and* cosh (1.5) = 2.3524.

Solution

Let $f(x) = \cosh x$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots$$

Putting $x = 1.5$, $h = 0.005$,

$f(x+h) = \cosh(x+h) = \cosh(1.5 + 0.005)$

$$= f(1.5) + (0.005) f'(1.5) + \frac{(0.005)^2}{2!} f''(1.5) + \frac{(0.005)^3}{3!} f'''(1.5) + \cdots \quad \text{... (1)}$$

$$f(x) = \cosh x, \qquad f(1.5) = \cosh(1.5) = 2.3524$$

$$f'(x) = \sinh x, \qquad f'(1.5) = \sinh(1.5) = 2.1293$$

$$f''(x) = \cosh x, \qquad f''(1.5) = \cosh(1.5) = 2.3524$$ and so on

Substituting in Eq. (1) and considering only first three terms,

$$\begin{aligned}\cosh(1.505) &= \cosh(1.5) + (0.005)\sinh(1.5) + \frac{(0.005)^2}{2!}\cosh(1.5) + \ldots \\ &= 2.3524 + (0.005)(2.1293) + (12.5)(10^{-6})(2.3524) \\ &= 2.3631 \text{ approx.}\end{aligned}$$

Example 28

Find the approximate value of sin (30°30′).

Solution

Let $f(x) = \sin x$

$$\begin{aligned}\sin(30°30') = \sin(30° + 30') &= \sin\left(\frac{\pi}{6} + \frac{30}{60}\cdot\frac{\pi}{180}\right) \\ &= \sin\left(\frac{\pi}{6} + 0.0087\right)\end{aligned}$$

By Taylor's series,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$$

$$\sin(x+h) = \sin x + h\cos x + \frac{h^2}{2!}(-\sin x) + \cdots$$

Putting $x = \frac{\pi}{6}, h = 0.0087,$

$$\sin\left(\frac{\pi}{6} + 0.0087\right) = \sin\frac{\pi}{6} + (0.0087)\left(\cos\frac{\pi}{6}\right) + \frac{(0.0087)^2}{2!}\left(-\sin\frac{\pi}{6}\right)$$

[Considering first 3 terms]

$\sin 30°\, 30' = 0.50752$ approx.

EXERCISE 6.1

1. Expand e^x in powers of $(x - 1)$.

$$\left[\textbf{Ans.: } e\left(1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \ldots\right)\right]$$

2. Expand $2x^3 + 7x^2 + x - 1$ in powers of $(x - 2)$.

[**Ans.:** $45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$]

3. Expand $x^5 - 5x^4 + 6x^3 - 7x^2 + 8x - 9$ in powers of $(x - 1)$.

$$\left[\textbf{Ans.: } -6-3(x-1)-9(x-1)^2-4(x-1)^3+(x-1)^5\right]$$

4. Expand $x^4 - 3x^3 + 2x^2 - x + 1$ in powers of $(x - 3)$.

$$\left[\textbf{Ans.: } 16+38(x-3)+29(x-3)^2+9(x-3)^3+(x-3)^4\right]$$

5. Expand $x^3 - 2x^2 + 3x - 5$ in powers of $(x - 2)$.

$$\left[\textbf{Ans.: } 11 + 7(x-2) + 4(x-2)^2 + (x-2)^3\right]$$

6. Expand $2x^3 + 3x^2 - 8x + 7$ in terms of $(x - 2)$.

$$\left[\textbf{Ans.: } 19 + 28(x-2) + 15(x-2)^2 + 2(x-2)^3\right]$$

7. Expand $2x^3 + 5x^2 + 3x - 4$ in powers of $(x + 3)$.

$$\left[\textbf{Ans.: } -22 + 27(x+3) - 13(x+3)^2 + 2(x+3)^3\right]$$

8. Expand $\sqrt{x}$ in powers of $(x - a)$.

$$\left[\textbf{Ans.: } \sqrt{a}+\frac{(x-a)}{2\sqrt{a}}-\frac{(x-a)^2}{8a\sqrt{a}}-\ldots\right]$$

9. Expand $\sqrt{1+x+2x^2}$ in powers of $(x - 1)$.

$$\left[\textbf{Ans.: } 2+\frac{5}{4}(x-1)+\frac{7}{32}(x-1)^2+\ldots\right]$$

10. Expand $\sin x$ in powers of $(x - a)$.

$$\left[\textbf{Ans.: } \sin a+(x-a)\cos a-\frac{(x-a)^2}{2!}\sin a-\frac{(x-a)^3}{3!}\cos a+\ldots\right]$$

11. Expand $\cos x$ in powers of $\left(x-\frac{\pi}{2}\right)$.

$$\left[\textbf{Ans.: } -\left(x-\frac{\pi}{2}\right)+\frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3-\frac{1}{5!}\left(x-\frac{\pi}{2}\right)^5+\ldots\right]$$

12. Expand $\tan x$ in powers of $\left(x-\frac{\pi}{4}\right)$.

$$\left[\textbf{Ans.: } 1+2\left(x-\frac{\pi}{4}\right)+2\left(x-\frac{\pi}{4}\right)^2+\ldots\right]$$

13. Expand $\sin\left(\frac{\pi}{6}+x\right)$ in powers of x up to x^4.

$$\left[\textbf{Ans.: } \frac{1}{2}+\frac{\sqrt{3}}{2}x-\frac{1}{2}\cdot\frac{x^2}{2!}-\frac{\sqrt{3}}{2}\cdot\frac{x^3}{3!}+\frac{1}{2}\cdot\frac{x^4}{4!}+\ldots\right]$$

14. Expand $\tan\left(\frac{\pi}{4}+x\right)$ in powers of x up to x^4 and hence, find the value of tan (46° 36').

$$\left[\text{Ans.: } \left(1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4+\ldots\right), 1.0574\right]$$

15. Find the approximate value of cos 64°.

[Ans.: 0.4384]

16. Expand log x in powers of $(x-2)$.

$$\left[\text{Ans.: } \log 2+\frac{1}{2}(x-2)-\frac{1}{2!}\cdot\frac{(x-2)^2}{4}+\frac{1}{3!}\cdot\frac{(x-2)^3}{4}+\ldots\right]$$

17. Expand $\log\tan\left(\frac{\pi}{4}+x\right)$ in powers of x.

$$\left[\text{Ans.: } 2x+\frac{4}{3}x^3+\frac{4}{3}x^5+\ldots\right]$$

18. Expand $7+(x+2)+3(x+2)^3+(x+2)^4$ in powers of x.

[Ans.: $49+69x+42x^2+11x^3+x^4$]

19. Expand $17+6(x+2)+3(x+2)^3+(x+2)^4-(x+2)^5$ in powers of x.

[Ans.: $37-6x-38x^2-29x^3-9x^4-x^5$]

20. Expand $(x-2)^4-3(x-2)^3+4(x-2)^2+5$ in powers of x.

[Ans.: $61-84x+46x^2-11x^3+x^4$]

21. Expand $(x+2)^4+5(x+2)^3+6(x+2)^2+7(x+2)+8$ in powers of $(x+1)$.

$$\left[\begin{array}{l}\text{Hint: } f(x)=x^4+5x^3+6x^2+7x+8, f[(x+1)+1]=f(1)\\ \qquad +(x+1)f'(1)+\frac{(x+1)^2}{2!}f''(1)+\ldots\end{array}\right]$$

$$\left[\text{Ans.: } 27+38(x+1)+27(x+1)^2+9(x+1)^3+(x+1)^4\right]$$

22. Prove that $\sinh(x+a)=\sinh a+x\cosh a+\frac{x^2}{2!}\sinh a+\ldots$ If $\sinh(1.5)=2.1293$, $\cosh(1.5)=2.3524$, find the value of sinh (1.505).

[Ans.: 2.1411]

6.3 MACLAURIN'S SERIES

Statement If $f(x)$ is a given function of x which can be expanded in positive ascending integral powers of x then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots\ldots + \frac{x^n}{n!} f^n(0) + \ldots\ldots\ldots$$

Proof Let $f(x)$ be a function of x which can be expanded into positive ascending integral powers of x.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots\ldots\ \ldots \qquad \ldots (1)$$

Differentiating w.r.t. x successively,

$$f'(x) = a_1 + a_2 \cdot 2x + a_3 \cdot 3x^2 + a_4 \cdot 4x^3 + \ldots\ldots\ \ldots\ldots\ldots \qquad \ldots (2)$$

$$f''(x) = a_2 \cdot 2 + a_3 \cdot 6x + a_4 \cdot 12x^2 + \ldots\ldots\ \ldots\ldots\ldots \qquad \ldots (3)$$

$$f'''(x) = a_3 \cdot 6 + a_4 \cdot 24x + \ldots\ldots\ \ldots\ldots\ldots \qquad \ldots (4)$$

and so on.

Putting $x = 0$ in Eq. (1), (2), (3) and (4),

$$a_0 = f(0)$$

$$a_1 = f'(0)$$

$$a_2 = \frac{1}{2!} f''(0)$$

$$a_3 = \frac{1}{3!} f'''(0) \quad \text{and so on.}$$

Substituting a_0, a_1, a_2 and a_3 in Eq. (1),

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots\ldots + \frac{x^n}{n!} f^n(0) + \ldots\ldots\ldots$$

This is known as **Maclaurin's series.**

This series can also be written as,

$$y = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \ldots\ldots + \frac{x^n}{n!} y_n(0) + \ldots\ldots\ldots$$

Standard Expansions

Using Maclaurin's series, expansion of some standard functions can be obtained. These expansions can be directly used while solving the examples.

(i) Expansion of e^x (Exponential series)

Proof Let $y = e^x$, $y(0) = e^0 = 1$

Now, $y_n = \frac{d^n}{dx^n}(e^x) = e^x, \; y_n(0) = e^0 = 1$

Substituting in Maclaurin's series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots\dots$$

This series is known as the exponential series.

(a) Replacing x by $-x$ in the above series,

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\dots\dots$$

(b) Replacing x by ax in the above series,

$$e^{ax} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \dots\dots\dots$$

(ii) Expansion of sin x (Sine series)

Proof Let $y = \sin x, \; y(0) = \sin 0 = 0$

Now, $y_n = \frac{d^n}{dx^n}(\sin x) = \sin\left(x + \frac{n\pi}{2}\right), \; y_n(0) = \sin\left(\frac{n\pi}{2}\right)$

Putting $n = 1, 2, 3, 4, 5, \dots\dots$,

$y_1(0) = 1, \; y_2(0) = 0, \; y_3(0) = -1, \; y_4(0) = 0, \; y_5(0) = 1$, and so on.

Substituting in Maclaurin's series,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\dots\dots$$

This series is known as the sine series.

(iii) Expansion of cos x (Cosine series)

Proof Let $y = \cos x, \; y(0) = \cos 0 = 1$

Now, $y_n = \frac{d^n}{dx^n}(\cos x) = \cos\left(x + \frac{n\pi}{2}\right), \; y_n(0) = \cos\left(\frac{n\pi}{2}\right)$,

Putting $n = 1, 2, 3, 4, \dots\dots$,

$y_1(0) = 0, \; y_2(0) = -1, \; y_3(0) = 0, \; y_4(0) = 1$, and so on.

Substituting in Maclaurin's series,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\dots\dots$$

This series is known as the cosine series.

(iv) Expansion of tan x (Tangent series)

Proof Let $y = \tan x, \quad y(0) = 0$

$$y_1 = \sec^2 x = 1 + \tan^2 x = 1 + y^2, \quad y_1(0) = 1$$

$$y_2 = 2yy_1, \quad y_2(0) = 2y(0)y_1(0) = 2(0)(1) = 0$$

$$y_3 = 2y_1^2 + 2yy_2, \quad y_3(0) = 2(1)^2 + 2(0)(0) = 2$$

$$y_4 = 4y_1y_2 + 2y_1y_2 + 2yy_3 = 6y_1y_2 + 2yy_3, \quad y_4(0) = 6(1)(0) + 2(0)(2) = 0$$

$$y_5 = 6y_2^2 + 6y_1y_3 + 2y_1y_3 + 2yy_4 = 6y_2^2 + 8y_1y_3 + 2yy_4, \quad y_5(0) = 0 + 8(1)(2) + 0 = 16$$

Substituting in Maclaurin's series,

$$\tan x = x + \frac{x^3}{3!}(2) + \frac{x^5}{5!}(16) + \dots\dots\dots$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots\dots\dots$$

This series is known as the tangent series.

***Note*:** This series can also be obtained by dividing the sine and cosine series since $\tan x = \frac{\sin x}{\cos x}$.

(v) Expansion of sinh x

Proof $\sinh x = \frac{e^x - e^{-x}}{2}$

Substituting exponential series e^x and e^{-x},

$$\sinh x = \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right)}{2}$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\dots\dots$$

(vi) Expansion of cosh x

Proof $\cosh x = \frac{e^x + e^{-x}}{2}$

Substituting exponential series e^x and e^{-x},

$$\cosh x = \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right)}{2}$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\dots\dots$$

(vii) Expansion of tanh x

Proof Expansion of tanh x can be obtained by dividing the series of sinh x and cosh x.

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$= \frac{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots}{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots}$$

$$= x - \frac{x^3}{3} + \frac{2}{15}x^5 - \dots\dots\dots$$

***Note*:** This series can also be obtained by using Maclaurin's series (refer tangent series).

(viii) Expansion of log (1 + x) (Logarithmic series)

Proof Let $y = \log(1 + x)$, $y(0) = \log 1 = 0$

Now, $$y_n = \frac{d^n}{dx^n}[\log(1+x)] = (-1)^{n-1} \cdot \frac{(n-1)!}{(x+1)^n}$$

$$y_n(0) = (-1)^{n-1} \cdot (n-1)!$$

Putting $n = 1, 2, 3, 4, \dots..$

$$y_1(0) = 1,\ y_2(0) = -1,\ y_3(0) = 2! \quad \text{and so on}$$

Substituting in Maclaurin's series,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\dots\dots$$

This series is known as the logarithmic series and is valid for $-1 < x < 1$.

***Note*:** In the above series, replacing x by $-x$, we get expansion of log $(1 - x)$.

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \dots$$

(ix) Expansion of (1 + x)^m (Binomial series)

Proof Let $$y = (1+x)^m,\ y(0) = (1+0)^m = 1$$

Now, $$y_n = m(m-1)(m-2)\dots\dots\dots(m-n+1)(1+x)^{m-n}$$

$$y_n(0) = m(m-1)(m-2)\dots\dots\dots(m-n+1)$$

Putting $n = 1, 2, 3, 4, \dots..$

$$y_1(0) = m,\ y_2(0) = m(m-1),\ y_3(0) = m(m-1)(m-2) \text{ and so on}$$

Substituting in Maclaurin's series,

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \dots\dots\dots$$

This series is known as the binomial series and is valid for $-1 < x < 1$.

By Definition

Example 1

Expand 5^x *up to the first three non-zero terms of the series.*

Solution

Let $f(x) = 5^x$

By Maclaurin's series,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots\dots\dots \qquad \dots(1)$$

$$f(x) = 5^x, \qquad f(0) = 5^0 = 1$$

$$f'(x) = 5^x \log 5, \qquad f'(0) = 5^0 \log 5 = \log 5$$

$$f''(x) = 5^x (\log 5)^2, \qquad f''(0) = 5^0 (\log 5)^2 = (\log 5)^2$$

Substituting in Eq. (1),

$$5^x = 1 + x\log 5 + \frac{x^2}{2!}(\log 5)^2 + \dots\dots\dots$$

Aliter: $f(x) = 5^x = e^{\log 5^x} = e^{x \log 5}$

$$= 1 + x\log 5 + \frac{(x\log 5)^2}{2!} + \dots\dots\dots$$ [Using exponential series]

Example 2

Obtain the series $\log(1+x)$ *and find the series* $\log\left(\frac{1+x}{1-x}\right)$ *and hence, find the value of* $\log_e\left(\frac{11}{9}\right)$. **[Winter 2016]**

Solution

Let $y = \log(1+x)$

By Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \dots\dots\dots \qquad \dots(1)$$

$$y = \log(1+x), \qquad y(0) = 0$$
$$y_1 = \frac{1}{1+x}, \qquad y_1(0) = 1$$
$$y_2 = -\frac{1}{(1+x)^2}, \qquad y_2(0) = -1$$
$$y_3 = \frac{(2!)}{(1+x)^3}, \qquad y_3(0) = 2!$$
$$y_4 = -\frac{(3!)}{(1+x)^4}, \qquad y_4(0) = -(3!)$$

Substituting in Eq. (1),

$$y = 0 + x - \frac{x^2}{2!} + \frac{x^3}{3!}(2!) - \frac{x^4}{4!}(3!) + \dots\dots\dots$$
$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots\dots$$

Replacing x by $-x$,

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots\dots\dots$$

Now, $\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$$

Putting $x = \frac{1}{10}$, and considering first three terms,

$$\log_e\left(\frac{11}{9}\right) = 2\left[\frac{1}{10} + \frac{1}{3}\cdot\frac{1}{(10)^3} + \frac{1}{5}\cdot\frac{1}{(10)^5}\right] = 0.20067$$

Example 3

If $x^3 + y^3 + xy - 1 = 0$, *prove that* $y = 1 - \frac{x}{3} - \frac{26}{81}x^3 - \cdots$.

Solution

By Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \dots\dots \qquad \dots(1)$$

$$x^3 + y^3 + xy - 1 = 0 \qquad \dots(2)$$

Putting $x = 0, y(0) = 1$

Differentiating Eq. (2) w.r.t. x,

$$3x^2 + 3y^2y_1 + xy_1 + y = 0 \quad ...(3)$$

Putting $x = 0$, $y_1(0) = -\frac{1}{3}$

Differentiating Eq. (3) w.r.t. x,

$$6x + 6yy_1^2 + 3y^2y_2 + 2y_1 + xy_2 = 0 \quad ...(4)$$

Putting $x = 0$,

$$6\left(-\frac{1}{3}\right)^2 + 3y_2(0) + 2\left(-\frac{1}{3}\right) = 0$$

$$y_2(0) = 0$$

Differentiating Eq. (4) w.r.t. x,

$$6 + 6y_1^3 + 12yy_1y_2 + 3y^2y_3 + 6yy_1y_2 + 3y_2 + xy_3 = 0$$

Putting $x = 0$,

$$6 + 6\left(-\frac{1}{27}\right) + 0 + 3y_3(0) = 0$$

$$y_3(0) = \frac{-52}{27} \text{ and so on.}$$

Substituting in Eq. (1),

$$y = 1 - \frac{x}{3} + \frac{x^2}{2!}(0) + \frac{x^3}{3!}\left(-\frac{52}{27}\right) + \cdots\cdots$$

$$= 1 - \frac{x}{3} - \frac{26}{81}x^3 - \cdots\cdots$$

Example 4

If $x^3 + 2xy^2 - y^3 + x - 1 = 0$, expand y in ascending powers of x.

Solution

By Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \cdots\cdots \quad ...(1)$$

$$x^3 + 2xy^2 - y^3 + x - 1 = 0 \quad ...(2)$$

Putting $x = 0, y(0) = -1$

Differentiating Eq. (2) w.r.t. x,

$$3x^2 + 2y^2 + 4xyy_1 - 3y^2y_1 + 1 = 0 \quad ...(3)$$

Putting $x = 0$,

$$2 - 3y_1(0) + 1 = 0$$
$$y_1(0) = 1$$

Differentiating Eq. (3) w.r.t. x,

$$6x + 4yy_1 + 4yy_1 + 4xy_1^2 + 4xyy_2 - 6yy_1^2 - 3y^2y_2 = 0$$

Putting $x = 0$,

$$-8 + 6 - 3y_2(0) = 0$$
$$y_2(0) = -\frac{2}{3} \text{ and so on.}$$

Substituting in Eq. (1),

$$y = -1 + x + \frac{x^2}{2!}\left(-\frac{2}{3}\right) + \cdots\cdots$$
$$= -1 + x - \frac{x^2}{3} + \cdots\cdots$$

Example 5

If $x = y(1 + y^2)$, *prove that* $y = x - x^3 + 3x^5 + \ldots$.

Solution

By Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \frac{x^5}{5!}y_5(0) + \cdots\cdots \quad \ldots (1)$$

$$x = y(1 + y^2) \quad \ldots. (2)$$

Putting $x = 0$, $y(0) = 0$

Differentiating Eq. (2) w.r.t. x,

$$1 = y_1 + 3y^2y_1 \quad \ldots (3)$$

Putting $x = 0$,

$$1 = y_1(0)$$

$$y_1(0) = 1$$

Differentiating Eq. (3) w.r.t. x,

$$0 = y_2 + 6yy_1^2 + 3y^2y_2 \quad \ldots (4)$$

Putting $x = 0$, $y_2(0) = 0$,

Differentiating Eq. (4) w.r.t. x,

$$0 = y_3 + 12yy_1y_2 + 6y_1^3 + 6yy_1y_2 + 3y^2y_3$$

$$0 = y_3(1 + 3y^2) + 18yy_1y_2 + 6y_1^3 \quad \dots (5)$$

Putting $x = 0$,

$$0 = y_3(0) + 6$$

$$y_3(0) = -6$$

Differentiating Eq. (5) w.r.t. x,

$$0 = (1 + 3y^2)y_4 + 6yy_1y_3 + 18y_1^2 y_2 + 18yy_2^2 + 18yy_1y_3 + 18y_1^2y_2$$

$$= (1 + 3y^2)y_4 + 24yy_1y_3 + 36y_1^2 y_2 + 18yy_2^2 \quad \dots (6)$$

Putting $x = 0$, $y_4(0) = 0$,

Differentiating Eq. (6) w.r.t. x,

$$0 = (1 + 3y^2)y_5 + 6yy_1y_4 + 24y_1^2 y_3 + 24yy_2y_3 + 24yy_1y_4 + 72y_1y_2^2$$
$$+ 36{y_1}^2y_3 + 36yy_2y_3 + 18y_1{y_2}^2$$

Putting $x = 0$,

$$0 = y_5(0) + 24(-6) + 36(-6)$$

$$y_5(0) = 360 \quad \text{and so on.}$$

Substituting in Eq. (1),

$$y = 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!}(-6) + \frac{x^4}{4!} \cdot 0 + \frac{x^5}{5!} \cdot 360 + \cdots\cdots$$
$$= x - x^3 + 3x^5 + \cdots\cdots$$

By Standard Expansion

Example 1

Obtain the expansion of $\dfrac{1+x^2}{1+x^4}$.

Solution

$$\frac{1+x^2}{1+x^4} = (1+x^2)(1+x^4)^{-1}$$

$$= (1 + x^2)(1 - x^4 + x^8 - x^{12} + x^{16} - \dots)$$
$$= 1 + x^2 - x^4 - x^6 + x^8 + x^{10} - \dots$$

Example 2

If $x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots\cdots$, *prove that*

$y = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots\cdots$ *and conversely.*

Solution

$$
\begin{aligned}
x &= \log(1+y) \\
1+y &= e^x \\
y &= e^x - 1 \\
&= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots
\end{aligned}
$$

Conversely,

$$
\begin{aligned}
y &= e^x - 1 \\
e^x &= 1+y \\
x &= \log(1+y) \\
&= y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots\cdots
\end{aligned}
$$

Example 3

Expand $\sqrt{1+\sin x}$.

Solution

$$
\begin{aligned}
\sqrt{1+\sin x} &= \sin\frac{x}{2} + \cos\frac{x}{2} \\
&= \left[\frac{x}{2} - \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \cdots\right] + \left[1 - \frac{1}{2!}\left(\frac{x}{2}\right)^2 + \frac{1}{4}\left(\frac{x}{2}\right)^4 - \cdots\right] \\
&= 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \cdots
\end{aligned}
$$

Example 4

Prove that $\cos^2 x = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \cdots$.

Solution

$$
\cos^2 x = \frac{1}{2}(1+\cos 2x)
$$

$$= \frac{1}{2}\left[1 + 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right]$$

$$= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \cdots$$

Example 5

Show that $\sin^2 x = x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 + \cdots$.

Solution

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$= \frac{1}{2}\left(1 - 1 + \frac{4x^2}{2!} - \frac{16x^4}{4!} + \frac{64x^6}{6!} - \cdots\right)$$

$$= x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 - \cdots$$

Example 6

Prove that $\cosh^3 x = \frac{1}{4}\sum_{n=0}^{\infty} \frac{3^{2n} + 3}{(2n)!} x^{2n}$.

Solution

$$\cosh^3 x = \frac{1}{4}(\cosh 3x + 3\cosh x)$$

$$= \frac{1}{4}\left[\left(1 + \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} + \cdots\right) + 3\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\right]$$

$$= \frac{1}{4}\left[(1+3) + \frac{3^2 + 3}{2!}x^2 + \frac{3^4 + 3}{4!}x^4 + \cdots\right]$$

$$= \frac{1}{4}\sum_{n=0}^{\infty} \frac{3^{2n} + 3}{(2n)!} x^{2n}$$

Example 7

Prove that $\sin x \sinh x = x^2 - \frac{8}{6!}x^6 + \frac{32}{10!}x^{10} - \ldots$.

Solution

$$\sin x \sinh x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots\right)\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots\right)$$

$$= x^2 + x^6\left[\frac{2}{5!} - \frac{1}{(3!)^2}\right] + x^{10}\left[\frac{2}{9!} - \frac{2}{7!3!} + \frac{1}{(5!)^2}\right] + \cdots$$

$$= x^2 - \frac{8}{6!}x^6 + \frac{32}{10!}x^{10} - \ldots$$

Example 8

Prove that $\cos x \cosh x = 1 - \frac{2^2 x^4}{4!} + \frac{2^2 x^8}{8!} - \cdots$.

Solution

$$\cos x \cosh x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right)\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots\right)$$

$$= 1 + x^4\left(\frac{2}{4!} - \frac{1}{(2!)^2}\right) + x^8\left[\frac{2}{8!} - \frac{2}{6!2!} + \frac{1}{(4!)^2}\right] + \cdots$$

$$= 1 - \frac{2^2 x^4}{4!} + \frac{2^2 x^8}{8!} - \cdots$$

Example 9

Expand $\sin x \cosh x$ *in ascending powers of* x *up to* x^5.

Solution

$$\sin x \cosh x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$

$$= x\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!}\right) - \frac{x^3}{3!}\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{x^5}{5!}\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \cdots$$

$$= x + \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^3}{6} - \frac{x^5}{6.2} + \frac{x^5}{120} + \cdots$$

[Considering the terms only up to x^5]

$$= x + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$$

Example 10

Prove that $\log x = \log 2 + \left(\frac{x}{2} - 1\right) - \frac{1}{2}\left(\frac{x}{2} - 1\right)^2 + \frac{1}{3}\left(\frac{x}{2} - 1\right)^3 + \cdots$.

Solution

$$\log x = \log\left(2 \cdot \frac{x}{2}\right)$$

$$= \log 2 + \log \frac{x}{2}$$

$$= \log 2 + \log\left[1 + \left(\frac{x}{2} - 1\right)\right]$$

$$= \log 2 + \left(\frac{x}{2} - 1\right) - \frac{1}{2}\left(\frac{x}{2} - 1\right)^2 + \frac{1}{3}\left(\frac{x}{2} - 1\right)^3 - \dots$$

Example 11

Expand $\log(1 + x + x^2 + x^3)$ *up to* x^8.

Solution

$$\log(1 + x + x^2 + x^3) = \log[(1 + x)(1 + x^2)]$$

$$= \log(1 + x) + \log(1 + x^2)$$

$$= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \dots\right] + \left[x^2 - \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3} - \frac{(x^2)^4}{4} + \dots\right]$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3}{4}x^4 + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} - \frac{3}{8}x^8 + \dots$$

Example 12

Prove that $\log(1 + x + x^2 + x^3 + x^4) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \cdots$.

Solution

$$\log(1 + x + x^2 + x^3 + x^4) = \log\left(\frac{1 - x^5}{1 - x}\right) \qquad \text{[Using sum of G.P.]}$$

$$= \log(1 - x^5) - \log(1 - x)$$

$$= \left(-x^5 - \frac{x^{10}}{2} - \frac{x^{15}}{3} - \frac{x^{20}}{4} - \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots\right)$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \cdots$$

Example 13

Prove that $\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$.

Solution

$$\log(1+\sin x) = \sin x - \frac{\sin^2 x}{2} + \frac{\sin^3 x}{3} - \frac{\sin^4 x}{4} + \cdots$$

$$= \left(x - \frac{x^3}{3!} + \cdots\right) - \frac{1}{2}\left(x - \frac{x^3}{3!} + \cdots\right)^2 + \frac{1}{3}\left(x - \frac{x^3}{3!} + \cdots\right)^3 - \frac{1}{4}\left(x - \frac{x^3}{3!} + \cdots\right)^4 + \cdots$$

$$= x - \frac{x^2}{2} + x^3\left(-\frac{1}{6} + \frac{1}{3}\right) + x^4\left(\frac{1}{6} - \frac{1}{4}\right) + \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$$

Example 14

Prove that $\log(1+\tan x) = x - \dfrac{x^2}{2} + \dfrac{2x^3}{3} + \cdots$.

Solution

$$\log(1+\tan x) = \tan x - \frac{\tan^2 x}{2} + \frac{\tan^3 x}{3} - \cdots$$

$$= \left(x + \frac{x^3}{3} + \cdots\right) - \frac{1}{2}\left(x + \frac{x^3}{3} + \cdots\right)^2 + \frac{1}{3}(x + \cdots)^3 \cdots\cdots$$

$$= x - \frac{x^2}{2} + x^3\left(\frac{1}{3} + \frac{1}{3}\right) + \cdots$$

$$= x - \frac{x^2}{2} + \frac{2}{3}x^3 - \cdots$$

Example 15

Prove that $\log\left(\dfrac{\tan x}{x}\right) = \dfrac{x^2}{3} + \dfrac{7}{90}x^4 + \cdots$.

Solution

$$\log\left(\frac{\tan x}{x}\right) = \log\left[\frac{1}{x}\left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots\right)\right]$$

$$= \log\left(1 + \frac{x^2}{3} + \frac{2}{15}x^4 + \cdots\right)$$

$$= \left(\frac{x^2}{3} + \frac{2}{15}x^4 + \cdots\right) - \frac{1}{2}\left(\frac{x^2}{3} + \frac{2}{15}x^4 + \cdots\right)^2 + \cdots$$

$$= \frac{x^2}{3} + x^4\left(\frac{2}{15} - \frac{1}{18}\right) + \cdots$$

$$= \frac{x^2}{3} + \frac{7}{90}x^4 + \cdots$$

Example 16

Prove that $\log\left(\frac{\sinh x}{x}\right) = \frac{x^2}{6} - \frac{x^4}{180} + \ldots$.

Solution

$$\log\left(\frac{\sinh x}{x}\right) = \log\left[\frac{1}{x}\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)\right]$$

$$= \log\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots\right)$$

$$= \left(\frac{x^2}{3!} + \frac{x^4}{5!} \cdots\right) - \frac{1}{2}\left(\frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)^2 + \cdots\cdots$$

$$= x^2 + x^4\left(\frac{1}{120} - \frac{1}{72}\right) + \cdots$$

$$= \frac{x^2}{6} - \frac{x^4}{180} + \cdots\cdots$$

Example 17

Prove that $\log(x\cot x) = -\frac{x^2}{3} - \frac{7}{90}x^4 + \cdots$.

Solution

$$\log(x\cot x) = -\log\left(\frac{1}{x\cot x}\right)$$

$$= -\log\left(\frac{\tan x}{x}\right)$$

$$= -\log\left(1 + \frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right)$$

$$= -\left[\left(\frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right) - \frac{1}{2}\left(\frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right)^2 + \ldots\right]$$

$$= -\left[\frac{x^2}{3} + x^4\left(\frac{2}{15} - \frac{1}{18}\right) + \ldots\right]$$

$$= -\frac{x^2}{3} - \frac{7}{90}x^4 + \ldots$$

Example 18

Expand $[\log(1+x)]^2$ *in ascending powers of x.*

Solution

$$[\log(1+x)]^2 = [\log(1+x)]\cdot[\log(1+x)]$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)$$

$$= x\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) - \frac{x^2}{2}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)$$

$$+ \frac{x^3}{3}\left(x - \frac{x^2}{2} + \cdots\right) - \frac{x^4}{4}(x - \cdots)$$

[Considering the terms only up to x^5]

$$= x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} - \frac{x^3}{2} + \frac{x^4}{4} - \frac{x^5}{6} + \frac{x^4}{3} - \frac{x^5}{6} - \frac{x^5}{4} + \cdots$$

$$= x^2 - x^3 + \frac{11}{12}x^4 - \frac{10}{12}x^5 + \cdots$$

Example 19

Prove that $\log\left(\frac{1+e^{2x}}{e^x}\right) = \log 2 + \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{45} - \cdots$.

Solution

$$\log\left(\frac{1+e^{2x}}{e^x}\right) = \log(e^{-x} + e^x)$$

$$= \log(2\cosh x)$$

$$= \log 2 + \log\cosh x$$

$$= \log 2 + \log\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\right)$$

$$= \log 2 + \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\right) - \frac{1}{2}\left(\frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right)^2 + \frac{1}{3}\left(\frac{x^2}{2!} + \ldots\right)^3 + \ldots$$

$$= \log 2 + \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\right) - \frac{1}{2}\left(\frac{x^4}{4} + 2\cdot\frac{x^6}{48} + \ldots\right) + \frac{1}{3}\left(\frac{x^6}{8} + \ldots\right) + \ldots$$

$$= \log 2 + \frac{x^2}{2} + x^4\left(\frac{1}{24} - \frac{1}{8}\right) + x^6\left(\frac{1}{720} - \frac{1}{48} + \frac{1}{24}\right) + \ldots$$

$$= \log 2 + \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{45} - \ldots$$

Example 20

Expand $\log\left(\frac{xe^x}{e^x-1}\right)$ *in ascending powers of x up to the terms in* x^4.

Solution

$$\log\left(\frac{xe^x}{e^x-1}\right) = -\log\left(\frac{e^x-1}{xe^x}\right)$$

$$= -\log\left(\frac{1-e^{-x}}{x}\right)$$

$$= -\log\left[\frac{1}{x}\left\{1-\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\cdots\right)\right\}\right]$$

$$= -\log\left[\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{24}+\frac{x^5}{120}+\cdots\right)\right]$$

$$= -\log\left[1-\left(\frac{x}{2}-\frac{x^2}{6}+\frac{x^3}{24}-\frac{x^4}{120}+\cdots\right)\right]$$

$$= -\left[-\left(\frac{x}{2}-\frac{x^2}{6}+\frac{x^3}{24}-\frac{x^4}{120}+\cdots\right)-\frac{1}{2}\left(\frac{x}{2}-\frac{x^2}{6}+\frac{x^3}{24}-\cdots\right)^2-\frac{1}{3}\left(\frac{x}{2}-\frac{x^2}{6}+\cdots\right)^3\right.$$

$$\left.-\frac{1}{4}\left(\frac{x}{2}-\cdots\right)^4-\cdots\right] \qquad \text{[Considering the terms only up to } x^4\text{]}$$

$$= \left(\frac{x}{2}-\frac{x^2}{6}+\frac{x^3}{24}-\frac{x^4}{120}+\cdots\right)+\frac{1}{2}\left[\frac{x^2}{4}+\frac{x^4}{36}+2\left(\frac{x}{2}\right)\left(-\frac{x^2}{6}\right)+2\left(\frac{x}{2}\right)\left(\frac{x^3}{24}\right)+\cdots\right]$$

$$+\frac{1}{3}\left[\frac{x^3}{8}+3\left(\frac{x}{2}\right)^2\left(-\frac{x^2}{6}\right)+\cdots\right]+\frac{1}{4}\cdot\frac{x^4}{16}+\cdots$$

$$= \frac{x}{2}-\frac{x^2}{6}+\frac{x^3}{24}-\frac{x^4}{120}+\frac{x^2}{8}+\frac{x^4}{72}-\frac{x^3}{12}+\frac{x^4}{48}+\frac{x^3}{24}-\frac{x^4}{24}+\frac{x^4}{64}+\cdots$$

$$= \frac{x}{2}-\frac{x^2}{24}+\frac{x^4}{2880}+\cdots$$

Example 21

Prove that $\log(1+e^x) = \log 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+\cdots.$

Solution

$$\log(1+e^x) = \log\left(1+1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)$$

$$= \log\left[2\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\frac{x^4}{48}+\ldots\right)\right]$$

$$= \log 2+\log\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\frac{x^4}{48}+\ldots\right)$$

$$= \log 2+\left(\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\frac{x^4}{48}+\ldots\right)-\frac{1}{2}\left(\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\ldots\right)^2+\frac{1}{3}\left(\frac{x}{2}+\frac{x^2}{4}+\ldots\right)^3$$

$$-\frac{1}{4}\left(\frac{x}{2}+\ldots\right)^4+\ldots$$

$$= \log 2+\left(\frac{x}{2}\right)+x^2\left(\frac{1}{4}-\frac{1}{8}\right)+x^3\left(\frac{1}{12}-\frac{1}{8}+\frac{1}{24}\right)+x^4\left(\frac{1}{48}-\frac{1}{32}-\frac{1}{24}+\frac{1}{16}-\frac{1}{64}\right)+\ldots$$

$$= \log 2+\frac{x}{2}+\frac{x^2}{8}+0+\left(-\frac{1}{192}\right)x^4+\ldots$$

$$= \log 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+\ldots$$

Example 22

Prove that $\log\left[\log(1+x)^{\frac{1}{x}}\right] = -\frac{x}{2}+\frac{5x^2}{24}-\frac{x^3}{8}+\frac{251}{2880}x^4+\ldots.$

Solution

$$\log(1+x)^{\frac{1}{x}} = \frac{1}{x}\log(1+x)$$

$$= \frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots\right)$$

$$= 1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}-\cdots$$

$$= 1-\left(\frac{x}{2}-\frac{x^2}{3}+\frac{x^3}{4}-\frac{x^4}{5}+\cdots\right)$$

$$= 1-y$$

Now, $\log\left[\log(1+x)^{\frac{1}{x}}\right] = \log(1-y)$

$$= -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \ldots$$

$$= -\left(\frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4} - \frac{x^4}{5} + \ldots\right) - \frac{1}{2}\cdot\left(\frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4} - \ldots\right)^2$$

$$-\frac{1}{3}\left(\frac{x}{2} - \frac{x^2}{3} - \ldots\right)^3 - \frac{1}{4}\left(\frac{x}{2} - \frac{x^2}{3} + \ldots\right)^4 - \ldots$$

$$= -\frac{x}{2} + x^2\left(\frac{1}{3} - \frac{1}{8}\right) - x^3\left(\frac{1}{4} - \frac{1}{6} + \frac{1}{24}\right) + x^4\left(\frac{1}{5} - \frac{1}{18} - \frac{1}{8} + \frac{1}{12} - \frac{1}{64}\right) + \ldots$$

$$= -\frac{x}{2} + \frac{5x^2}{24} - \frac{x^3}{8} + \frac{251}{2880}x^4 + \ldots$$

Example 23

Expand $\left(\frac{1+e^x}{2e^x}\right)^{\frac{1}{2}}$ *up to the term containing* x^2.

Solution

$$\left(\frac{1+e^x}{2e^x}\right)^{\frac{1}{2}} = \left(\frac{1}{2}e^{-x} + \frac{1}{2}\right)^{\frac{1}{2}}$$

$$= \left[\frac{1}{2}\left(1 - x + \frac{x^2}{2!} - \cdots\right) + \frac{1}{2}\right]^{\frac{1}{2}}$$

$$= \left(1 - \frac{1}{2}x + \frac{x^2}{4} - \cdots\right)^{\frac{1}{2}}$$

$$= \left[1 - \left(\frac{x}{2} - \frac{x^2}{4} + \cdots\right)\right]^{\frac{1}{2}}$$

$$= 1 - \frac{1}{2}\left(\frac{x}{2} - \frac{x^2}{4} + \cdots\right) + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}\left(\frac{x}{2} - \frac{x^2}{4} + \cdots\right)^2 - \cdots$$

$$= 1 - \frac{x}{4} + \frac{x^2}{8} - \frac{1}{8}\cdot\frac{x^2}{4} + \cdots$$

$$= 1 - \frac{x}{4} + \frac{3}{32}x^2 + \cdots$$

Example 24

Expand $e^{\cos x}$ *up to* x^4.

Solution

$$\begin{aligned}
y &= e^{\cos x}\\
&= e^{\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right]}\\
&= e\, e^{\left(-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)}\\
&= e\left[1+\left(-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)+\frac{1}{2!}\left(-\frac{x^2}{2!}+\cdots\right)^2+\cdots\right]\\
&= e\left(1-\frac{x^2}{2!}+\frac{x^4}{24}+\frac{x^4}{8}+\cdots\right)\\
&= e\left(1-\frac{x^2}{2!}+\frac{x^4}{6}-\cdots\right)
\end{aligned}$$

Example 25

Prove that $e^x \cos x = 1 + x - \dfrac{x^3}{3} - \dfrac{x^4}{6} + \cdots$.

Solution

$$\begin{aligned}
e^x \cos x &= \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)\\
&= 1\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)+x\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)+\frac{x^2}{2!}\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\\
&\quad+\frac{x^3}{3!}\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)+\frac{x^4}{4!}\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)+\cdots\\
&= 1-\frac{x^2}{2}+\frac{x^4}{24}+x-\frac{x^3}{2}+\frac{x^2}{2}-\frac{x^4}{4}+\frac{x^3}{6}+\frac{x^4}{24}+\cdots
\end{aligned}$$

[Considering the terms only up to x^4]

$$= 1+x-\frac{x^3}{3}-\frac{x^4}{6}+\cdots$$

Example 26

Prove that $e^{x\cos x} = 1 + x + \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{11}{24}x^4 - \dfrac{x^5}{5} + \cdots$.

Solution

$$e^{x\cos x} = e^{x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)}$$

$$= 1 + \left(x - \frac{x^3}{2!} + \frac{x^5}{4!} - \cdots\right) + \frac{1}{2!}\left(x - \frac{x^3}{2!} + \cdots\right)^2 + \frac{1}{3!}\left(x - \frac{x^3}{2!} + \cdots\right)^3$$

$$+ \frac{1}{4!}\left(x - \frac{x^3}{2!} - \cdots\right)^4 + \frac{1}{5!}\left(x - \frac{x^3}{2!} - \cdots\right)^5$$

$$= 1 + x + \frac{x^2}{2} + x^3\left(-\frac{1}{2} + \frac{1}{6}\right) + x^4\left(-\frac{1}{2} + \frac{1}{24}\right) + x^5\left(\frac{1}{24} - \frac{1}{4} + \frac{1}{120}\right) + \cdots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11}{24}x^4 - \frac{x^5}{5} + \cdots$$

Example 27

Prove that $e^{x\sin x} = 1 + x^2 + \dfrac{x^4}{3} + \dfrac{x^6}{120} + \cdots$.

Solution

$$e^{x\sin x} = \left[1 + x\sin x + \frac{(x\sin x)^2}{2!} + \frac{(x\sin x)^3}{3!} + \cdots\right]$$

$$= 1 + x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) + \frac{x^2}{2!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2 + \frac{x^3}{3!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^3 + \cdots$$

$$= 1 + x^2 + x^4\left(-\frac{1}{6} + \frac{1}{2}\right) + x^6\left(\frac{1}{120} - \frac{1}{6} + \frac{1}{6}\right) + \cdots$$

$$= 1 + x^2 + \frac{x^4}{3} + \frac{x^6}{120} + \cdots$$

Example 28

Prove that $e^{e^x} = e\left(1 + x + x^2 + \dfrac{5x^3}{6} + \cdots\right)$.

Solution

$$\begin{aligned}
e^{e^x} &= e^{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)}\\
&= e\,e^{x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots}\\
&= e\left[1+\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)+\frac{1}{2!}\left(x+\frac{x^2}{2!}+\cdots\right)^2+\frac{1}{3!}(x+\ldots)^3+\cdots\right]\\
&= e\left[1+x+x^2\left(\frac{1}{2}+\frac{1}{2}\right)+x^3\left(\frac{1}{6}+\frac{1}{2}+\frac{1}{6}\right)+\cdots\right]\\
&= e\left(1+x+x^2+\frac{5}{6}x^3+\cdots\right)
\end{aligned}$$

Example 29

Expand $(1 + x)^x$ *in a series up to the term in* x^4.

Solution

$$\begin{aligned}
(1+x)^x &= e^{\log(1+x)^x}\\
&= e^{x\log(1+x)}\\
&= e^{x\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right)}\\
&= e^{\left(x^2-\frac{x^3}{2}+\frac{x^4}{3}-\cdots\right)}\\
&= 1+\left(x^2-\frac{x^3}{2}+\frac{x^4}{3}-\cdots\right)+\frac{1}{2!}\left(x^2-\frac{x^3}{2}+\frac{x^4}{3}-\cdots\right)^2+\cdots\\
&= 1+\left(x^2-\frac{x^3}{2}+\frac{x^4}{3}-\cdots\right)+\frac{1}{2}(x^4+\cdots)+\cdots
\end{aligned}$$

[Considering the terms only up to x^4]

$$= 1+x^2-\frac{x^3}{2}+\frac{5}{6}x^4+\cdots$$

Example 30

Prove that $(1+x)^{\frac{1}{x}} = e-\frac{e}{2}x+\frac{11e}{24}x^2+\cdots\cdots.$

Solution

$$(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\log(1+x)}$$

$$= e^{\frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)}$$

$$= e^{\left(1 - \frac{x}{2} + \frac{x^2}{3} - \cdots\right)}$$

$$= e e^{\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \cdots\right)}$$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \cdots\right) + \frac{1}{2!}\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right)^2 + \cdots\right]$$

$$= e\left[1 - \frac{x}{2} + x^2\left(\frac{1}{3} + \frac{1}{8}\right) + \cdots\right]$$

$$= e - \frac{e}{2}x + \frac{11e}{24}x^2 + \cdots$$

Example 31

Expand $(1 + x)^{(1+x)}$ *up to the term containing* x^3.

Solution

$$(1+x)^{(1+x)} = e^{\log(1+x)^{(1+x)}}$$

$$= e^{(1+x)\log(1+x)}$$

$$= e^{(1+x)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)}$$

$$= e^{\left(x - \frac{x^2}{2} + \frac{x^3}{3} + x^2 - \frac{x^3}{2} + \cdots\right)} \qquad \text{[Considering the terms only up to } x^3\text{]}$$

$$= e^{\left(x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right)}$$

$$= 1 + \left(x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right) + \frac{1}{2!}\left(x + \frac{x^2}{2} - \cdots\right)^2 + \frac{1}{3!}(x + \cdots)^3 + \cdots$$

[Considering the terms only up to x^3]

$$= 1 + \left(x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right) + \frac{1}{2}\left(x^2 + 2 \cdot x \cdot \frac{x^2}{2} + \cdots\right) + \frac{1}{6}\left(x^3 + \cdots\right) + \cdots$$

$$= 1 + x + x^2 + \frac{x^3}{2} + \cdots$$

Example 32

Prove that $\sin(e^x - 1) = x + \dfrac{x^2}{2} - \dfrac{5}{24}x^4 + \cdots$.

Solution

$$\sin(e^x - 1) = \sin\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$$

$$= \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \frac{1}{3!}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^3 + \cdots$$

$$= x + \frac{x^2}{2} + x^3\left(\frac{1}{6} - \frac{1}{6}\right) + x^4\left(\frac{1}{24} - \frac{1}{4}\right) + \cdots$$

$$= x + \frac{x^2}{2} - \frac{5}{24}x^4 + \cdots$$

Example 33

Prove that x cosec $x = 1 + \dfrac{x^2}{6} + \dfrac{7}{360}x^4 + \cdots$

Solution

$$x \operatorname{cosec} x = \frac{x}{\sin x}$$

$$= \frac{x}{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}$$

$$= \frac{1}{\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots\right)}$$

$$= \left[1 - \left(\frac{x^2}{6} - \frac{x^4}{120} - \cdots\right)\right]^{-1}$$

$$= \left[1 + \left(\frac{x^2}{6} - \frac{x^4}{120} - \cdots\right) + \left(\frac{x^2}{6} - \cdots\right)^2 + \cdots\right]$$

[Using $(1-x)^{-1} = 1 + x + x^2 + \cdots$]

$$= 1 + \left(\frac{x^2}{6} - \frac{x^4}{120} - \cdots\right) + \left(\frac{x^4}{36} - \cdots\right) + \cdots$$

[Considering the terms only up to x^4]

$$= 1 + \frac{x^2}{6} + \left(-\frac{1}{120} + \frac{1}{36}\right)x^4 + \cdots$$

$$= 1 + \frac{x^2}{6} + \frac{7}{360}x^4$$

Example 34

Expand $\dfrac{x}{e^x - 1}$ *up to* x^4 *and hence, prove that*

$$\frac{x}{2}\frac{e^x + 1}{e^x - 1} = 1 + \frac{x^2}{12} - \frac{x^4}{720} + \cdots.$$

Solution

$$\frac{x}{e^x - 1} = \frac{x}{\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\right) - 1\right]}$$

$$= \frac{x}{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right)}$$

$$= \left[1 + \left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots\right)\right]^{-1}$$

$$= 1 - \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \cdots\right) + \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \cdots\right)^2 - \left(\frac{x}{2} + \frac{x^2}{6} + \cdots\right)^3 + \left(\frac{x}{2} + \cdots\right)^4$$

$$= 1 - \frac{x}{2} + x^2\left(-\frac{1}{6} + \frac{1}{4}\right) + x^3\left(-\frac{1}{24} + \frac{1}{6} - \frac{1}{8}\right) + x^4\left(-\frac{1}{120} + \frac{1}{36} + \frac{1}{24} - \frac{1}{8} + \frac{1}{16}\right) + \cdots$$

$$= 1 - \frac{x}{2} + \frac{x^2}{12} + x^3(0) - \frac{x^4}{720} + \cdots \qquad \ldots(1)$$

$$\frac{x}{2}\frac{e^x + 1}{e^x - 1} = \frac{x}{2}\left(1 + \frac{2}{e^x - 1}\right)$$

$$= \frac{x}{2} + \frac{x}{e^x - 1}$$

$$= \frac{x}{2} + 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots \qquad \text{[Using Eq. (1)]}$$

$$= 1 + \frac{x^2}{12} - \frac{x^4}{720} + \cdots$$

Example 35

Prove that

$$\tan^{-1}\left(\frac{x\sin\theta}{1 - x\cos\theta}\right) = x\sin\theta + \frac{x^2}{2}\sin 2\theta + \frac{x^3}{3}\sin 3\theta + \cdots.$$

Solution

Let
$$y = \tan^{-1}\left(\frac{x\sin\theta}{1 - x\cos\theta}\right)$$

$$\tan y = \frac{x\sin\theta}{1 - x\cos\theta}$$

$$\frac{e^{iy} - e^{-iy}}{i(e^{iy} + e^{-iy})} = \frac{x\sin\theta}{1 - x\cos\theta}$$

$$\frac{e^{iy} - e^{-iy}}{e^{iy} + e^{-iy}} = \frac{ix\sin\theta}{1 - x\cos\theta}$$

Applying componendo–dividendo,

$$\frac{e^{iy}}{e^{-iy}} = \frac{1 - x(\cos\theta - i\sin\theta)}{1 - x(\cos\theta + i\sin\theta)}$$

$$e^{2iy} = \frac{1 - xe^{-i\theta}}{1 - xe^{i\theta}}$$

$$2iy = \log(1 - xe^{-i\theta}) - \log(1 - xe^{i\theta})$$

$$= \left(-xe^{-i\theta} - \frac{x^2e^{-2i\theta}}{2} - \frac{x^3e^{-3i\theta}}{3} - \ldots\right) - \left(-xe^{i\theta} - \frac{x^2e^{2i\theta}}{2} - \frac{x^3e^{3i\theta}}{3} - \ldots\right)$$

$$= x(e^{i\theta} - e^{-i\theta}) + \frac{x^2}{2}(e^{2i\theta} - e^{-2i\theta}) + \frac{x^3}{3}(e^{3i\theta} - e^{-3i\theta}) + \ldots$$

$$= x \cdot 2i\sin\theta + \frac{x^2}{2}\cdot 2i\sin 2\theta + \frac{x^3}{3}\cdot 2i\sin 3\theta + \ldots$$

$$y = x\sin\theta + \frac{x^2}{2}\sin 2\theta + \frac{x^3}{3}\sin 3\theta + \ldots$$

Example 36

Prove that $e^{ax}\cos bx = 1 + ax + \dfrac{(a^2 - b^2)}{2!}x^2 + \dfrac{a(a^2 - 3b^2)}{3!}x^3 + \cdots$

and hence, deduce $e^{x\cos\alpha}\cos(x\sin\alpha) = \sum_{n=0}^{\infty}\dfrac{x^n}{n!}\cos n\alpha.$

Solution

$e^{ax}\cos bx = e^{ax}$ · Real Part of (e^{ibx})

$= \text{RP of } e^{(a+ib)x}$

$$= \text{RP of}\left[1 + (a+ib)x + \frac{(a+ib)^2}{2!}x^2 + \frac{(a+ib)^3}{3!}x^3 + \cdots\right]$$

$$= \text{RP of}\left[1 + (a+ib)x + \frac{(a^2 - b^2 + 2aib)}{2!}x^2 + \frac{(a^3 - ib^3 + 3ia^2b - 3ab^2)}{3!}x^3 + \ldots\right]$$

$$= 1 + ax + \frac{(a^2 - b^2)}{2!}x^2 + \frac{a(a^2 - 3b^2)}{3!}x^3 + \ldots$$

Putting $a = \cos\alpha$ and $b = \sin\alpha$,

$$e^{x\cos\alpha}\cos(x\sin\alpha) = 1 + x\cos\alpha + \frac{(\cos^2\alpha - \sin^2\alpha)}{2!}x^2 + \frac{\cos^3\alpha - 3\cos\alpha\cdot\sin^2\alpha}{3!}x^3 + \ldots$$

$$= 1 + x\cos\alpha + \frac{\cos 2\alpha}{2!}x^2 + \frac{\cos^3\alpha - 3\cos\alpha(1 - \cos^2\alpha)}{3!}x^3 + \ldots$$

$$= 1 + x\cos\alpha + \frac{x^2}{2!}\cos 2\alpha + \frac{x^3}{3!}\cos 3\alpha + \ldots.$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{n!}\cos n\alpha$$

Example 37

Prove that $e^x = 1 + \tan x + \dfrac{1}{2!}\tan^2 x - \dfrac{1}{3!}\tan^3 x - \dfrac{7}{4!}\tan^4 x + \cdots.$

Solution

Let $e^x = a_0 + a_1\tan x + a_2\tan^2 x + a_3\tan^3 x + a_4\tan^4 x + \cdots\cdots\cdots$ ... (1)

$$= a_0 + a_1\left(x + \frac{x^3}{3} + \dots\right) + a_2\left(x + \frac{x^3}{3} + \dots\right)^2 + a_3\left(x + \frac{x^3}{3} + \dots\right)^3 + a_4\left(x + \frac{x^3}{3} + \dots\right)^4 + \dots\dots$$

$$= a_0 + a_1\left(x + \frac{x^3}{3} + \dots\right) + a_2\left(x^2 + \frac{2x^4}{3} + \dots\right) + a_3(x^3 + \dots) + a_4(x^4 + \dots) + \dots\dots$$

$$= a_0 + a_1 x + a_2 x^2 + \left(\frac{a_1}{3} + a_3\right)x^3 + \left(\frac{2}{3}a_2 + a_4\right)x^4 + \dots\dots \qquad \dots(2)$$

But $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\dots \qquad \dots(3)$$

From Eqs (2) and (3),

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\dots = a_0 + a_1 x + a_2 x^2 + \left(\frac{a_1}{3} + a_3\right)x^3 + \left(\frac{2}{3}a_2 + a_4\right)x^4 + \dots\dots$$

Comparing coefficients of x, x^2, x^3 and x^4 on both the sides,

$$a_0 = 1, a_1 = 1, a_2 = \frac{1}{2!} = \frac{1}{2}, \frac{a_1}{3} + a_3 = \frac{1}{3!} = \frac{1}{6}$$

$$a_3 = \frac{1}{6} - \frac{a_1}{3} = \frac{1}{6} - \frac{1}{3} = -\frac{1}{6} = -\frac{1}{3!}$$

$$\frac{2}{3}a_2 + a_4 = \frac{1}{4!} = \frac{1}{24}, \quad a_4 = \frac{1}{24} - \frac{2}{3}\cdot\frac{1}{2} = -\frac{7}{24} = -\frac{7}{4!}$$

Substituting in Eq. (1),

$$e^x = 1 + \tan x + \frac{1}{2!}\tan^2 x - \frac{1}{3!}\tan^3 x - \frac{7}{4!}\tan^4 x + \dots$$

Example 38

Find the values of a and b such that the expansion of

$\log(1+x) - \dfrac{x(1+ax)}{1+bx}$ *in ascending powers of x begins with the term* x^4 *and prove that this term is* $-\dfrac{x^4}{36}$.

Solution

Let $$f(x) = \log(1+x) - \frac{x(1+ax)}{1+bx}$$

$$=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)-(x+ax^2)(1+bx)^{-1}$$

$$=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)-(x+ax^2)(1-bx+b^2x^2-b^3x^3+\cdots)$$

$$=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)-(x-bx^2+b^2x^3-b^3x^4+ax^2-abx^3+ab^2x^4-ab^3x^5+\cdots\cdots)$$

$$=\left(-\frac{1}{2}+b-a\right)x^2+\left(\frac{1}{3}-b^2+ab\right)x^3+\left(-\frac{1}{4}+b^3-ab^2\right)x^4+\cdots\cdots$$

If the expansion begins with the term x^4, the coefficients of x^2 and x^3 must be zero.

$$-\frac{1}{2}+b-a=0,\quad b=a+\frac{1}{2}\quad \text{and}\quad \frac{1}{3}-b^2+ab=0 \qquad \ldots (1)$$

Substituting b in Eq. (1),

$$\frac{1}{3}-\left(a+\frac{1}{2}\right)^2+a\left(a+\frac{1}{2}\right)=0$$

$$\frac{1}{3}-a^2-\frac{1}{4}-a+a^2+\frac{1}{2}a=0$$

$$\frac{1}{2}a=\frac{1}{12},\quad a=\frac{1}{6}$$

$$b=\frac{1}{6}+\frac{1}{2}=\frac{4}{6}=\frac{2}{3}$$

Coefficient of $x^4=-\frac{1}{4}+b^3-ab^2=-\frac{1}{4}+\left(\frac{2}{3}\right)^3-\frac{1}{6}\left(\frac{2}{3}\right)^2=-\frac{1}{36}$

Hence, the expansion begins with the term $-\frac{x^4}{36}$.

EXERCISE 6.2

1. Expand $e^x \sec x$ in powers of x using Maclaurin's series.

   [Ans.: $1 + x + x^2 + \ldots$**]**

2. Using Maclaurin's series, prove that $e^{\sin x}=1+x+\frac{x^2}{2}+\cdots$

3. Using Maclaurin's series, prove that $a^x=1+x\log a+\frac{x^2}{2!}(\log a)^2+\cdots$

4. Prove that $\sin^2 x=x^2-\frac{x^4}{3}+\frac{2}{45}x^6+\cdots$

5. Prove that $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots$

$$\left[\text{Hint: } \sec x = \frac{1}{\cos x} = (\cos x)^{-1} = \left[1 - \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \ldots\right)\right]^{-1}\right]$$

6. Prove that $e^x \sin 2x = 2x + 2x^2 - \frac{x^3}{3} + \cdots$

7. Prove that $e^x \cos x = 1 + x - \frac{x^3}{3} + \cdots$

8. Prove that $\cos x \cosh x = 1 - \frac{2^2 x^4}{4!} + \frac{2^4 x^8}{8!} - \cdots$

9. Prove that $\sin(e^x - 1) = x + \frac{x^2}{2} - \frac{5}{24}x^4 + \cdots$

10. Prove that $\cos^n x = 1 - n \cdot \frac{x^2}{2!} + n(3n-2) \cdot \frac{x^4}{4!} - \cdots$

Hence, deduce that $\cos^3 x = 1 - \frac{3x^2}{2} + \frac{15x^4}{48} - \cdots$

11. Prove that $\sinh^3 x = \sum \frac{(3^n - 3) - [1 - (-1)^n]x^n}{8 \cdot n!}$.

12. Prove that $e^{x \sin x} = 1 + x^2 + \frac{x^4}{3} + \frac{x^6}{120} + \cdots\cdots\cdots$

13. Prove that $\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots$

14. Prove that $\log(1 - x + x^2) = -x + \frac{x^2}{2} + \frac{2x^3}{3} - \ldots$

15. Prove that $\log \cosh x = \frac{1}{2}x^2 - \frac{1}{12}x^4 + \frac{1}{45}x^6 - \ldots$

16. Prove that $\log(1 + \tan x) = x - \frac{x^2}{2} + \frac{2x^3}{3} + \cdots\cdots\cdots$

17. Prove that $\log\left(\frac{\sin x}{x}\right) = -\left(\frac{x^2}{6} + \frac{x^4}{180} + \frac{x^6}{2835} + \cdots\right)$

18. Prove that $\log\left(\frac{\tan x}{x}\right) = \frac{x^2}{3} + \frac{7}{90}x^4 + \cdots\cdots\cdots$

19. Prove that $e^x \log(1 + x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

20. Expand $\log\tan\left(\frac{\pi}{4}+x\right)$ upto x^5.

$$\left[\text{Hint}: \log\tan\left(\frac{\pi}{4}+x\right)=\log\left(\frac{1+\tan x}{1-\tan x}\right)=\log(1+\tan x)-\log(1-\tan x)\right]$$

$$\left[\text{Ans.}: 2x+\frac{4}{3}x^3+\frac{4}{3}x^5+\ldots\right]$$

21. Prove that $x=y+\frac{y^2}{2!}+\frac{y^3}{3!}+\frac{y^4}{4!}+\cdots$ if $y=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$

By Differentiation and Integration

Example 1

Prove that $\log(\sec x)=\frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{45}+\cdots$. **[Summer 2017]**

Solution

Let

$$y = \log(\sec x)$$

$$\frac{dy}{dx}=\frac{1}{\sec x}\cdot\sec x\tan x$$

$$=\tan x$$

$$=x+\frac{x^3}{3}+\frac{2}{15}x^5+\cdots\cdots \qquad \ldots (1)$$

Integrating Eq. (1),

$$y=c+\frac{x^2}{2}+\frac{x^4}{12}+\frac{2}{15}\cdot\frac{x^6}{6}+\cdots\cdots$$

$$\log(\sec x)=c+\frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{45}+\cdots\cdots$$

Putting $x=0$,

$$\log(\sec 0)=c+0$$

$$c=\log 1,\quad c=0$$

Hence,

$$\log(\sec x)=\frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{45}+\cdots\cdots$$

Example 2

Prove that $\sin^{-1}x=x+\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}+\cdots$.

Solution

Let $y = \sin^{-1} x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$= (1-x^2)^{-\frac{1}{2}}$$

$$= 1 + \frac{1}{2}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3 + \ldots$$

$$= 1 + \frac{x^2}{2} + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \ldots \quad \ldots (1)$$

Integrating Eq. (1),

$$y = c + x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \ldots\ldots$$

$$\sin^{-1} x = c + x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \ldots\ldots$$

Putting $x = 0$,

$$\sin^{-1} 0 = c$$

$$c = 0$$

Hence, $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \ldots\ldots$

Example 3

Prove that $\cos^{-1} x = \frac{\pi}{2} - \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \cdots\right)$.

Solution

Let $y = \cos^{-1} x$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Proceeding as in Example 2,

$$\cos^{-1} x = c - \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \ldots\right)$$

Putting $x = 0$,

$$\cos^{-1} 0 = c$$

$$c = \frac{\pi}{2}$$

Hence,
$$\cos^{-1} x = \frac{\pi}{2} - \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \cdots \right)$$

Example 4

Prove that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$.

Solution

Let
$$y = \tan^{-1} x$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \ldots \quad \ldots (1)$$

Integrating Eq. (1),

$$y = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

$$\tan^{-1} x = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$$

Putting $x = 0$,

$$\tan^{-1} 0 = c$$

$$c = 0$$

Hence,
$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

Example 5

Prove that $\sinh^{-1} x = x - \frac{x^3}{6} + \frac{3x^5}{40} + \cdots$.

Solution

Let
$$y = \sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$$

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{2x}{2\sqrt{x^2+1}}\right)$$

$$= \frac{1}{\sqrt{x^2+1}}$$

$$= (1+x^2)^{-\frac{1}{2}}$$

$$= 1 - \frac{1}{2}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(x^2)^2 - \ldots$$

$$= 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \ldots \qquad \ldots (1)$$

Integrating Eq. (1),

$$y = c + x - \frac{x^3}{6} + \frac{3}{8}\cdot\frac{x^5}{5} - \ldots$$

$$\sinh^{-1} x = c + x - \frac{x^3}{6} + \frac{3x^5}{40} - \ldots$$

Putting $x = 0$,

$$\sinh^{-1} 0 = c, \ c = 0$$

$$\sinh^{-1} x = x - \frac{x^3}{6} + \frac{3x^5}{40} - \ldots$$

Example 6

Prove that $\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$.

Solution

Let $y = \tanh^{-1} x = \frac{1}{2}\log\frac{(1+x)}{(1-x)} = \frac{1}{2}[\log(1+x) - \log(1-x)]$

$$\frac{dy}{dx} = \frac{1}{1-x^2}$$

$$= (1-x^2)^{-1}$$

$$= 1 + x^2 + x^4 + x^6 + \cdots \qquad \ldots (1)$$

Integrating Eq. (1),

$$y = c + x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

$$\tanh^{-1} x = c + x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

Putting $x = 0$,

$$\tanh^{-1} 0 = c, \ c = 0$$

Hence, $\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$

Example 7

If $x = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + ...$, *find y in a series of x.*

Solution

$$x = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + ...$$

$$= \cos y$$

$$y = \cos^{-1} x$$

Proceeding as in Example 3,

$$y = \frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + ... \right)$$

Example 8

Show that $\tan^{-1}\sqrt{\frac{1-x}{1+x}} = \frac{\pi}{4} - \frac{1}{2}\left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots \right)$

Solution

Let $$y = \tan^{-1}\sqrt{\frac{1-x}{1+x}}$$

Putting $x = \cos 2\theta$,

$$y = \tan^{-1}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$$

$$= \tan^{-1}\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}$$

$$= \tan^{-1} \tan\theta$$

$$= \theta$$

$$= \frac{1}{2}\cos^{-1} x$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots \right) \right]$$

By Substitution

Example 1

Expand $\sin^{-1}(3x - 4x^3)$ *in ascending powers of x.*

Solution

Let $$y = \sin^{-1}(3x - 4x^3)$$

Putting $x = \sin\theta$,

$$\begin{aligned} y &= \sin^{-1}(3\sin\theta - 4\sin^3\theta) \\ &= \sin^{-1}(\sin 3\theta) \\ &= 3\theta \\ &= 3\sin^{-1} x \\ &= 3\left(x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots\right) \end{aligned}$$

Example 2

Prove that $\sinh^{-1}(3x + 4x^3) = 3\left(x - \frac{x^3}{6} + \frac{3}{40}x^5 + \cdots\right)$.

Solution

Let $$y = \sinh^{-1}(3x + 4x^3)$$

Putting $x = \sinh\theta$,

$$\begin{aligned} y &= \sinh^{-1}(3\sinh\theta + 4\sinh^3\theta) \\ &= \sinh^{-1}(\sinh 3\theta) \\ &= 3\theta \\ &= 3\sinh^{-1} x \\ &= 3\left(x - \frac{x^3}{6} + \frac{3x^5}{40} - \cdots\right) \end{aligned}$$

Example 3

Prove that $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)$.

Solution

Let $$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Putting $x = \tan\theta$,

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$
$$= \sin^{-1}(\sin 2\theta)$$
$$= 2\theta$$
$$= 2\tan^{-1}x$$
$$= 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right)$$

Example 4

Expand $\sec^{-1}\left(\frac{1}{1-2x^2}\right)$.

Solution

Let $$y = \sec^{-1}\left(\frac{1}{1-2x^2}\right)$$

Putting $x = \sin\theta$,

$$y = \sec^{-1}\left(\frac{1}{1-2\sin^2\theta}\right)$$
$$= \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$$
$$= \sec^{-1}(\sec 2\theta)$$
$$= 2\theta$$
$$= 2\sin^{-1}x$$
$$= 2\left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots\right)$$

Example 5

Prove that $\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right) = \pi - 2\left(n\pi + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right)$.

Solution

Let $$y = \cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right) = \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$$

Putting $x = \tan\theta$,

$$y = \cos^{-1}\left(\frac{\tan^2\theta - 1}{\tan^2\theta + 1}\right)$$

$$= \cos^{-1}(-\cos 2\theta)$$

$$= \cos^{-1}[-\cos(2n\pi + 2\theta)]$$

[Considering general value of $\cos 2\theta$]

$$= \cos^{-1}[\cos\{\pi - (2n\pi + 2\theta)\}]$$

$$= \pi - 2(n\pi + \theta)$$

$$= \pi - 2(n\pi + \tan^{-1} x)$$

$$= \pi - 2\left(n\pi + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right)$$

Example 6

Prove that $\cos^{-1}[\tanh(\log x)] = \pi - 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)$.

Solution

Let

$$y = \cos^{-1}[\tanh(\log x)]$$

$$= \cos^{-1}\left(\frac{e^{\log x} - e^{-\log x}}{e^{\log x} + e^{-\log x}}\right)$$

$$= \cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right)$$

$$= \cos^{-1}\left(\frac{x^2 - 1}{x^2 + 1}\right)$$

Putting $x = \tan\theta$,

$$y = \cos^{-1}\left(\frac{\tan^2\theta - 1}{\tan^2\theta + 1}\right)$$

$$= \cos^{-1}(-\cos 2\theta)$$

$$= \cos^{-1}[\cos(\pi - 2\theta)]$$

$$= \pi - 2\theta$$

$$= \pi - 2\tan^{-1} x$$

$$= \pi - 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)$$

Example 7

Prove that $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)=\frac{1}{2}\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\dots\right).$ **[Winter 2013]**

Solution

Let $$y=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$$

Putting $x=\tan\theta$,

$$\begin{aligned}
y&=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)\\
&=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)\\
&=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)\\
&=\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)\\
&=\tan^{-1}\left(\tan\frac{\theta}{2}\right)\\
&=\frac{\theta}{2}\\
&=\frac{1}{2}\tan^{-1}x\\
&=\frac{1}{2}\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\dots\right)
\end{aligned}$$

Example 8

Prove that $\tan^{-1}\left(\frac{p-qx}{q+px}\right)=\tan^{-1}\frac{p}{q}-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots\right).$

Solution

Let $$y=\tan^{-1}\left(\frac{\frac{p}{q}-x}{1+\frac{p}{q}x}\right)$$

Putting $x=\tan\theta,\ \frac{p}{q}=\tan A$

$$y = \tan^{-1}\left(\frac{\tan A - \tan\theta}{1 + \tan A \cdot \tan\theta}\right)$$
$$= \tan^{-1}[\tan(A-\theta)]$$
$$= A - \theta$$
$$= \tan^{-1}\frac{p}{q} - \tan^{-1} x$$
$$= \tan^{-1}\frac{p}{q} - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right)$$

EXERCISE 6.3

1. Prove that $\frac{\tan^{-1} x}{1+x^2} = x - \frac{4}{3}x^3 + \frac{23}{15}x^5 - \ldots.$

2. Prove that $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right).$

3. Prove that $\tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right) = 3\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right).$

4. Prove that $\cot^{-1}\left(\frac{3x - x^3}{1-3x^2}\right) = \frac{\pi}{2} - 3\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right).$

5. Prove that $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \ldots.$

6. Prove that $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{\pi}{4} - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right).$

7. Prove that $\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right) = \frac{\pi}{4} - \frac{1}{2}\left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots\right).$

8. Prove that $\cot^{-1} x = \frac{\pi}{2} - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right).$

9. Prove that $\cos^{-1}(4x^3 - 3x) = 3\left[\frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3}{40}x^5 + \ldots\right)\right].$

10. Prove that $\sec^{-1}\left(\sqrt{1+x^2}\right) = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots.$

11. Prove that $\tan^{-1}\left(\frac{2-3x}{3+2x}\right) = \tan^{-1}\frac{2}{3} - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)$.

Points to Remember

Taylor's Series

(i) $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots + \frac{h^n}{n!}f^n(x) + \ldots$

(ii) $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a)$

$$+ \ldots + \frac{(x-a)^n}{n!}f^n(a) + \ldots$$

Maclaurin's Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots$$

List of Expansion of Some Standard Functions

(i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

(ii) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

(iii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

(iv) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$

(v) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$

(vi) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$

(vii) $\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \ldots$

(viii) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$

(ix) $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots$

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. The expansion of log (1 + x) in Maclaurin's series is

(a) $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$ (b) $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots$

(c) $x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$ (d) $x-\frac{x^2}{2}+\frac{x^3}{3!}-\frac{x^4}{4!}+\cdots$

2. The expansion of tan x in Maclaurin's series is

(a) $x+\frac{x^3}{2!}+\frac{x^5}{3!}+\cdots$ (b) $x-\frac{x^3}{3}+\frac{2}{15}x^5-\cdots$

(c) $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$ (d) $x+\frac{x^3}{3}+\frac{2}{15}x^5+\cdots$

3. The Maclaurin's series of sin x is

(a) $x-\frac{x^3}{3!}+\frac{x^5}{5!}x^5-\cdots$ (b) $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots$

(c) $1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots$ (d) $x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots$

4. The nth term in Maclaurin's series expansion is

(a) $\frac{f^n(x)}{n!}$ (b) $\frac{f^n(0)}{n!}$ (c) $\frac{f(x)}{n!}$ (d) $\frac{f(0)}{n!}$

5. Taylor's series expansion of $y=\frac{1}{x}$ about $x = 1$ is

(a) $1-(x-1)+(x-1)^2-(x-1)^3+\cdots$

(b) $1+(x-1)+(x-1)^2+(x-1)^3+\cdots$

(c) $1-(x-1)+\frac{(x-1)^2}{2!}-\frac{(x-1)^3}{3!}+\cdots$

(d) $1+(x-1)+\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+\cdots$

6. Taylor's series expansion of $y = \sin x$ about $x=\frac{\pi}{2}$ is

(a) $1-\left(x-\frac{\pi}{2}\right)^2+\left(x-\frac{\pi}{2}\right)^4-\cdots$

(b) $1-\frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2+\frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4-\cdots$

(c) $\left(x-\frac{\pi}{2}\right)-\frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3+\frac{1}{5!}\left(x-\frac{\pi}{2}\right)^5-\cdots$

(d) $\left(x-\frac{\pi}{2}\right)-\left(x-\frac{\pi}{2}\right)^3+\left(x-\frac{\pi}{2}\right)^5-\cdots$

7. Maclaurin's series of $f(x)$ is

(a) $f(x)+\frac{x}{1!}f'(x)+\frac{x^2}{2!}f''(x)+\cdots+\frac{x^n}{n!}f^n(x)+\cdots$

(b) $f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\cdots+\frac{x^n}{n!}f^n(0)+\cdots$

(c) $1+\frac{x}{1!}f'(1)+\frac{x^2}{2!}f''(1)+\cdots+\frac{x^n}{n!}f^n(1)+\cdots$

(d) $1+\frac{x}{1!}f'(x)+\frac{x^2}{2!}f''(x)+\cdots+\frac{x^n}{n!}f^n(x)+\cdots$

8. The Taylor's series expansion of log x in $(x-1)$ is

(a) $(x-1)+\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3+\cdots$

(b) $(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\cdots$

(c) $1+(x-1)+\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3+\cdots$

(d) $1-(x-1)+\frac{1}{2}(x-1)^2-\frac{1}{3}(x-1)^3+\cdots$

9. Which of the following is the coefficient of x^{11} in the expansion of e^x?

(a) $\frac{1}{11}$ (b) $11!$ (c) $-\frac{1}{11}$ (d) $\frac{1}{11!}$

10. The coefficient of x^5 in the expansion of cos x is

(a) 0 (b) $\frac{1}{5!}$ (c) $-\frac{1}{5!}$ (d) $\frac{1}{5}$

11. The coefficient of x^{100} in the expansion of log $(1-x)^2$ is

(a) $\frac{1}{100}$ (b) $-\frac{1}{100}$ (c) $-\frac{1}{50}$ (d) $\frac{1}{50}$

12. The constant term in the expansion of $7+(x+2)+3(x+2)^3+(x+2)^4$ is

(a) 40 (b) 48 (c) 49 (d) 50

13. The coefficient of x^2 in the expansion of $e^x \cos x$ is

(a) $\frac{1}{2}$ (b) -1 (c) 0 (d) $-\frac{1}{2}$

14. The coefficients of x^4 and x^5 respectively, in the expansion of $(x-2)^4 - 3(x-2)^3 + 4(x-2)^2 + 5(x-1) - 100$ are

(a) (1, 1) (b) (0, 0) (c) (0, 1) (d) (1, 0)

15. The expansion of $x^4 - 3x^3 + 2x^2 - x + 1$ about 3 is

(a) $16 + 38(x+3) + 29(x+3)^2 + 9(x+3)^3 + (x+3)^4$

(b) $16 + 38x + 29x^2 + 9x^3 + x^4$

(c) $16 + 38(x-3) + 29(x-3)^2 + 9(x-3)^3 + (x-3)^4$

(d) $16 - 38(x+3) + 29(x+3)^2 - 9(x+3)^3 + (x+3)^4$

16. The Maclaurin's series of $\sin x$ is **[Summer 2015]**

(a) $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ (b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

(c) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ (d) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

17. The Maclaurin's series of e^{-x} is **[Winter 2015]**

(a) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ (c) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ (d) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!}$

18. The series $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ represent expansion of **[Summer 2016]**

(a) $\sin x$ (b) $\cos x$ (c) $\sinh x$ (d) $\cosh x$

19. The series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ represent expansion of **[Summer 2017]**

(a) e^x (b) $\log(1+x)$ (c) $\sin x$ (d) $\cos x$

20. The coefficient of x^5 in the expansion of e^x is **[Winter 2016]**

(a) $\frac{1}{5}$ (b) $\frac{1}{4!}$ (c) $\frac{1}{5!}$ (d) 5

Answers

1. (a)	2. (d)	3. (a)	4. (b)	5. (a)	6. (b)	7. (b)	8. (b)	9. (d)
10. (a)	11. (c)	12. (c)	13. (c)	14. (d)	15. (c)	16. (b)	17. (b)	18. (c)
19. (b)	20. (c)							

UNIT-3

Chapter 7. Fourier Series

CHAPTER 7

Fourier Series

Chapter Outline

7.1 INTRODUCTION

Fourier series is used in the analysis of periodic functions. Many of the phenomena studied in engineering and sciences are periodic in nature, e.g., current and voltage in an ac circuit. These periodic functions can be analyzed into their constituent components by a Fourier analysis. The Fourier series makes use of orthogonality relationships of the sine and cosine functions. It decomposes a periodic function into a sum of sine-cosine functions. The computation and study of Fourier series is known as *harmonic analysis*. It has many applications in electrical engineering, vibration analysis, acoustics, optics, signal processing, image processing, etc.

7.2 PERIODIC FUNCTIONS

A function $f(x)$ is said to be periodic with period $T > 0$, if $f(x) = f(x + T)$ for all real x. The function $f(x)$ repeats itself after each interval of T. If $f(x) = f(x + T) = f(x + 2T) = f(x + 3T) = \ldots$ then T is called the period of the function $f(x)$.

For example, $\sin x$ is a periodic function with period 2π. Hence, $\sin x = \sin(x + 2\pi)$.

7.3 ORTHOGONALITY OF TRIGONOMETRIC SYSTEM

7.3.1 Orthogonality of Functions

A set of functions $f_1(x), f_2(x), f_3(x), \ldots, f_n(x)$ is said to be orthogonal in the interval (a, b) if

$$\int_a^b f_m(x) f_n(x)\,\mathrm{d}x = 0, \quad \text{if } m \neq n$$

$$\neq 0, \quad \text{if } m = n$$

7.3.2 Orthogonality of Trigonometric Functions

Consider a set of trigonometric functions $\sin\frac{n\pi x}{l}$ and $\cos\frac{n\pi x}{l}$, where $n = 1, 2, 3, \ldots$ in the interval $(c, c + 2l)$ for any value of c.

Let $\quad f_n(x) = \sin\frac{n\pi x}{l}, \quad n = 1, 2, 3, \ldots$

$$g_n(x) = \cos\frac{n\pi x}{l}, \quad n = 1, 2, 3, \ldots$$

Consider

(i) $\displaystyle\int_c^{c+2l} f_m(x) g_n(x)\,\mathrm{d}x$

$$= \int_c^{c+2l} \sin\frac{m\pi x}{l} \cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_c^{c+2l} \left[\sin\frac{(m+n)\pi x}{l} + \sin\frac{(m-n)\pi x}{l}\right]\mathrm{d}x$$

$$= \frac{1}{2}\left| -\left\{\cos\frac{(m+n)\pi x}{l}\right\}\frac{l}{(m+n)\pi} - \left\{\cos\frac{(m-n)nx}{l}\right\}\frac{l}{(m-n)\pi}\right|_c^{c+2l}, \quad m \neq n$$

$$= -\frac{l}{2(m+n)\pi}\left[\cos\frac{(m+n)(c+2l)\pi}{l} - \cos\frac{(m+n)c\pi}{l}\right]$$

$$- \frac{l}{2(m-n)\pi}\left[\cos\frac{(m-n)(c+2l)\pi}{l} - \cos\frac{(m-n)c\pi}{l}\right], \quad m \neq n$$

$$= -\frac{l}{2(m+n)\pi}\left[\cos\left\{\frac{(m+n)c\pi}{l} + 2(m+n)\pi\right\} - \cos\frac{(m+n)c\pi}{l}\right]$$

$$- \frac{l}{2(m-n)\pi}\left[\cos\left\{\frac{(m-n)c\pi}{l} + 2(m-n)\pi\right\} - \cos\frac{(m-n)c\pi}{l}\right], \quad m \neq n$$

$$= -\frac{l}{2(m+n)\pi}\left[\cos\frac{(m+n)c\pi}{l} - \cos\frac{(m+n)c\pi}{l}\right]$$
$$-\frac{l}{2(m-n)\pi}\left[\cos\frac{(m-n)c\pi}{l} - \cos\frac{(m-n)c\pi}{l}\right],\ m \neq n$$
$$\left[\because \cos(2n\pi+\theta) = \cos\theta\right]$$
$$= 0,\ m \neq n$$

If $m = n$,

$$\int_c^{c+2l} f_n(x) g_n(x)\,dx = \int_c^{c+2l} \sin\frac{n\pi x}{l}\cos\frac{n\pi x}{l}\,dx$$
$$= \frac{1}{2}\int_c^{c+2l} \sin\frac{2n\pi x}{l}\,dx$$
$$= \frac{1}{2}\left| -\left(\cos\frac{2n\pi x}{l}\right)\frac{l}{2n\pi}\right|_c^{c+2l}$$
$$= -\frac{l}{4n\pi}\left[\cos\frac{2n(c+2l)\pi}{l} - \cos\frac{2nc\pi}{l}\right]$$
$$= -\frac{l}{4n\pi}\left[\cos\left(\frac{2nc\pi}{l} + 4n\pi\right) - \cos\frac{2nc\pi}{l}\right]$$
$$= -\frac{l}{4n\pi}\left[\cos\frac{2nc\pi}{l} - \cos\frac{2nc\pi}{l}\right] \quad \left[\because \cos(4n\pi+\theta) = \cos\theta\right]$$
$$= 0$$

Hence, $\int_c^{c+2l} \sin\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = 0$ for all m, n

Now consider,

(ii) $\int_c^{c+2l} f_m(x) f_n(x)\,dx$

$$= \int_c^{c+2l} \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx$$
$$= \frac{1}{2}\int_c^{c+2l}\left[\cos\frac{(m-n)\pi x}{l} - \cos\frac{(m+n)\pi x}{l}\right]dx$$
$$= \frac{1}{2}\left[\left\{\sin\frac{(m-n)\pi x}{l}\right\}\frac{l}{(m-n)\pi} - \left\{\sin\frac{(m+n)\pi x}{l}\right\}\frac{l}{(m+n)\pi}\right]_c^{c+2l},\ m \neq n$$

$$= \frac{l}{2(m-n)\pi}\left[\sin\frac{(m-n)(c+2l)\pi}{l} - \sin\frac{(m-n)c\pi}{l}\right]$$
$$- \frac{l}{2(m+n)\pi}\left[\sin\frac{(m+n)(c+2l)\pi}{l} - \sin\frac{(m+n)c\pi}{l}\right], \quad m \neq n$$
$$= \frac{l}{2(m-n)\pi}\left[\sin\left\{\frac{(m-n)c\pi}{l} + 2(m-n)\pi\right\} - \sin\frac{(m-n)c\pi}{l}\right]$$
$$- \frac{l}{2(m+n)\pi}\left[\sin\left\{\frac{(m+n)c\pi}{l} + 2(m+n)\pi\right\} - \sin\frac{(m+n)c\pi}{l}\right], \quad m \neq n$$
$$= \frac{l}{2(m-n)\pi}\left[\sin\frac{(m-n)c\pi}{l} - \sin\frac{(m-n)c\pi}{l}\right]$$
$$- \frac{l}{2(m+n)\pi}\left[\sin\frac{(m+n)c\pi}{l} - \sin\frac{(m+n)c\pi}{l}\right], \quad m \neq n$$
$$[\because \sin(2n\pi + \theta) = \sin\theta]$$
$$= 0, \quad m \neq n$$

If $m = n$,

$$\int_c^{c+2l} f_n(x) f_n(x)\,\mathrm{d}x = \int_c^{c+2l} \sin^2\frac{n\pi x}{l}\,\mathrm{d}x$$
$$= \frac{1}{2}\int_c^{c+2l}\left(1 - \cos\frac{2n\pi x}{l}\right)\mathrm{d}x$$
$$= \frac{1}{2}\left| x - \left(\sin\frac{2n\pi x}{l}\right)\frac{l}{2n\pi}\right|_c^{c+2l}$$
$$= \frac{1}{2}\left[\{(c+2l) - c\} - \frac{l}{2n\pi}\left\{\sin\frac{2n(c+2l)\pi}{l} - \sin\frac{2nc\pi}{l}\right\}\right]$$
$$= \frac{1}{2}\left[2l - \frac{l}{2n\pi}\left\{\sin\left(\frac{2nc\pi}{l} + 4n\pi\right) - \sin\frac{2nc\pi}{l}\right\}\right]$$
$$= \frac{1}{2}\left[2l - \frac{l}{2n\pi}\left(\sin\frac{2nc\pi}{l} - \sin\frac{2nc\pi}{l}\right)\right]$$
$$[\because \sin(4n\pi + \theta) = \sin\theta]$$
$$= l \neq 0$$

Hence, $\displaystyle\int_c^{c+2l} \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,\mathrm{d}x = 0, \quad m \neq n$

$$= l, \quad m = n$$

Again consider,

(iii) $\int_c^{c+2l} g_m(x)g_n(x)\,\mathrm{d}x$

$$= \int_c^{c+2l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_c^{c+2l}\left[\cos\frac{(m+n)\pi x}{l} + \cos\frac{(m-n)\pi x}{l}\right]\mathrm{d}x$$

$$= \frac{1}{2}\left|\left\{\sin\frac{(m+n)\pi x}{l}\right\}\frac{l}{(m+n)\pi x} + \left\{\sin\frac{(m-n)\pi x}{l}\right\}\frac{l}{(m-n)\pi x}\right|_c^{c+2l}, \; m \neq n$$

Proceeding same as in part (ii),

$$\int_c^{c+2l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,\mathrm{d}x = 0, \;\; m \neq n$$

If $m = n$,

$$\int_c^{c+2l} g_n(x)g_n(x)\,\mathrm{d}x = \int_c^{c+2l} \cos^2\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_c^{c+2l}\left(1+\cos\frac{2n\pi x}{l}\right)\mathrm{d}x$$

Proceeding same as in part (ii),

$$\int_c^{c+2l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,\mathrm{d}x = l, \;\; m = n$$

Hence, $\int_c^{c+2l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,\mathrm{d}x = 0, \;\; m \neq n$

$$= l, \;\; m = n$$

From (i), (ii), and (iii) it is evident that the set of trigonometric functions $\sin\frac{n\pi x}{l}$ and $\cos\frac{n\pi x}{l}$, where $n = 1, 2, 3, \ldots$ is orthogonal in the interval $(c, c + 2l)$.

7.4 DIRICHLET'S CONDITIONS FOR REPRESENTATION BY A FOURIER SERIES

Representation of a function over a certain interval by a linear combination of mutually orthogonal functions is called *Fourier series representation*.

A function $f(x)$ can be represented by a complete set of orthogonal functions within the interval $(c, c + 2l)$. The Fourier series of the function $f(x)$ exists only if the following conditions are satisfied:

(i) $f(x)$ is periodic, i.e., $f(x) = f(x + 2l)$, where $2l$ is the period of the function $f(x)$.

(ii) $f(x)$ and its integrals are finite and single-valued.

(iii) $f(x)$ has a finite number of discontinuities, i.e., $f(x)$ is piecewise continuous in the interval $(c, c + 2l)$.

(iv) $f(x)$ has a finite number of maxima and minima.

These conditions are known as *Dirichlet's conditions*.

7.5 TRIGONOMETRIC FOURIER SERIES

We know that the set of functions $\sin\frac{n\pi x}{l}$ and $\cos\frac{n\pi x}{l}$ are orthogonal in the interval $(c, c + 2l)$ for any value of c, where $n = 1, 2, 3, \ldots$.

i.e., $$\int_c^{c+2l} \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = 0 \qquad m \neq n$$
$$= l \qquad m = n$$

$$\int_c^{c+2l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = 0 \qquad m \neq n$$
$$= l \qquad m = n$$

$$\int_c^{c+2l} \sin\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = 0 \text{ for all } m, n$$

Hence, any function $f(x)$ can be represented in terms of these orthogonal functions in the interval $(c, c + 2l)$ for any value of c.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

This series is known as a *trigonometric Fourier series* or simply, a *Fourier series*. For example, a square function can be constructed by adding orthogonal sine components (Fig. 7.1).

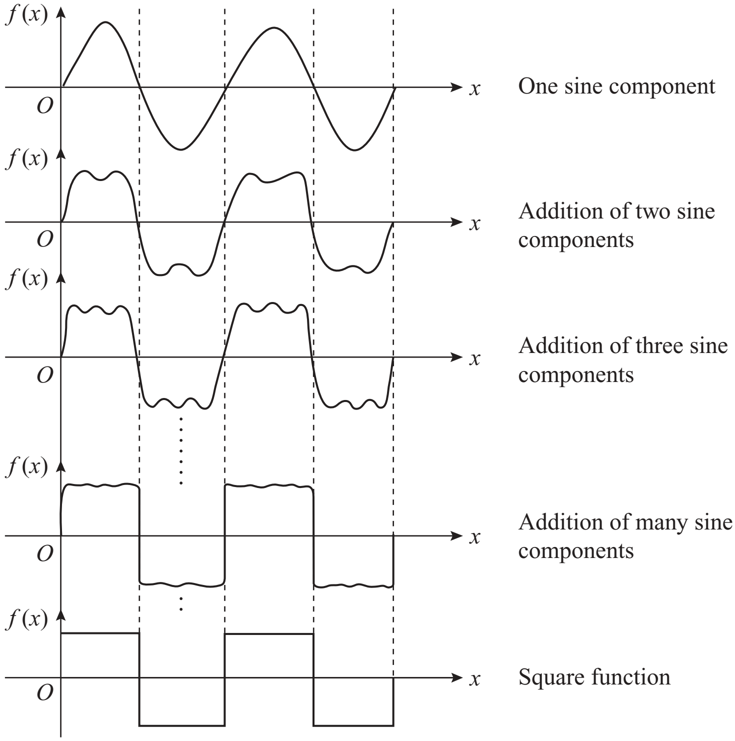


Fig. 7.1 ***Representation of a function in terms of sine components***

7.6 FOURIER SERIES OF FUNCTIONS OF PERIOD 2*l*

Let $f(x)$ be a periodic function with period $2l$ in the interval $(c, c + 2l)$. Then the Fourier series of $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots(1)$$

Determination of a_0

Integrating both the sides of Eq. (1) w.r.t. x in the interval $(c, c + 2l)$,

$$\int_c^{c+2l} f(x)\mathrm{d}x = a_0\int_c^{c+2l}\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}\right)\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\right)\mathrm{d}x$$

$$= a_0(c + 2l - c) + 0 + 0$$

$$= 2la_0$$

Hence,
$$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\mathrm{d}x \qquad \ldots(2)$$

Determination of a_n

Multiplying both the sides of Eq. (1) by $\cos\frac{n\pi x}{l}$ and integrating w.r.t. x in the interval $(c, c + 2l)$,

$$\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = a_0\int_c^{c+2l}\cos\frac{n\pi x}{l}\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}\right)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$+\int_c^{c+2l}\left(\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}\right)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$= 0 + la_n + 0$$

$$= l\, a_n$$

Hence,
$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x \qquad \ldots(3)$$

Determination of b_n

Multiplying both the sides of Eq. (1) by $\sin\frac{n\pi x}{l}$ and integrating w.r.t. x in the interval $(c, c + 2l)$,

$$\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x = a_0\int_c^{c+2l}\sin\frac{n\pi x}{l}\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}\right)\sin\frac{n\pi x}{l}\mathrm{d}x$$

$$+\int_c^{c+2l}\left(\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}\right)\sin\frac{n\pi x}{l}\mathrm{d}x$$

$$= 0 + 0 + lb_n$$

$$= l\, b_n$$

Hence,
$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x \qquad \ldots(4)$$

The formulae (2), (3), and (4) are known as *Euler's formulae* which give the values of coefficients a_0, a_n, and b_n. These coefficients are known as *Fourier coefficients*.

Corollary 1 **When $c = 0$ and $2l = 2\pi$**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

where
$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$

Corollary 2 **When $c = -\pi$ and $2l = 2\pi$**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Corollary 3 **When $c = 0$**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where
$$a_0 = \frac{1}{2l}\int_{0}^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{0}^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{0}^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Corollary 4 **When $c = -l$**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where
$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Fourier Series Expansion with Period 2π

Example 1

Find the Fourier series of $f(x) = x$ in the interval $(0, 2\pi)$.

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} x\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left|\frac{x^2}{2}\right|_0^{2\pi} \\
&= \frac{1}{2\pi}\left(\frac{4\pi^2}{2}\right) \\
&= \pi
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left| x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{2\pi} \\
&= \frac{1}{\pi}\left(\frac{\cos 2n\pi}{n^2} - \frac{\cos 0}{n^2}\right) && [\because\ \sin 2n\pi = \sin 0 = 0] \\
&= 0 && [\because\ \cos 2n\pi = \cos 0 = 1]
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left| x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{2\pi} \\
&= \frac{1}{\pi}\left[-2\pi\left(\frac{\cos 2n\pi}{n}\right)\right] && [\because\ \sin 2n\pi = \sin 0 = 0] \\
&= -\frac{2}{n} && [\because\ \cos 2n\pi = 1]
\end{aligned}$$

Hence, $$f(x) = \pi - 2\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$x = \pi - 2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right)$$

Example 2

Find the Fourier series of $f(x) = x^2$ *in the interval* $(0, 2\pi)$ *and, hence, deduce that* $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \cdots$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} x^2\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left|\frac{x^3}{3}\right|_0^{2\pi} \\
&= \frac{1}{2\pi}\left(\frac{8\pi^3}{3}\right) \\
&= \frac{4\pi^2}{3}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left|x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_0^{2\pi} \\
&= \frac{1}{\pi}\left[4\pi\left(\frac{\cos 2n\pi}{n^2}\right)\right] && [\because\ \sin 2n\pi = \sin 0 = 0] \\
&= \frac{1}{\pi}\left(\frac{4\pi}{n^2}\right) && [\because\ \cos 2n\pi = 1] \\
&= \frac{4}{n^2}
\end{aligned}$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left| x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right|_0^{2\pi} \\
&= \frac{1}{\pi}\left[4\pi^2\left(-\frac{\cos 2n\pi}{n}\right) + 2\left(\frac{\cos 2n\pi}{n^3}\right) - 2\left(\frac{\cos 0}{n^3}\right)\right] \\
&= \frac{1}{\pi}\left(-\frac{4\pi^2}{n}\right) \qquad [\because\ \cos 2n\pi = \cos 0 = 1] \\
&= -\frac{4\pi}{n}
\end{aligned}
$$

Hence,

$$f(x) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx - 4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$
\begin{aligned}
x^2 &= \frac{4\pi^2}{3} + 4\left(\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots\right) \\
&\quad - 4\pi\left(\frac{1}{1}\sin x + \frac{1}{2}\sin x + \frac{1}{3}\sin 3x + \cdots\right) \qquad \text{... (1)}
\end{aligned}
$$

Putting $x = \pi$ in Eq. (1),

$$
\begin{aligned}
\pi^2 &= \frac{4\pi^2}{3} + 4\left(\frac{1}{1^2}\cos\pi + \frac{1}{2^2}\cos 2\pi + \frac{1}{3^2}\cos 3\pi + \cdots\right) + 0 \\
&= \frac{4\pi^2}{3} + 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \ldots\right) \\
\frac{\pi^2}{12} &= \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots
\end{aligned}
$$

Example 3

Find the Fourier series of $f(x) = \frac{1}{2}(\pi - x)$ *in the interval* $(0, 2\pi)$.

Hence, deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ **[Winter 2013]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\mathrm{d}x \\
&= \frac{1}{4\pi}\left|\pi x - \frac{x^2}{2}\right|_0^{2\pi} \\
&= \frac{1}{4\pi}(2\pi^2 - 2\pi^2) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{2\pi} \\
&= \frac{1}{2\pi}\left[-\frac{\cos 2n\pi}{n^2} + \frac{\cos 0}{n^2}\right] \\
&= 0 \qquad [\because\ \cos 2n\pi = \cos 0 = 1]
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left|(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{2\pi} \\
&= \frac{1}{2\pi}\left[(-\pi)\left(-\frac{\cos 2n\pi}{n}\right) - \pi\left(-\frac{\cos 0}{n}\right)\right] \qquad [\because\ \sin 2n\pi = \sin 0 = 0] \\
&= \frac{1}{2\pi}\left(\frac{\pi}{n} + \frac{\pi}{n}\right) \qquad [\because\ \cos 2n\pi = \cos 0 = 1] \\
&= \frac{1}{n}
\end{aligned}
$$

Hence, $$f(x) = \sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$
\frac{1}{2}(\pi - x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x + \frac{1}{6}\sin 6x + \frac{1}{7}\sin 7x + \cdots \qquad \text{...(1)}
$$

Putting $x = \frac{\pi}{2}$ in Eq. (1),

$$\frac{1}{2}\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \frac{1}{2}\sin\pi + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{4}\sin 2\pi + \frac{1}{5}\sin\frac{5\pi}{2} + \frac{1}{6}\sin 3\pi + \frac{1}{7}\sin\frac{7\pi}{2} + \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Example 4

Obtain the Fourier series of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ *in the interval* $0 \le x \le 2\pi$.

Hence, deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$ **[Winter 2014]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 dx \\
&= \frac{1}{8\pi}\left|\frac{(\pi - x)^3}{-3}\right|_0^{2\pi} \\
&= -\frac{1}{24\pi}(-\pi^3 - \pi^3) \\
&= \frac{\pi^2}{12}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \cos nx\,dx \\
&= \frac{1}{4\pi}\left|(\pi - x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\cos nx}{n^2}\right) + 2(-1)(-1)\left(-\frac{\sin nx}{n^3}\right)\right|_0^{2\pi}
\end{aligned}$$

$$= \frac{1}{4\pi}\left[2\pi\left(\frac{\cos 2n\pi}{n^2}\right) - \left\{-2\pi\left(\frac{\cos 0}{n^2}\right)\right\}\right] \quad [\because \sin 2n\pi = \sin 0 = 0]$$

$$= \frac{1}{4\pi}\left(\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right) \quad [\because \cos 2n\pi = \cos 0 = 1]$$

$$= \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \sin nx\,dx$$

$$= \frac{1}{4\pi}\left|(\pi - x)^2\left(-\frac{\cos nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\sin nx}{n^2}\right) + 2(-1)(-1)\left(\frac{\cos nx}{n^3}\right)\right|_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left\{\pi^2\left(-\frac{\cos 2n\pi}{n}\right) + \frac{2\cos 2n\pi}{n^3}\right\} - \left\{\pi^2\left(-\frac{\cos 0}{n}\right) + 2\left(\frac{\cos 0}{n^3}\right)\right\}\right]$$

$$[\because \sin 2n\pi = \sin 0 = 0]$$

$$= \frac{1}{4\pi}\left(-\frac{\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right) \quad [\because \cos 2n\pi = \cos 0 = 1]$$

$$= 0$$

Hence, $f(x) = \dfrac{\pi^2}{12} + \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx$

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots \qquad \text{....(1)}$$

Putting $x = \pi$ in Eq. (1),

$$0 = \frac{\pi^2}{12} - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

Example 5

Find the Fourier series for $f(x) = e^{ax}$ in $(0, 2\pi)$, $a > 0$. **[Summer 2018]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x$$
$$= \frac{1}{2\pi}\int_0^{2\pi} e^{ax}\mathrm{d}x$$
$$= \frac{1}{2\pi}\left|\frac{e^{ax}}{a}\right|_0^{2\pi}$$
$$= \frac{1}{2a\pi}(e^{2a\pi}-1)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x$$
$$= \frac{1}{\pi}\int_0^{2\pi} e^{ax}\cos nx\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx)\right|_0^{2\pi}$$
$$= \frac{1}{\pi}\left[\frac{e^{2a\pi}}{a^2+n^2}(a\cos 2n\pi) - \frac{a}{a^2+n^2}\right] \quad \left[\begin{array}{l}\because \sin 2n\pi = \sin 0 = 0\\ \cos 0 = 1\end{array}\right]$$
$$= \frac{a}{\pi(a^2+n^2)}(e^{2a\pi}-1) \qquad [\because \cos 2n\pi = 1]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$
$$= \frac{1}{\pi}\int_0^{2\pi} a^{ax}\sin nx\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right|_0^{2\pi}$$
$$= \frac{1}{\pi}\left[\frac{e^{2a\pi}}{a^2+n^2}(-n\cos 2n\pi) + \frac{n}{a^2+n^2}\right] \quad \left[\begin{array}{l}\because \sin 2n\pi = \sin 0 = 0\\ \cos 0 = 1\end{array}\right]$$
$$= \frac{n}{\pi(a^2+n^2)}(1-e^{2a\pi}) \qquad [\because \cos 2n\pi = 1]$$

Hence,
$$f(x) = \frac{1}{2a\pi}(e^{2a\pi}-1) + \frac{a(e^{2a\pi}-1)}{\pi}\sum_{n=1}^{\infty}\frac{1}{a^2+n^2}\cos nx + \frac{1-e^{2a\pi}}{\pi}\sum_{n=1}^{\infty}\frac{n}{a^2+n^2}\sin nx$$

Example 6

Find the Fourier series of $f(x)=\dfrac{3x^2-6x\pi+2\pi^2}{12}$ *in the interval* $(0, 2\pi)$

Hence, deduce that $\dfrac{\pi^2}{6}=1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\ldots$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x\\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{3x^2-6x\pi+2\pi^2}{12}\mathrm{d}x\\
&= \frac{1}{24\pi}\left|3\left(\frac{x^3}{3}\right)-6\pi\left(\frac{x^2}{2}\right)+2\pi^2x\right|_0^{2\pi}\\
&= \frac{1}{24\pi}\left[3\left(\frac{8\pi^3}{3}\right)-6\pi\left(\frac{4\pi^2}{2}\right)+4\pi^3\right]\\
&= 0
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x\\
&= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3x^2-6x\pi+2\pi^2}{12}\right)\cos nx\,\mathrm{d}x\\
&= \frac{1}{12\pi}\left|(3x^2-6x\pi+2\pi^2)\left(\frac{\sin nx}{n}\right)-(6x-6\pi)\left(-\frac{\cos nx}{n^2}\right)+6\left(-\frac{\sin nx}{n^3}\right)\right|_0^{2\pi}\\
&= \frac{1}{12\pi}\left[(6\pi)\left(\frac{\cos 2n\pi}{n^2}\right)-(-6\pi)\left(\frac{\cos 0}{n^2}\right)\right] \qquad [\because\ \sin 2n\pi=\sin 0=0]\\
&= \frac{1}{12\pi}\left(\frac{6\pi}{n^2}+\frac{6\pi}{n^2}\right) \qquad [\because\ \cos 2n\pi=\cos 0=1]\\
&= \frac{1}{n^2}
\end{aligned}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3x^2 - 6x\pi + 2\pi^2}{12}\right)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{12\pi}\left|(3x^2 - 6x\pi + 2\pi^2)\left(-\frac{\cos nx}{n}\right) - (6x - 6\pi)\left(-\frac{\sin nx}{n^2}\right) + 6\left(\frac{\cos nx}{n^3}\right)\right|_0^{2\pi}$$

$$= \frac{1}{12\pi}\left[(12\pi^2 - 12\pi^2 + 2\pi^2)\left(-\frac{\cos 2n\pi}{n}\right) + 6\left(\frac{\cos 2n\pi}{n^3}\right) - (2\pi)^2\left(-\frac{\cos 0}{n}\right) - 6\left(\frac{\cos 0}{n^3}\right)\right] \quad [\because \sin 2n\pi = \sin 0 = 0]$$

$$= 0 \qquad [\because \cos 2n\pi = \cos 0 = 1]$$

Hence, $$f(x) = \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

$$\frac{3x^2 - 6x\pi + 2\pi^2}{12} = \cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$\frac{\pi^2}{6} = \cos 0 + \frac{1}{2^2}\cos 0 + \frac{1}{3^2}\cos 0 + \cdots$$

$$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Example 7

Find the Fourier series of $f(x) = e^{-x}$ in the interval $(0, 2\pi)$.

Hence, deduce that $\dfrac{\pi}{2}\dfrac{1}{\sinh\pi} = \sum_{n=2}^{\infty}\dfrac{(-1)^n}{n^2+1}$. **[Summer 2014]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-x}\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|-e^{-x}\right|_0^{2\pi}$$

$$= \frac{-e^{-2\pi} + e^0}{2\pi}$$

$$= \frac{1 - e^{-2\pi}}{2\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{e^{-x}}{n^2+1}(-\cos nx + n\sin nx)\right|_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{n^2+1}(-\cos 2n\pi) - \frac{1}{n^2+1}(-\cos 0)\right] \qquad [\because \sin 2n\pi = \sin 0 = 0]$$

$$= \frac{1}{\pi(n^2+1)}(1 - e^{-2\pi}) \qquad [\because \cos 2n\pi = \cos 0 = 1]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{e^{-x}}{n^2+1}(-\sin nx - n\cos nx)\right|_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{n^2+1}(-n\cos 2n\pi) - \frac{1}{n^2+1}(-n\cos 0)\right] \qquad [\because \sin 2n\pi = \sin 0 = 0]$$

$$= \frac{n}{\pi(n^2+1)}(1 - e^{-2\pi}) \qquad [\because \cos 2n\pi = \cos 0 = 1]$$

Hence, $$f(x) = \frac{1 - e^{-2\pi}}{2\pi} + \frac{1 - e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2+1}\cos nx + \frac{1 - e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{n}{n^2+1}\sin nx \qquad \ldots (1)$$

Putting $x = \pi$ in Eq. (1),

$$f(\pi) = \frac{1 - e^{-2\pi}}{2\pi} + \frac{1 - e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1} \qquad [\because \cos n\pi = (-1)^n, \sin n\pi = 0]$$

$$e^{-\pi} = \frac{1 - e^{-2\pi}}{2\pi} + \frac{1 - e^{-2\pi}}{\pi}\left[-\frac{1}{2} + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}\right]$$

$$= \frac{1 - e^{-2\pi}}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$\frac{\pi}{e^{\pi}(1-e^{-2\pi})}=\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$\frac{\pi}{e^{\pi}-e^{-\pi}}=\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Hence, $$\frac{\pi}{2}\frac{1}{\sinh\pi}=\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Example 8

Find the Fourier series of $f(x)=\sqrt{1-\cos x}$ *in the interval* $(0, 2\pi)$. *Hence, deduce that* $\frac{1}{2}=\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$.

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$f(x)=\sqrt{1-\cos x}=\sqrt{2}\sin\frac{x}{2}$$

$$\begin{aligned}a_0&=\frac{1}{2\pi}\int_0^{2\pi}f(x)\mathrm{d}x\\&=\frac{1}{2\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\mathrm{d}x\\&=\frac{\sqrt{2}}{2\pi}\left|-2\cos\frac{x}{2}\right|_0^{2\pi}\\&=\frac{\sqrt{2}}{2\pi}(-2\cos\pi+2\cos 0)\\&=\frac{2\sqrt{2}}{\pi}\qquad[\because\ \cos\pi=-1, \cos 0=1]\end{aligned}$$

$$\begin{aligned}a_n&=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,\mathrm{d}x\\&=\frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\cos nx\,\mathrm{d}x\\&=\frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\sin\left(\frac{2n+1}{2}\right)x-\sin\left(\frac{2n-1}{2}\right)x\right]\mathrm{d}x\end{aligned}$$

$$= \frac{\sqrt{2}}{2\pi}\left| -\frac{2}{2n+1}\cos\left(\frac{2n+1}{2}\right)x + \frac{2}{2n-1}\cos\left(\frac{2n-1}{2}\right)x \right|_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2\pi}\left[-\frac{2}{2n+1}\cos(2n\pi+\pi) + \frac{2\cos 0}{2n+1} + \frac{2}{2n-1}\cos(2n\pi-\pi) - \frac{2\cos 0}{2n-1} \right]$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{4}{2n+1} - \frac{4}{2n-1} \right] \qquad [\because \cos(2n+1)\pi = \cos(2n-1)\pi = -1, \cos 0 = 1]$$

$$= -\frac{4\sqrt{2}}{\pi}\frac{1}{4n^2-1}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\sin nx \, dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\cos\left(\frac{2n-1}{2}\right)x - \cos\left(\frac{2n+1}{2}\right)x \right] dx$$

$$= \frac{\sqrt{2}}{2\pi}\left| \frac{2}{2n-1}\sin\left(\frac{2n-1}{2}\right)x - \frac{2}{2n+1}\sin\left(\frac{2n+1}{2}\right)x \right|_0^{2\pi}$$

$$= 0 \quad [\because \sin(2n-1)\pi = \sin(2n+1)\pi = \sin 0 = 0]$$

Hence,
$$f(x) = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos nx \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

Example 9

Find the Fourier series of $f(x) = -1 \qquad 0 < x < \pi$

$$= 2 \qquad \pi < x < 2\pi$$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x$$
$$= \frac{1}{2\pi}\left[\int_0^{\pi}(-1)\mathrm{d}x + \int_{\pi}^{2\pi} 2\mathrm{d}x\right]$$
$$= \frac{1}{2\pi}\left[\left|-x\right|_0^{\pi} + \left|2x\right|_{\pi}^{2\pi}\right]$$
$$= \frac{1}{2\pi}[(-\pi)+(4\pi-2\pi)]$$
$$= \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left[\int_0^{\pi}(-1)\cos nx\,\mathrm{d}x + \int_{\pi}^{2\pi} 2\cos nx\,\mathrm{d}x\right]$$
$$= \frac{1}{\pi}\left[-\left|\frac{\sin nx}{n}\right|_0^{\pi} + 2\left|\frac{\sin nx}{n}\right|_{\pi}^{2\pi}\right]$$
$$= 0 \quad [\because\ \sin 2n\pi = \sin n\pi = \sin 0 = 0]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left[\int_0^{\pi}(-1)\sin nx\,\mathrm{d}x + \int_{\pi}^{2\pi} 2\sin nx\,\mathrm{d}x\right]$$
$$= \frac{1}{\pi}\left[\left|\frac{\cos nx}{n}\right|_0^{\pi} + \left|-\frac{2\cos nx}{n}\right|_{\pi}^{2\pi}\right]$$
$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n} - \frac{\cos 0}{n} - \frac{2\cos 2n\pi}{n} + \frac{2\cos n\pi}{n}\right]$$
$$= \frac{3}{n\pi}[(-1)^n - 1] \quad [\because\ \cos 2n\pi = \cos 0 = 1,\ \cos n\pi = (-1)^n]$$

Hence, $$f(x) = \frac{1}{2} + \frac{3}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n}\right]\sin nx$$
$$= \frac{1}{2} + \frac{3}{\pi}\left(-2\sin x - \frac{2}{3}\sin 3x - \frac{2}{5}\sin 5x - \cdots\right)$$
$$= \frac{1}{2} - \frac{6}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

Example 10

Find the Fourier series of $f(x) = x^2 \quad 0 < x < \pi$ **[Winter 2012]**

$\qquad\qquad\qquad\qquad = 0 \quad \pi < x < 2\pi$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} x^2 \mathrm{d}x + \int_{\pi}^{2\pi} 0\cdot \mathrm{d}x\right]$$

$$= \frac{1}{2\pi}\left|\frac{x^3}{3}\right|_0^{\pi}$$

$$= \frac{1}{2\pi}\left(\frac{\pi^3}{3}\right)$$

$$= \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x^2 \cos nx\,\mathrm{d}x + \int_{\pi}^{2\pi} 0\cdot\cos nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left|x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left(2\pi\frac{\cos n\pi}{n^2}\right) \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{2}{n^2}(-1)^n \qquad \left[\because\ \cos n\pi = (-1)^n\right]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x^2 \sin nx\,\mathrm{d}x + \int_{\pi}^{2\pi} 0\cdot\sin nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left|x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\pi^2\left(\frac{\cos n\pi}{n}\right) + 2\left(\frac{\cos n\pi}{n^3}\right) - \frac{2\cos 0}{n^3}\right] \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{\pi}\left[-\pi^2 \frac{(-1)^n}{n} + 2\frac{(-1)^n}{n^3} - \frac{2}{n^3}\right] \quad \left[\because \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

Hence, $f(x) = \dfrac{\pi^2}{6} + 2\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right]\sin nx$

Example 11

Expand f(x) in Fourier series in the interval (0, 2π) *if*

$$f(x) = \begin{cases} -\pi & 0 < x < \pi \\ x - \pi & \pi < x < 2\pi \end{cases}$$

and hence, show that $\displaystyle\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$. **[Winter 2016; Summer 2018]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi}(-\pi)dx + \int_{\pi}^{2\pi}(x-\pi)\,dx\right]$$

$$= \frac{1}{2\pi}\left[(-\pi)\left| x\right|_0^{\pi} + \left|\frac{x^2}{2} - \pi x\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[-\pi^2 + 2\pi^2 - 2\pi^2 - \frac{\pi^2}{2} + \pi^2\right]$$

$$= -\frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi}(-\pi)\cos nx\,dx + \int_{\pi}^{2\pi}(x-\pi)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left|\frac{\sin nx}{n}\right|_0^{\pi} + \left|(x-\pi)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \left|(x-\pi)\left(\frac{\sin nx}{n}\right) + \left(\frac{\cos nx}{n^2}\right)\right|_{\pi}^{2\pi}\right] \quad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right] \quad \begin{matrix}[\because \sin 2n\pi = \sin n\pi = 0] \\ [\cos 2n\pi = 1, \cos n\pi = (-1)^n]\end{matrix}$$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^n}{n^2}\right]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_0^{\pi}(-\pi)\sin nx\,\mathrm{d}x + \int_{\pi}^{2\pi}(x-\pi)\sin nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left|-\frac{\cos nx}{n}\right|_0^{\pi} + \left|(x-\pi)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\left\{\frac{(-1)^n}{n} - \frac{1}{n}\right\} + \left|-(x-\pi)\left(\frac{\cos nx}{n}\right) + \left(\frac{\sin nx}{n^2}\right)\right|_{\pi}^{2\pi}\right] \quad [\because \cos n\pi = (-1)^n]$$

$$= \frac{(-1)^n}{n} - \frac{1}{n} - \frac{1}{n} \quad [\because \cos 2n\pi = 1, \cos n\pi = (-1)^n, \sin 2n\pi = \sin n\pi = 0]$$

$$= \frac{(-1)^n}{n} - \frac{2}{n}$$

$$= \frac{1}{n}[(-1)^n - 2]$$

Hence, $$f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx + \sum_{n=1}^{\infty}\frac{1}{n}[(-1)^n - 2]\sin nx$$

$$= -\frac{\pi}{4} + \frac{2}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right] - 3\sin x - \frac{1}{2}\sin 2x - \sin 3x - \cdots$$

$$= -\frac{\pi}{4} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2} - \sum_{n=1}^{\infty}\left[\frac{2-(-1)^n}{n}\right]\sin nx \quad \text{...(1)}$$

Putting $x = \pi$ in Eq. (1),

$$f(\pi) = -\frac{\pi}{4} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)}{(2n+1)^2} - 0$$

$$\frac{1}{2}\left[\lim_{x\to\pi^-} f(x) + \lim_{x\to\pi^+} f(x)\right] = -\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$\frac{1}{2}[-\pi + 0] + \frac{\pi}{4} = -\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi}{4}$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

Example 12

Find the Fourier series of $f(x) = x + x^2$ in the interval $(-\pi, \pi)$, and hence, deduce that

(i) $$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

(ii) $$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

[Winter 2017, 2012]

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} (x + x^2)\mathrm{d}x \\
&= \frac{1}{2\pi}\left|\frac{x^2}{2} + \frac{x^3}{3}\right|_{-\pi}^{\pi} \\
&= \frac{1}{2\pi}\left(\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right) \\
&= \frac{1}{2\pi}\left(\frac{2\pi^3}{3}\right) \\
&= \frac{\pi^2}{3}
\end{aligned}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left|(x+x^2)\left(\frac{\sin nx}{n}\right) - (1+2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1+2\pi)\left(\frac{\cos n\pi}{n^2}\right) - (1-2\pi)\left\{\frac{\cos(-n\pi)}{n^2}\right\}\right]$$

$$= \frac{1}{\pi}\left[4\pi\left(\frac{\cos n\pi}{n^2}\right)\right] \qquad [\because\ \cos(-n\pi) = \cos(n\pi)]$$

$$= \frac{4(-1)^n}{n^2} \qquad [\because\ \cos n\pi = (-1)^n]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left|(x+x^2)\left(-\frac{\cos nx}{n}\right) - (1+2x)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi+\pi^2)\left(-\frac{\cos n\pi}{n}\right) + 2\left(\frac{\cos n\pi}{n^3}\right) + (-\pi+\pi^2)\left\{\frac{\cos(-n\pi)}{n}\right\} - 2\left\{\frac{\cos(-n\pi)}{n^3}\right\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos n\pi\right] \qquad [\because\ \cos(-n\pi) = \cos n\pi]$$

$$= \frac{-2(-1)^n}{n} \qquad [\because\ \cos n\pi = (-1)^n]$$

Hence,
$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$x + x^2 = \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right)$$
$$-2\left(-\frac{1}{1}\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right) \qquad \ldots(1)$$

(i) Putting $x = 0$ in Eq. (1),

$$0 = \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2}\cos 0 + \frac{1}{2^2}\cos 0 - \frac{1}{3^2}\cos 0 + \cdots\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$$

(ii) Putting $x = \pi$ in Eq. (1),

$$\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos\pi + \frac{1}{2^2}\cos 2\pi - \frac{1}{3^2}\cos 3\pi + \cdots\right]$$

$$= \frac{\pi^2}{3} + 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \qquad \ldots (2)$$

Putting $x = -\pi$ in Eq. (1),

$$-\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos(-\pi) + \frac{1}{2^2}\cos(-2\pi) - \frac{1}{3^2}\cos(-3\pi) + \cdots\right]$$

$$= \frac{\pi^2}{3} + 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \qquad \ldots (3)$$

Adding Eqs (2) and (3),

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 13

Find the Fourier series expansion of the periodic function $f(x) = x - x^2$ in the interval $-\pi \le x \le \pi$ and show that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$$

[Summer 2017]

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (x - x^2)\,dx$$

$$= \frac{1}{2\pi}\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\pi^2}{2} - \frac{\pi^3}{3} - \frac{\pi^2}{2} - \frac{\pi^3}{3}\right]$$

$$= \frac{1}{2\pi}\left(-\frac{2\pi^3}{3}\right)$$

$$= -\frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\cos nx \, dx$$

$$= \frac{1}{\pi}\left|(x - x^2)\left(\frac{\sin nx}{n}\right) - (1-2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left|(x - x^2)\left(\frac{\sin nx}{n}\right) + (1-2x)\left(\frac{\cos nx}{n^2}\right) + \frac{2\sin nx}{n^3}\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1-2\pi)\frac{(-1)^n}{n^2} - (1+2\pi)\frac{(-1)^n}{n^2}\right] \qquad \begin{bmatrix}\sin n\pi = \sin(-n\pi) = 0 \\ \cos n\pi = 0\end{bmatrix}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{2\pi(-1)^n}{n^2} - \frac{(-1)^n}{n^2} - \frac{2\pi(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{4\pi(-1)^n}{n^2}\right]$$

$$= -\frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\sin nx \, dx$$

$$= \frac{1}{\pi}\left|(x - x^2)\left(-\frac{\cos nx}{n}\right) - (1-2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left|-(x - x^2)\left(\frac{\cos nx}{n}\right) + (1-2x)\left(\frac{\sin x}{n^2}\right) - 2\left(\frac{\cos nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi^2 - \pi)\frac{(-1)^n}{n} - 2\frac{(-1)^n}{n^3} + (-\pi - \pi^2)\frac{(-1)^n}{n} + 2\frac{(-1)^n}{n^3}\right]$$

$$[\because \cos n\pi = (-1)^n, \sin n\pi = \sin(-n\pi) = 0]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2(-1)^n}{n} - \frac{\pi(-1)^n}{n} - \frac{\pi(-1)^n}{n} - \frac{\pi^2(-1)^n}{n}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi(-1)^n}{n}\right]$$

$$= -\frac{2(-1)^n}{n}$$

Hence, $f(x) = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$

$$x - x^2 = -\frac{\pi^2}{3} - 4\left(-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right)$$
$$-2\left(-\frac{1}{1}\sin nx + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right) \qquad ...(1)$$

Putting $x = 0$ in Eq. (1),

$$0 = -\frac{\pi^2}{3} - 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right)$$

$$\frac{\pi^2}{3} = 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right)$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$$

Example 14

Find the Fourier series of $f(x) = x + |x|$ in the interval $-\pi < x < \pi$.

[Winter 2015, 2014]

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (x + |x|)\,\mathrm{d}x$$
$$= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} x\,\mathrm{d}x + \int_{-\pi}^{\pi} |x|\,\mathrm{d}x\right]$$
$$= \frac{1}{2\pi}\left[0 + 2\int_{0}^{\pi} |x|\,\mathrm{d}x\right] \qquad \left[\because \int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x, \text{ if } f(x) \text{ is even function}; \; = 0, \text{ if } f(x) \text{ is odd function}\right]$$
$$= \frac{1}{\pi}\int_0^{\pi} x\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{2}\right)$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + |x|)\cos nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nx \, dx + \int_{-\pi}^{\pi} |x|\cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left[0 + 2\int_{0}^{\pi} |x|\cos nx \, dx\right] \quad \left[\begin{array}{l}\because \ x\cos nx \text{ is odd function} \\ \quad \text{and } |x|\cos nx \text{ is even function}\end{array}\right]$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\cos nx \, dx$$

$$= \frac{2}{\pi}\left|x\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] \quad [\because \ \sin n\pi = \sin 0 = 0]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right] \quad \left[\because \ \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + |x|)\sin nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx \, dx + \int_{-\pi}^{\pi} |x|\sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[2\int_{0}^{\pi} x\sin nx \, dx + 0\right] \quad \left[\begin{array}{l}\because \ x\sin nx \text{ is an even function} \\ |x|\sin x \text{ is an odd function}\end{array}\right]$$

$$= \frac{2}{\pi}\left|x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] \quad [\because \ \sin n\pi = \sin 0 = 0]$$

$$= -\frac{2}{n}(-1)^n \quad \left[\because \ \cos n\pi = (-1)^n\right]$$

Hence, $f(x)=\dfrac{\pi}{2}+\dfrac{2}{\pi}\sum_{n=1}^{\infty}\left[\dfrac{(-1)^n-1}{n^2}\right]\cos nx-2\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\sin nx$

$$x+|x|=\frac{\pi}{2}+\frac{2}{\pi}\left[-\frac{2}{1^2}\cos x-\frac{2}{3^2}\cos 3x-\frac{2}{5^2}\cos 5x-\cdots\right]$$
$$-2\left[-\frac{1}{1}\sin x+\frac{1}{2}\sin 2x-\frac{1}{3}\sin 3x+\cdots\right]$$

$$=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{1}{1^2}\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+\cdots\right)$$
$$+2\left(\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\cdots\right)$$

Example 15

Find the Fourier series of $f(x)=e^{ax}$ *in the interval* $(-\pi,\pi)$.

[Winter 2013]

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ax}\,dx$$
$$=\frac{1}{2\pi}\left|\frac{e^{ax}}{a}\right|_{-\pi}^{\pi}$$
$$=\frac{1}{2\pi a}(e^{a\pi}-e^{-a\pi})$$
$$=\frac{\sinh a\pi}{\pi a}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$
$$=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{ax}\cos nx\,dx$$

$$= \frac{1}{\pi}\left| \frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx) \right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(a\cos n\pi) - \frac{e^{-a\pi}}{a^2+n^2}\{a\cos(-n\pi)\} \right] \quad [\because \sin n\pi = \sin(-n\pi) = 0]$$

$$= \frac{a\cos n\pi}{\pi(a^2+n^2)}(e^{a\pi} - e^{-a\pi}) \quad [\because \cos(-n\pi) = \cos n\pi]$$

$$= \frac{(-1)^n 2a\sinh a\pi}{\pi(a^2+n^2)} \quad [\because \cos n\pi = (-1)^n]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin nx\,dx$$

$$= \frac{1}{\pi}\left| \frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx) \right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(-n\cos n\pi) + \frac{e^{-a\pi}}{a^2+n^2}\{n\cos(-n\pi)\} \right]$$

$$= -\frac{n\cos n\pi}{\pi(a^2+n^2)}(e^{a\pi} - e^{-a\pi}) \quad [\because \cos(-n\pi) = \cos n\pi]$$

$$= -\frac{2n(-1)^n \sinh a\pi}{\pi(a^2+n^2)} \quad [\because \cos n\pi = (-1)^n]$$

Hence,
$$f(x) = \frac{\sinh a\pi}{a\pi} + \frac{2a\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx - \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{a^2+n^2}\sin nx$$

$$= \frac{\sinh a\pi}{a\pi} + \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}(a\cos nx - n\sin nx)$$

Example 16

Find the Fourier series of $f(x) = 0 \quad -\pi < x < 0$

$= x \quad 0 < x < \pi$ **[Summer 2013]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\,\mathrm{d}x + \int_{0}^{\pi} x\,\mathrm{d}x\right]$$

$$= \frac{1}{2\pi}\left|\frac{x^2}{2}\right|_0^{\pi}$$

$$= \frac{1}{2\pi}\left(\frac{\pi^2}{2} - 0\right)$$

$$= \frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|x\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] \quad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n}\right] \quad \left[\because\ \cos n\pi = (-1)^n\right]$$

$$= -\frac{(-1)^n}{n}$$

Hence, $f(x) = \dfrac{\pi}{4} + \dfrac{1}{\pi}\sum_{n=1}^{\infty}\left[\dfrac{(-1)^n}{n^2} - \dfrac{1}{n^2}\right]\cos nx - \sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\sin nx$

$$= \frac{\pi}{4} + \frac{1}{\pi}\left(-\frac{2}{1^2}\cos x - \frac{2}{3^2}\cos 3x - \frac{2}{5^2}\cos 5x - \cdots\right)$$
$$-\left(-\frac{1}{1}\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right)$$
$$= \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right)$$
$$+\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right)$$

Example 17

Find the Fourier series of $f(x) = -\pi \qquad -\pi < x < 0$

$$= x \qquad 0 < x < \pi$$

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dots.$ **[Summer 2016, 2014]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x$$
$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} (-\pi)\mathrm{d}x + \int_0^{\pi} x\,\mathrm{d}x\right]$$
$$= \frac{1}{2\pi}\left[\left|-\pi x\right|_{-\pi}^{0} + \left|\frac{x^2}{2}\right|_0^{\pi}\right]$$
$$= \frac{1}{2\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right]$$
$$= -\frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\cos nx\,\mathrm{d}x + \int_0^{\pi} x\cos nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[-\pi \left| \frac{\sin nx}{n} \right|_{-\pi}^{0} + \left| x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right) \right|_{0}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2} \right] \qquad \left[\because \sin n\pi = \sin 0 = 0\right]$$

$$= \frac{1}{\pi n^2}\left[(-1)^n - 1 \right] \qquad \left[\because \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, \mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\sin nx \, \mathrm{d}x + \int_{0}^{\pi} x\sin nx \, \mathrm{d}x \right]$$

$$= \frac{1}{\pi}\left[-\pi \left| -\frac{\cos nx}{n} \right|_{-\pi}^{0} + \left| x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right) \right|_{0}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[\pi\left\{ \frac{\cos 0}{n} - \frac{\cos(-n\pi)}{n} \right\} + \pi\left(-\frac{\cos n\pi}{n} \right) \right] \qquad \left[\because \sin n\pi = \sin 0 = 0\right]$$

$$= \frac{1}{n}[1 - 2\cos n\pi] \qquad \left[\because \cos 0 = 1, \cos(-n\pi) = \cos n\pi\right]$$

$$= \frac{1}{n}[1 - 2(-1)^n] \qquad \left[\because \cos n\pi = (-1)^n\right]$$

Hence, $$f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2} \right]\cos nx + \sum_{n=1}^{\infty}\left[\frac{1 - 2(-1)^n}{n} \right]\sin nx \qquad \ldots (1)$$

At $x = 0$,

$$f(0) = \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x) \right] = \frac{-\pi + 0}{2} = -\frac{\pi}{2}$$

Putting $x = 0$ in Eq. (1),

$$f(0) = -\frac{\pi}{2} = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2} \right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 18

Find the Fourier series of $$f(x) = -x - \pi \qquad -\pi < x < 0$$
$$= x + \pi \qquad 0 < x < \pi$$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
&= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-x-\pi)dx + \int_{0}^{\pi}(x+\pi)\,dx\right] \\
&= \frac{1}{2\pi}\left[\left|-\frac{x^2}{2}-\pi x\right|_{-\pi}^{0} + \left|\frac{x^2}{2}+\pi x\right|_{0}^{\pi}\right] \\
&= \frac{1}{2\pi}\left[\left(\frac{\pi^2}{2}-\pi^2\right)+\left(\frac{\pi^2}{2}+\pi^2\right)\right] \\
&= \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x-\pi)\cos nx\,dx + \int_{0}^{\pi}(x+\pi)\cos nx\,dx\right] \\
&= \frac{1}{\pi}\left[\left|(-x-\pi)\left(\frac{\sin nx}{n}\right)-(-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{-\pi}^{0}\right. \\
&\qquad \left. + \left|(x+\pi)\left(\frac{\sin nx}{n}\right)-(1)\left(-\frac{\cos nx}{n^2}\right)\right|_{0}^{\pi}\right] \\
&= \frac{1}{\pi}\left[\left\{-\frac{\cos 0}{n^2}+\frac{\cos(-n\pi)}{n^2}\right\}+\left(\frac{\cos n\pi}{n^2}-\frac{\cos 0}{n^2}\right)\right] \qquad \left[\because \begin{aligned}\sin n\pi &= \sin(-n\pi)\\ &= \sin 0 = 0\end{aligned}\right] \\
&= \frac{2}{\pi n^2}[(-1)^n - 1] \quad \left[\because \cos(-n\pi) = \cos n\pi = (-1)^n, \cos 0 = 1\right]
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x-\pi)\sin nx\,dx + \int_{0}^{\pi}(x+\pi)\sin nx\,dx\right]
\end{aligned}$$

$$= \frac{1}{\pi}\left[\left|(-x-\pi)\left(-\frac{\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right|_{-\pi}^{0}\right.$$

$$\left.+\left|(x+\pi)\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{(-\pi)\left(-\frac{\cos 0}{n}\right)\right\}+\left\{(2\pi)\left(-\frac{\cos n\pi}{n}\right)+\pi\left(\frac{\cos 0}{n}\right)\right\}\right]$$

$$\left[\begin{aligned}\because \sin n\pi &= \sin(-n\pi)\\ &= \sin 0 = 0\end{aligned}\right]$$

$$= \frac{2}{n}[1-(-1)^n] \qquad [\because \cos 0 = 1, \cos(n\pi) = (-1)^n]$$

Hence, $f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos nx + 2\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]\sin nx$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right)$$

$$+4\left(\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

Example 19

Find the Fourier series of $f(x) = 0 \qquad -\pi < x < 0$

$$= \sin x \qquad 0 < x < \pi$$

Hence, deduce that $\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} \sin x\,dx\right]$$

$$= \frac{1}{2\pi}\left|-\cos x\right|_0^{\pi}$$

$$= \frac{1}{2\pi}(-\cos\pi + \cos 0)$$

$$= \frac{1}{\pi} \quad [\because \cos\pi = -1, \cos 0 = 1]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\cos nx\,\mathrm{d}x + \int_{0}^{\pi}\sin x\cos nx\,\mathrm{d}x\right]$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}[\sin(n+1)x - \sin(n-1)x]\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right|_0^{\pi}, \quad n \neq 1$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos 0}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \frac{\cos 0}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{1}{n+1} + \frac{(-1)^{n-1}}{n-1} - \frac{1}{n-1}\right], \quad n \neq 1 \quad \left[\begin{array}{c}\because \cos(n+1)\pi = (-1)^{n+1} \\ \cos(n-1)\pi = (-1)^{n-1} \\ \cos 0 = 1\end{array}\right]$$

$$= -\frac{1}{\pi(n^2-1)}[1+(-1)^n], \quad n \neq 1$$

For $n = 1$,

$$a_1 = \frac{1}{\pi}\int_0^{\pi}\sin x\cos x\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\int_0^{\pi}\sin 2x\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|-\frac{\cos 2x}{2}\right|_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos 2\pi}{2} + \frac{\cos 0}{2}\right]$$

$$= 0 \quad [\because \cos 2\pi = \cos 0 = 1]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\sin nx\,\mathrm{d}x + \int_0^{\pi}\sin x\sin nx\,\mathrm{d}x\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi}[\cos(n-1)x - \cos(n+1)x]\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right|_0^\pi, \quad n \neq 1$$

$$= 0, \quad n \neq 1 \quad [\because \ \sin(n-1)\pi = \sin(n+1)\pi = \sin 0 = 0]$$

For $n = 1$,

$$b_1 = \frac{1}{\pi}\int_0^\pi \sin x \sin x \, dx$$

$$= \frac{1}{2\pi}\int_0^\pi (1-\cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left|x - \frac{\sin 2x}{2}\right|_0^\pi$$

$$= \frac{1}{2\pi}(\pi) \quad [\because \ \sin 2\pi = \sin 0 = 0]$$

$$= \frac{1}{2}$$

Hence, $$f(x) = \frac{1}{\pi} - \frac{1}{\pi}\sum_{n=2}^{\infty}\left[\frac{1+(-1)^n}{n^2-1}\right]\cos nx + \frac{1}{2}\sin x$$

$$= \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x + \cdots\right) + \frac{1}{2}\sin x \qquad \text{... (1)}$$

At $x = 0$,

$$f(0) = \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x \to 0^+} f(x)\right] = 0$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \ldots\right)$$

$$\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots$$

Example 20

Find the Fourier series of $$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x - \pi & 0 < x < \pi \end{cases}$$

[Summer 2015]

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-\pi)\,\mathrm{d}x + \int_{0}^{\pi}(x-\pi)\,\mathrm{d}x\right]$$

$$= \frac{1}{2\pi}\left[(-\pi)\left|x\right|_{-\pi}^{0} + \left|\frac{x^2}{2} - \pi x\right|_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[(-\pi)[-(-\pi)] + \left(\frac{\pi^2}{2} - \pi^2\right)\right]$$

$$= \frac{1}{2\pi}\left[-\pi^2 - \frac{\pi^2}{2}\right]$$

$$= \frac{1}{2\pi}\left(-\frac{3\pi^2}{2}\right)$$

$$= -\frac{3\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,\mathrm{d}x + \int_{0}^{\pi}(x-\pi)\cos nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left|\frac{\sin nx}{n}\right|_{-\pi}^{0} + \left|(x-\pi)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[(-\pi)(0) + \left|(x-\pi)\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right|_{0}^{\pi}\right] \quad [\because \sin(-n\pi) = \sin 0 = 0]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] \quad [\because \sin n\pi = \sin 0 = 0, \cos 0 = 1]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] \quad [\because \cos n\pi = (-1)^n]$$

$$= \frac{1}{n^2\pi}[(-1)^n - 1]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \int_{0}^{\pi}(x-\pi)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left|-\frac{\cos nx}{n}\right|_{-\pi}^{0} + \left|(x-\pi)\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\left\{\frac{1}{n}-\frac{\cos n\pi}{n}\right\}+\left|-(x-\pi)\left(\frac{\cos nx}{n}\right)+\frac{\sin nx}{n^2}\right|_{0}^{\pi}\right] \qquad [\ \cos 0 = 1]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{1}{n}-\frac{\cos n\pi}{n}\right)+(-\pi)\left(\frac{1}{n}\right)\right] \qquad \left[\begin{array}{c}\because \sin n\pi = \sin 0 = 0\\ \cos 0 = 1\end{array}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}-\frac{\pi(-1)^n}{n}-\frac{\pi}{n}\right] \qquad [\because \cos n\pi = (-1)^n]$$

$$= \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n}\right]$$

$$= -\frac{(-1)^n}{n}$$

$$= \frac{(-1)^{n+1}}{n}$$

Hence, $f(x) = -\frac{3\pi}{4}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n-1]\cos nx+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$

$$= -\frac{3\pi}{4}-\frac{2}{\pi}\left(\frac{1}{1^2}\cos x+\frac{1}{3^2}\cos 3x+\cdots\right)+\left(\frac{1}{1}\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\cdots\right)$$

Example 21

Find the Fourier series of $f(x) = x \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$= \pi - x \qquad \frac{\pi}{2} < x < \frac{3\pi}{2}$$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\mathrm{d}x$$

$$= \frac{1}{2\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)\mathrm{d}x\right]$$

$$= \frac{1}{2\pi}\left[\left|\frac{x^2}{2}\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left|\pi x - \frac{x^2}{2}\right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{\pi^2}{8} - \frac{\pi^2}{8}\right) + \left(\frac{3\pi^2}{2} - \frac{9\pi^2}{8}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos nx\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)\cos nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\left|x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2n}\left(\sin\frac{3n\pi}{2} + \sin\frac{n\pi}{2}\right) - \frac{1}{n^2}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\pi\cos\frac{n\pi}{2} + \frac{2}{n^2}\sin n\pi\sin\frac{n\pi}{2}\right]$$

$$\left[\because\ \sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \quad \cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{B-A}{2}\right]$$

$$= 0 \qquad [\because\ \sin n\pi = 0]$$

$$b_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin nx\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)\sin nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\left| x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left|(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{3}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{2n}\cos\frac{3n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2n}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right) + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\pi\sin\frac{n\pi}{2} + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi n^2}\left[3\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right] \qquad [\because\ \sin n\pi = 0]$$

Hence, $f(x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[3\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right]\sin nx$

Fourier Series Expansion with Period 2*l*

Example 22

Find the Fourier series of $f(x) = x^2$ *in the interval* (0, 4). *Hence, deduce that* $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$

Solution

The Fourier series of $f(x)$ with period $2l = 4$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\mathrm{d}x$$
$$= \frac{1}{4}\int_0^4 x^2 \mathrm{d}x$$
$$= \frac{1}{4}\left|\frac{x^3}{3}\right|_0^4$$
$$= \frac{1}{4}\left(\frac{64}{3}\right)$$
$$= \frac{16}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$
$$= \frac{1}{2}\int_0^4 x^2 \cos\frac{n\pi x}{2}\mathrm{d}x$$
$$= \frac{1}{2}\left|x^2\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (2x)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right) + 2\left(-\frac{8}{n^3\pi^3}\sin\frac{n\pi x}{2}\right)\right|_0^4$$
$$= \frac{1}{2}\left[8\left(\frac{4}{n^2\pi^2}\cos 2n\pi\right)\right] \qquad [\because\ \sin 2n\pi = \sin 0 = 0]$$
$$= \frac{1}{2}\left[8\left(\frac{4}{n^2\pi^2}\right)\right] \qquad [\because\ \cos 2n\pi = 1]$$
$$= \frac{16}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$
$$= \frac{1}{2}\int_0^4 x^2 \sin\frac{n\pi x}{2}\mathrm{d}x$$
$$= \frac{1}{2}\left|x^2\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - 2x\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right) + 2\left(\frac{8}{n^3\pi^3}\cos\frac{n\pi x}{2}\right)\right|_0^4$$
$$= \frac{1}{2}\left[16\left(-\frac{2}{n\pi}\cos 2n\pi\right) + 2\left(\frac{8}{n^3\pi^3}\cos 2n\pi\right) - 2\left(\frac{8}{n^3\pi^3}\cos 0\right)\right]$$
$$= \frac{1}{2}\left(-\frac{32}{n\pi}\right) \qquad [\because\ \cos 2n\pi = \cos 0 = 1]$$
$$= -\frac{16}{n\pi}$$

Hence,
$$f(x)=\frac{16}{3}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2}-\frac{16}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}$$
$$x^2=\frac{16}{3}+\frac{16}{\pi^2}\left(\frac{1}{1^2}\cos\frac{\pi x}{2}+\frac{1}{2^2}\cos\pi x+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\cdots\right)$$
$$-\frac{16}{\pi}\left(\frac{1}{1}\sin\frac{\pi x}{2}+\frac{1}{2}\sin\pi x+\frac{1}{3}\sin\frac{3\pi x}{2}+\cdots\right) \quad ...(1)$$

Putting $x = 0$ in Eq. (1),
$$0=\frac{16}{3}+\frac{16}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)$$
$$-\frac{1}{3}=\frac{1}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right) \quad \ldots (2)$$

Putting $x = 4$ in Eq. (1),
$$16=\frac{16}{3}+\frac{16}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)$$
$$\frac{2}{3}=\frac{1}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right) \quad \ldots (3)$$

Adding Eqs (2) and (3),
$$\frac{1}{3}=\frac{2}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)$$
$$\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$$

Example 23

Find the Fourier series of $f(x) = 4 - x^2$ *in the interval* (0, 2). *Hence, deduce that* $\dfrac{\pi^2}{6}=\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\ldots$

Solution

The Fourier series of $f(x)$ with period $2l = 2$ is given by
$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$$
$$=a_0+\sum_{n=1}^{\infty}a_n\cos n\pi x+\sum_{n=1}^{\infty}b_n\sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{2} (4-x^2)\mathrm{d}x$$

$$= \frac{1}{2}\left|4x - \frac{x^3}{3}\right|_0^2$$

$$= \frac{1}{2}\left(8 - \frac{8}{3}\right)$$

$$= \frac{8}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$= \int_0^{2} (4-x^2)\cos n\pi x\,\mathrm{d}x$$

$$= \left|(4-x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right|_0^2$$

$$= -4\left(\frac{\cos 2n\pi}{n^2\pi^2}\right) \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= -\frac{4}{n^2\pi^2} \qquad [\because\ \cos 2n\pi = 1]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$

$$= \int_0^{2} (4-x^2)\sin n\pi x\,\mathrm{d}x$$

$$= \left|(4-x^2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-2x)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right|_0^2$$

$$= -2\left(\frac{\cos 2n\pi}{n^3\pi^3}\right) + 4\left(\frac{\cos 0}{n\pi}\right) + 2\left(\frac{\cos 0}{n^3\pi^3}\right) \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{4}{n\pi} \qquad [\because\ \cos 2n\pi = \cos 0 = 1]$$

Hence, $$f(x) = \frac{8}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x$$

$$4 - x^2 = \frac{8}{3} - \frac{4}{\pi^2}\left(\frac{1}{1^2}\cos\pi x + \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x + \cdots\right)$$

$$+ \frac{4}{\pi}\left(\frac{1}{1}\sin\pi x + \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 2\pi x + \cdots\right) \qquad ...(1)$$

Putting $x = 0$ in Eq. (1),

$$4 = \frac{8}{3} - \frac{4}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$\frac{1}{3} = -\frac{1}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \qquad \ldots (2)$$

Putting $x = 2$ in Eq. (1),

$$0 = \frac{8}{3} - \frac{4}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$-\frac{2}{3} = -\frac{1}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \qquad \ldots (3)$$

Adding Eqs (2) and (3),

$$-\frac{1}{3} = -\frac{2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 24

Find the Fourier series of $f(x) = 2x - x^2$ *in the interval* (0, 3). *Hence, deduce that* $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ **[Summer 2016]**

Solution

The Fourier series of $f(x)$ with period $2l = 3$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{3}$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\mathrm{d}x$$

$$= \frac{1}{3}\int_0^{3} (2x - x^2)\mathrm{d}x$$

$$= \frac{1}{3}\left| x^2 - \frac{x^3}{3} \right|_0^3$$

$$= \frac{1}{3}\left(9 - \frac{27}{3}\right)$$

$$= 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{3}\int_0^{3}(2x-x^2)\cos\frac{2n\pi x}{3}\,\mathrm{d}x$$

$$= \frac{2}{3}\left|(2x-x^2)\left(\frac{3}{2n\pi}\sin\frac{2n\pi x}{3}\right)-(2-2x)\left(-\frac{9}{4n^2\pi^2}\cos\frac{2n\pi x}{3}\right)\right.$$

$$\left.+(-2)\left(-\frac{27}{8n^3\pi^3}\sin\frac{2n\pi x}{3}\right)\right|_0^3$$

$$= \frac{2}{3}\left[4\left(-\frac{9}{4n^2\pi^2}\cos 2n\pi\right)+2\left(-\frac{9}{4n^2\pi^2}\cos 0\right)\right] \quad [\because\ \sin 2n\pi = \sin 0 = 0]$$

$$= \frac{2}{3}\left[\frac{9}{4n^2\pi^2}(-4-2)\right] \quad [\because\ \cos 2n\pi = \cos 0 = 1]$$

$$= -\frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{3}\int_0^{3}(2x-x^2)\sin\frac{2n\pi x}{3}\,\mathrm{d}x$$

$$= \frac{2}{3}\left|(2x-x^2)\left(-\frac{3}{2n\pi}\cos\frac{2n\pi x}{3}\right)-(2-2x)\left(-\frac{9}{4n^2\pi^2}\sin\frac{2n\pi x}{3}\right)\right.$$

$$\left.+(-2)\left(\frac{27}{8n^3\pi^3}\cos\frac{2n\pi x}{3}\right)\right|_0^3$$

$$= \frac{2}{3}\left[(-3)\left(-\frac{3}{2n\pi}\cos 2n\pi\right)-(2)\left(\frac{27}{8n^3\pi^3}\cos 2n\pi\right)\right.$$

$$\left.+2\left(\frac{27}{8n^3\pi^3}\cos 0\right)\right] \quad [\because\ \sin 2n\pi = \sin 0 = 0]$$

$$= \frac{2}{3}\left(\frac{9}{2n\pi}\right) \quad [\because\ \cos 2n\pi = \cos 0 = 1]$$

$$= \frac{3}{n\pi}$$

Hence, $$f(x) = -\frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{3}+\frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{3}$$

$$2x - x^2 = -\frac{9}{\pi^2}\left(\frac{1}{1^2}\cos\frac{2\pi x}{3} + \frac{1}{2^2}\cos\frac{4\pi x}{3} + \frac{1}{3^2}\cos\frac{6\pi x}{3} + \cdots\right)$$
$$+\frac{3}{\pi}\left(\frac{1}{1}\sin\frac{2\pi x}{3} + \frac{1}{2}\sin\frac{4\pi x}{3} + \frac{1}{3}\sin\frac{6\pi x}{3} + \cdots\right) \quad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$0 = -\frac{9}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$
$$0 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \quad \ldots (2)$$

Putting $x = 3$ in Eq. (1),

$$-3 = -\frac{9}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$
$$\frac{\pi^2}{3} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \quad \ldots (3)$$

Adding Eqs (2) and (3),

$$\frac{\pi^2}{3} = 2\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$
$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 25

For the function $f(x) = \begin{cases} x & 0 \le x \le 2 \\ 4 - x & 2 \le x \le 4 \end{cases}$, *find its Fourier series. Hence, show that* $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$. **[Winter 2015]**

Solution

The Fourier series of $f(x)$ with period $2l = 4$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$
$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$
$$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\,\mathrm{d}x$$

$$= \frac{1}{4}\int_0^4 f(x)\,\mathrm{d}x$$

$$= \frac{1}{4}\left[\int_0^2 x\,\mathrm{d}x + \int_2^4 (-x+4)\,\mathrm{d}x\right]$$

$$= \frac{1}{4}\left[\left|\frac{x^2}{2}\right|_0^2 + \left|\frac{-x^2}{2} + 4x\right|_2^4\right]$$

$$= \frac{1}{4}\left[(2-0) + \left\{(-8+16) - (-2+8)\right\}\right]$$

$$= \frac{1}{4}\left[2 + (8-6)\right]$$

$$= 1$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^4 f(x)\cos\frac{n\pi x}{2}\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\int_0^2 x\cos\frac{n\pi x}{2}\,\mathrm{d}x + \int_2^4 (4-x)\cos\frac{n\pi x}{2}\,\mathrm{d}x\right]$$

$$= \frac{1}{2}\left[\left|(x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (1)\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right|_0^2 + \left|(4-x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right|_2^4\right]$$

$$= \frac{1}{2}\left[\left|\frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right|_0^2 + \left|\frac{2(4-x)}{n\pi}\sin\left(\frac{n\pi x}{2}\right) - \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right|_2^4\right]$$

$$= \frac{1}{2}\left[\frac{4}{n^2\pi^2}\cos(n\pi) - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos(2n\pi) + \frac{4}{n^2\pi^2}\cos(n\pi)\right]$$

$$= \frac{1}{2}\left[\frac{8}{n^2\pi^2}\cos n\pi - \frac{8}{n^2\pi^2}\right] \qquad [\because \cos 2n\pi = 1]$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right] \qquad [\because \cos n\pi = (-1)^n]$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{2}\int_0^4 f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_0^2 x\sin\frac{n\pi x}{2}\,dx + \int_2^4 (4-x)\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\left|(x)\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (1)\left(-\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right|_0^2 + \left|(4-x)\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(-\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right|_2^4\right]$$

$$= \frac{1}{2}\left[\left|\frac{-2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\right|_0^2 + \left|\frac{-2(4-x)}{n\pi}\cos\left(\frac{n\pi x}{2}\right) - \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\right|_2^4\right]$$

$$= \frac{1}{2}\left[-\frac{4}{n\pi}\cos n\pi + \frac{4}{n\pi}\cos n\pi\right]$$

$$= 0$$

Hence, $$f(x) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{\left\{(-1)^n - 1\right\}}{n^2}\right]\cos\frac{n\pi x}{2}$$

$$= 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right]$$

$$= 1 - \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{(2n+1)\pi x}{2} \quad ...(1)$$

Putting $x = 2$ in Eq. (1),

$$2 = 1 - \frac{8}{\pi^2}\sum_{n=0}^{\infty}\left[\frac{1}{(2n+1)^2}\right]\cos(2n+1)\pi$$

$$2 = 1 + \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$1 = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Example 26

Find the Fourier series of $f(x) = \pi x \qquad 0 < x < 1$

$\qquad\qquad\qquad\qquad\qquad\quad = 0 \qquad 1 < x < 2$

Solution

The Fourier series of $f(x)$ with period $2l = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{2}\left(\int_0^1 \pi x\,dx + \int_1^2 0\cdot dx\right)$$

$$= \frac{1}{2}\left|\frac{\pi x^2}{2}\right|_0^1$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{4}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \int_0^1 \pi x\cos n\pi x\,\mathrm{d}x + \int_1^2 0\cdot\cos n\pi x\,\mathrm{d}x$$

$$= \left|\pi x\left(\frac{\sin n\pi x}{n\pi}\right) - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_0^1$$

$$= \left[\pi\left(\frac{\cos n\pi}{n^2\pi^2}\right) - \pi\left(\frac{\cos 0}{n^2\pi^2}\right)\right] \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{n^2\pi}[(-1)^n - 1] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \int_0^1 \pi x\sin n\pi x\,\mathrm{d}x + \int_1^2 0\cdot\sin n\pi x\,\mathrm{d}x$$

$$= \left|\pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1$$

$$= -\frac{\pi\cos n\pi}{n\pi} \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= -\frac{(-1)^n}{n} \qquad \left[\because\ \cos n\pi = (-1)^n\right]$$

Hence,

$$f(x) = \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos n\pi x - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi x$$

$$= \frac{\pi}{4} + \frac{1}{\pi}\left(-\frac{2}{1^2}\cos\pi x - \frac{2}{3^2}\cos 3\pi x - \cdots\right)$$

$$-\left(-\frac{1}{1}\sin\pi x + \frac{1}{2}\sin 2\pi x - \frac{1}{3}\sin 3\pi x + \cdots\right)$$

Example 27

Find the Fourier series of the periodic function with a period 2 of

$$f(x) = \pi \qquad 0 \le x \le 1$$
$$= \pi(2-x) \qquad 1 \le x \le 2$$

[Summer 2013]

Solution

The Fourier series of $f(x)$ with period $2l = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\int_0^1 \pi\,\mathrm{d}x + \int_1^2 \pi(2-x)\,\mathrm{d}x\right]$$

$$= \frac{\pi}{2}\left[|x|_0^1 + \left|2x - \frac{x^2}{2}\right|_1^2\right]$$

$$= \frac{\pi}{2}\left[(1) + \left(4-2-2+\frac{1}{2}\right)\right]$$

$$= \frac{3\pi}{4}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos n\pi x\,\mathrm{d}x$$

$$= \int_0^1 \pi\cos n\pi x\,\mathrm{d}x + \int_1^2 \pi(2-x)\cos n\pi x\,\mathrm{d}x$$

$$= \left|\pi\left(\frac{\sin n\pi x}{n\pi}\right)\right|_0^1 + \left|\pi(2-x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_1^2$$

$$= \left(-\frac{\cos 2n\pi}{n^2\pi} + \frac{\cos n\pi}{n^2\pi}\right) \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{n^2\pi}\left[(-1)^n - 1\right] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 2n\pi = 1\right]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin n\pi x\,\mathrm{d}x$$

$$= \int_0^1 \pi\sin n\pi x\,\mathrm{d}x + \int_1^2 \pi(2-x)\sin n\pi x\,\mathrm{d}x$$

$$= \left| \pi\left(-\frac{\cos n\pi x}{n\pi}\right) \right|_0^1 + \left| \pi(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) \right|_1^2$$

$$= \left[\pi\left(-\frac{\cos n\pi}{n\pi}\right) + \pi\left(\frac{\cos 0}{n\pi}\right) + \pi\left(-\frac{\cos n\pi}{n\pi}\right) \right] \quad [\because \sin n\pi = \sin 0 = 0]$$

$$= \left(-\frac{2\cos n\pi}{n}\right) + \left(\frac{\cos 0}{n}\right)$$

$$= \frac{1-2(-1)^n}{n} \quad \left[\because \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

Hence, $f(x) = \dfrac{3\pi}{4} + \dfrac{1}{\pi}\sum_{n=1}^{\infty}\left[\dfrac{(-1)^n - 1}{n^2}\right]\cos n\pi x + \sum_{n=1}^{\infty}\left[\dfrac{1-2(-1)^n}{n}\right]\sin n\pi x$

$$= \frac{3\pi}{4} - \frac{2}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right)$$
$$+ \left(\frac{3}{1}\sin \pi x - \frac{1}{2}\sin 2\pi x + \frac{3}{3}\sin 3\pi x - \cdots\right)$$

Example 28

Find the Fourier series of

$$f(x) = \begin{cases} \pi x & 0 \le x < 1 \\ 0 & x = 1 \\ \pi(x-2) & 1 < x \le 2 \end{cases}$$

Hence, deduce that $\dfrac{\pi}{4} = \dfrac{1}{1} - \dfrac{1}{3} + \dfrac{1}{5} - \ldots$

Solution

The Fourier series of $f(x)$ with period $2l = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{2}\left[\int_0^1 \pi x\,dx + \int_1^2 \pi(x-2)\,dx\right]$$

$$= \frac{1}{2}\left[\pi\left|\frac{x^2}{2}\right|_0^1 + \pi\left|\frac{x^2}{2} - 2x\right|_1^2\right]$$

$$= 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \int_0^1 \pi x \cos n\pi x\,\mathrm{d}x + \int_1^2 \pi(x-2)\cos n\pi x\,\mathrm{d}x$$

$$= \pi\left[\left|x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_0^1 + \left|(x-2)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_1^2\right]$$

$$= \pi\left[\frac{\cos n\pi}{n^2\pi^2} - \frac{\cos 0}{n^2\pi^2} + \frac{\cos 2n\pi}{n^2\pi^2} - \frac{\cos n\pi}{n^2\pi^2}\right] \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= 0 \quad [\because\ \cos 0 = \cos 2n\pi = 1]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \int_0^1 \pi x \sin n\pi x\,\mathrm{d}x + \int_1^2 \pi(x-2)\sin n\pi x\,\mathrm{d}x$$

$$= \pi\left[\left|x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1 + \left|(x-2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_1^2\right]$$

$$= \pi\left[-\frac{\cos n\pi}{n\pi} - \frac{\cos n\pi}{n\pi}\right] \quad [\because\ \sin 2n\pi = \sin n\pi = \sin 0 = 0]$$

$$= -\frac{2(-1)^n}{n} \quad \left[\because\ \cos n\pi = (-1)^n\right]$$

Hence, $f(x) = 2\sum_{n=1}^{\infty}\left[-\frac{(-1)^n}{n}\right]\sin n\pi x$

$$= 2\left(\frac{1}{1}\sin \pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x - \frac{1}{4}\sin 4\pi x + \frac{1}{5}\sin 5\pi x - \cdots\right) \quad \ldots (1)$$

Putting $x = \frac{1}{2}$ in Eq. (1),

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{1}\sin\frac{\pi}{2} - \frac{1}{2}\sin \pi + \frac{1}{3}\sin\frac{3\pi}{2} - \cdots\right)$$

$$\frac{\pi}{2} = 2\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \ldots\right)$$

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \ldots$$

Example 29

Find the Fourier series of $f(x) = x \qquad -1 < x < 0$

$= 2 \qquad 0 < x < 1$ **[Winter 2012]**

Solution

The Fourier series of $f(x)$ with period $2l = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,\mathrm{d}x$$

$$= \frac{1}{2}\left(\int_{-1}^{0} x\,\mathrm{d}x + \int_{0}^{1} 2\,\mathrm{d}x\right)$$

$$= \frac{1}{2}\left[\left|\frac{x^2}{2}\right|_{-1}^{0} + |2x|_0^1\right]$$

$$= \frac{1}{2}\left[-\frac{1}{2} + 2\right]$$

$$= \frac{3}{4}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{1}{1}\left[\int_{-1}^{0} x\cos n\pi x\,\mathrm{d}x + \int_{0}^{1} 2\cos n\pi x\,\mathrm{d}x\right]$$

$$= \left|x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_{-1}^{0} + 2\left|\frac{\sin n\pi x}{n\pi}\right|_0^1$$

$$= \frac{\cos 0}{n^2\pi^2} - \frac{\cos n\pi}{n^2\pi^2} \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{n^2\pi^2}\left[1 - (-1)^n\right] \quad \left[\because\ \cos 0 = 1, \cos n\pi = (-1)^n\right]$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{1}{1}\left[\int_{-1}^{0} x\sin n\pi x\,\mathrm{d}x + \int_{0}^{1} 2\sin n\pi x\,\mathrm{d}x\right]$$

$$= \left| x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_{-1}^{0} + 2\left| -\frac{\cos n\pi x}{n\pi}\right|_{0}^{1}$$

$$= -\frac{\cos n\pi}{n\pi} - \frac{2\cos n\pi}{n\pi} + \frac{2\cos 0}{n\pi} \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{n\pi}\left[-3(-1)^n + 2\right] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

$$= \frac{1}{n\pi}\left[2 - 3(-1)^n\right]$$

Hence,

$$f(x) = \frac{3}{4} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos n\pi x + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{2-3(-1)^n}{n}\right]\sin n\pi x$$

$$= \frac{3}{4} + \frac{1}{\pi^2}\left(\frac{2}{1^2}\cos\pi x + \frac{2}{3^2}\cos 3\pi x + \frac{2}{5^2}\cos 5\pi x + \cdots\right)$$

$$+ \frac{1}{\pi}\left(\frac{5}{1}\sin\pi x - \frac{1}{2}\sin 2\pi x + \frac{5}{3}\sin 3\pi x - \frac{1}{4}\sin 4\pi x + \cdots\right)$$

$$= \frac{3}{4} + \frac{2}{\pi^2}\left(\frac{1}{1^2}\cos\pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5nx + \cdots\right)$$

$$+ \frac{5}{\pi}\left(\sin\pi x + \frac{1}{3}\sin 3\pi x + \cdots\right) - \frac{1}{\pi}\left(\frac{1}{2}\sin 2\pi x + \frac{1}{4}\sin 4\pi x + \cdots\right)$$

Example 30

Find the Fourier series of $f(x) = 4 - x \qquad 3 < x < 4$

$\qquad\qquad\qquad\qquad\quad = x - 4 \qquad 4 < x < 5$

Solution

The Fourier series of $f(x)$ with period $2l = 5 - 3 = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\mathrm{d}x$$
$$= \frac{1}{2}\int_3^5 f(x)\mathrm{d}x$$
$$= \frac{1}{2}\left[\int_3^4 (4-x)\mathrm{d}x + \int_4^5 (x-4)\mathrm{d}x\right]$$
$$= \frac{1}{2}\left[\left|4x - \frac{x^2}{2}\right|_3^4 + \left|\frac{x^2}{2} - 4x\right|_4^5\right]$$
$$= \frac{1}{2}\left[\left\{\left(16 - \frac{16}{2}\right) - \left(12 - \frac{9}{2}\right)\right\} + \left\{\left(\frac{25}{2} - 20\right) - \left(\frac{16}{2} - 16\right)\right\}\right]$$
$$= \frac{1}{2}$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$
$$= \int_3^4 (4-x)\cos n\pi x\,\mathrm{d}x + \int_4^5 (x-4)\cos n\pi x\,\mathrm{d}x$$
$$= \left|(4-x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_3^4 + \left|(x-4)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_4^5$$
$$= -\frac{1}{n^2\pi^2}(\cos 4n\pi - \cos 3n\pi) + \frac{1}{n^2\pi^2}(\cos 5n\pi - \cos 4n\pi)\ [\because\ \sin 3n\pi = \sin 5n\pi = 0]$$
$$= -\frac{1}{n^2\pi^2}[(-1)^{4n} - (-1)^{3n} - (-1)^{5n} + (-1)^{4n}]$$
$$= \frac{2}{n^2\pi^2}[(-1)^n - 1]$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$
$$= \int_3^4 (4-x)\sin n\pi x\,\mathrm{d}x + \int_4^5 (x-4)\sin n\pi x\,\mathrm{d}x$$
$$= \left|(4-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_3^4$$
$$+ \left|(x-4)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_4^5$$
$$= \frac{1}{n\pi}\cos 3n\pi - \frac{1}{n\pi}\cos 5n\pi \quad [\because\ \sin 4n\pi = \sin 3n\pi = \sin 5n\pi = 0]$$
$$= 0 \quad \left[\because\ \cos 3n\pi = \cos 5n\pi = (-1)^n\right]$$

Hence, $f(x)=\frac{1}{2}+\frac{2}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\cos n\pi x$

$$=\frac{1}{2}+\frac{2}{\pi^2}\left(-\frac{2}{1^2}\cos \pi x-\frac{2}{3^2}\cos 3\pi x-\frac{2}{5^2}\cos 5\pi x-\cdots\right)$$

$$=\frac{1}{2}-\frac{4}{\pi^2}\left(\frac{1}{1^2}\cos \pi x+\frac{1}{3^2}\cos 3\pi x+\frac{1}{5^2}\cos 5\pi x+\cdots\right)$$

Example 31

Find the Fourier series of $f(x)=0 \qquad -5<x<0$

$\qquad\qquad\qquad\qquad\qquad\qquad = 3 \qquad 0<x<5$

Solution

The Fourier series of $f(x)$ with period $2l = 10$ is given by

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$$

$$=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{5}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{5}$$

$$a_0=\frac{1}{2l}\int_{-l}^{l}f(x)\mathrm{d}x$$

$$=\frac{1}{10}\left(\int_{-5}^{0}0\,\mathrm{d}x+\int_{0}^{5}3\mathrm{d}x\right)$$

$$=\frac{1}{10}\left|3x\right|_0^5$$

$$=\frac{1}{10}(15)$$

$$=\frac{3}{2}$$

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$=\frac{1}{5}\left(\int_{-5}^{0}0\cdot\cos\frac{n\pi x}{5}\mathrm{d}x+\int_{0}^{5}3\cos\frac{n\pi x}{5}\mathrm{d}x\right)$$

$$=\frac{3}{5}\left|\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right|_0^5$$

$$=0 \quad [\because\ \sin n\pi=\sin 0=0]$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{5}\left(\int_{-5}^{0} 0\cdot\sin\frac{n\pi x}{5}\,dx + \int_{0}^{5} 3\sin\frac{n\pi x}{5}\,dx\right)$$

$$= \frac{3}{5}\left|\frac{5}{n\pi}\left(-\cos\frac{n\pi x}{5}\right)\right|_0^5$$

$$= \frac{3}{n\pi}[-\cos n\pi + \cos 0]$$

$$= \frac{3}{n\pi}[1-(-1)^n] \qquad [\because \cos n\pi = (-1)^n, \cos 0 = 1]$$

Hence, $$f(x) = \frac{3}{2} + \frac{3}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]\sin\frac{n\pi x}{5}$$

$$= \frac{3}{2} + \frac{3}{\pi}\left(\frac{2}{1}\sin\frac{\pi x}{5} + \frac{2}{3}\sin\frac{3\pi x}{5} + \cdots\right)$$

Example 32

Find the Fourier series of $f(x) = x \qquad -1 < x < 0$

$\qquad\qquad = x + 2 \qquad 0 < x < 1$

Solution

The Fourier series of $f(x)$ with period $2l = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{1}{2}\left[\int_{-1}^{0} x\,dx + \int_{0}^{1}(x+2)\,dx\right]$$

$$= \frac{1}{2}\left[\left|\frac{x^2}{2}\right|_{-1}^{0} + \left|\frac{x^2}{2} + 2x\right|_0^1\right]$$

$$= \frac{1}{2}\left[-\frac{1}{2} + \left(\frac{1}{2} + 2\right)\right]$$

$$= 1$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$
$$= \left[\int_{-1}^{0} x\cos n\pi x\,dx + \int_{0}^{1}(x+2)\cos n\pi x\,dx\right]$$
$$= \left| x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_{-1}^{0} + \left|(x+2)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_{0}^{1}$$
$$= \left[\left\{\frac{\cos 0}{n^2\pi^2} - \frac{\cos(-n\pi)}{n^2\pi^2}\right\} + \left\{\frac{\cos n\pi}{n^2\pi^2} - \frac{\cos 0}{n^2\pi^2}\right\}\right] \quad [\because \sin n\pi = \sin(-n\pi) = \sin 0 = 0]$$
$$= 0 \quad [\because \cos(-n\pi) = \cos n\pi]$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$
$$= \left[\int_{-1}^{0} x\sin n\pi x\,dx + \int_{0}^{1}(x+2)\sin n\pi x\,dx\right]$$
$$= \left| x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_{-1}^{0} + \left|(x+2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_{0}^{1}$$
$$= \left[\left\{-\frac{\cos(-n\pi)}{n\pi}\right\} + \left\{3\left(-\frac{\cos n\pi}{n\pi}\right) - 2\left(-\frac{\cos 0}{n\pi}\right)\right\}\right] \quad \left[\begin{aligned}\because\ \sin n\pi &= \sin(-n\pi)\\ &= \sin 0 = 0\end{aligned}\right]$$
$$= \left[\frac{-(-1)^n}{n\pi} - \frac{3(-1)^n}{n\pi} + \frac{2}{n\pi}\right] \quad [\because \cos(-n\pi) = \cos n\pi = (-1)^n,\ \cos 0 = 1]$$
$$= \frac{2}{n\pi}[1 - 2(-1)^n]$$

Hence,
$$f(x) = 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-2(-1)^n}{n}\right]\sin n\pi x$$
$$= 1 + \frac{2}{\pi}\left(\frac{3}{1}\sin\pi x - \frac{1}{2}\sin 2\pi x + \frac{3}{3}\sin 3\pi x - \cdots\right)$$

EXERCISE 7.1

Find the Fourier series of the following functions:

1. $f(x) = e^x \qquad 0 < x < 2\pi$

$$\left[\textbf{Ans.:}\ \frac{e^{2\pi}-1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\cos nx - n\sin nx}{n^2+1}\right]\right]$$

2. $f(x) = 1 \qquad 0 < x < \pi$

$\quad = 2 \qquad \pi < x < 2\pi$

Hence, deduce that $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots$

$$\left[\textbf{Ans.:}\ \frac{3}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n}\right]\sin nx\right]$$

3. $f(x) = x \qquad 0 < x < \pi$

$\quad = 2\pi - x \qquad \pi < x < 2\pi$

$$\left[\textbf{Ans.:}\ \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos nx\right]$$

4. $f(x) = 1 \qquad -\pi < x \le 0$

$\quad = -2 \qquad 0 < x \le \pi$

$$\left[\textbf{Ans.:} -\frac{1}{2} - \frac{6}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}\right]$$

5. $f(x) = -x \qquad -\pi < x \le 0$

$\quad = 0 \qquad 0 < x \le \pi$

$$\left[\textbf{Ans.:}\ \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2} - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin nx\right]$$

6. $f(x) = \dfrac{1}{2} \qquad -\pi < x < 0$

$\quad = \dfrac{x}{\pi} \qquad 0 < x < \pi$

$$\left[\textbf{Ans.:}\ \frac{1}{2} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2nx\right]$$

7. $f(x) = x - \pi \qquad -\pi < x < 0$

$\quad = \pi - x \qquad 0 < x < \pi$

Hence, deduce that $\dfrac{\pi^2}{8}=\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\ldots$

$$\left[\textbf{Ans.}: -\frac{\pi}{2}+\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)x + 4\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin(2n+1)x\right]$$

8. $f(x)=\cos x \quad -\pi < x < 0$
$\quad = \sin x \quad 0 < x < \pi$

$$\left[\textbf{Ans.}: \frac{1}{\pi}+\frac{1}{2}(\cos x+\sin x) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2nx\right]$$

9. $f(x)=2-\dfrac{x^2}{2} \quad 0 \le x \le 2$

$$\left[\textbf{Ans.}: \frac{4}{3}-\frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{n^2}\cos n\pi x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x\right]$$

10. $f(x)=\dfrac{1}{2}(\pi - x) \quad 0 < x < 2$

$$\left[\textbf{Ans.}: (\pi-1)+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x\right]$$

11. $f(x)=1 \quad 0 < x < 1$
$\quad = 2 \quad 1 < x < 2$

$$\left[\textbf{Ans.}: 3-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)\pi x\right]$$

12. $f(x)=x \quad 0 < x < 1$
$\quad = 0 \quad 1 < x < 2$

$$\left[\textbf{Ans.}: \frac{1}{4}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\cos n\pi x + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin n\pi x\right]$$

13. $f(x)=2 \quad -2 < x < 0$
$\quad = x \quad 0 < x < 2$

$$\left[\textbf{Ans.}: \frac{3}{2}+\frac{2}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\frac{\cos n\pi x}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}\right]$$

7.7 FOURIER SERIES OF EVEN AND ODD FUNCTIONS

A function $f(x)$ is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$ for all x, (Fig. 7.2).

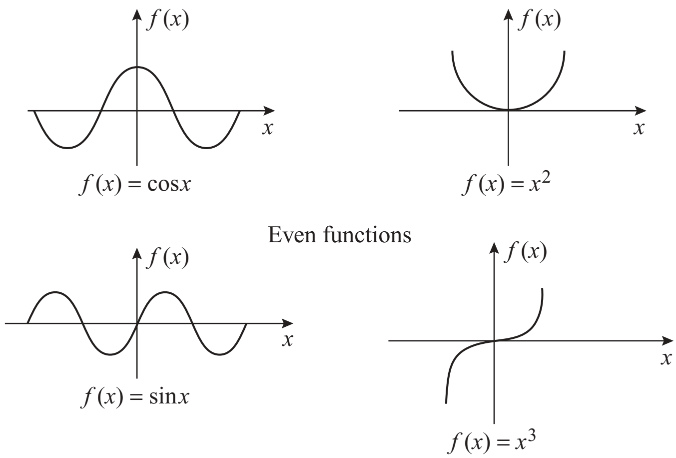


Fig. 7.2 ***Even and odd functions***

Properties of Even and Odd Functions

(i) The product of two even functions is even.
(ii) The product of two odd functions is even.
(iii) The product of an even function and an odd function is odd.
(iv) The sum or difference of two even functions is even.
(v) The sum or difference of two odd functions is odd.
(vi) If $f(x)$ is even, $\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx$
(viii) If $f(x)$ is odd, $\int_{-l}^{l} f(x)\,dx = 0$

We know that the Fourier series of a function $f(x)$ in the interval $(-l, l)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where

$$a_0 = \frac{1}{2l}\int_{l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Case I When $f(x)$ is an even function, $\int_{-l}^{l} f(x)\,dx = 2\int_0^l f(x)\,dx$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx$$

Since the product of two even functions is even,

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

Since the product of an even function and an odd function is odd,

$$b_n = 0$$

Corollary The Fourier series of an even function $f(x)$ in the interval $(-\pi, \pi)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where
$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Case II When $f(x)$ is an odd function,

$$a_0 = 0 \quad \text{and} \quad a_n = 0$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Corollary The Fourier series of an odd function $f(x)$ in the interval $(-\pi, \pi)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where
$$a_0 = 0 \quad \text{and} \quad a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Thus, the Fourier series of an even function consists entirely of cosine terms while the Fourier series of an odd function consists entirely of sine terms.

Example 1

Find the Fourier series of $f(x) = x$ in $-\pi < x < \pi$. **[Summer 2014]**

Solution

$$f(-x) = -x \qquad -\pi < -x < \pi$$
$$f(-x) = -f(x) \quad \pi > x > -\pi \qquad \text{or} \qquad -\pi < x < \pi$$

$f(x) = x$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx \\ &= \frac{2}{\pi}\int_0^{\pi} x\sin nx \, dx \\ &= \frac{2}{\pi}\left| x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi} \\ &= \frac{2}{\pi}\left[-\pi\frac{\cos n\pi}{n}\right] \quad [\because \sin n\pi = \sin 0 = 0] \\ &= -\frac{2}{n}(-1)^n \quad [\because \cos n\pi = (-1)^n] \end{aligned}$$

Hence,
$$f(x) = \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n \sin nx$$
$$x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

Example 2

Find the Fourier series of $f(x) = x^2$ *in the interval* $(-\pi, \pi)$*. Hence, deduce that* $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dots$ **[Summer 2016]**

Solution

$f(x) = x^2$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} x^2\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{x^3}{3}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^3}{3}\right)$$

$$= \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi}$$

$$= \frac{4}{n^2}\cos n\pi \quad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{4}{n^2}(-1)^n \quad \left[\because \cos n\pi = (-1)^n\right]$$

Hence, $$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right) \quad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$0 = \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \ldots\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$$

Example 3

Find the Fourier series of $f(x) = x^3$ in the interval $(-\pi, \pi)$.

Solution

$f(x) = x^3$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx\,dx \\
&= \frac{2}{\pi}\left| x^3\left(-\frac{\cos nx}{n}\right) - 3x^2\left(-\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right|_0^{\pi} \\
&= \frac{2}{\pi}\left(-\pi^3\frac{\cos n\pi}{n} + 6\pi\frac{\cos n\pi}{n^3}\right) \qquad [\because \sin n\pi = \sin 0 = 0] \\
&= 2(-1)^n\left(-\frac{\pi^2}{n} + \frac{6}{n^3}\right) \qquad [\because \cos n\pi = (-1)^n]
\end{aligned}$$

Hence, $$f(x) = 2\sum_{n=1}^{\infty}(-1)^n\left(-\frac{\pi^2}{n} + \frac{6}{n^3}\right)\sin nx$$

$$= -2\pi^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx + 6\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin nx$$

$$x^3 = 2\pi^2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right)$$
$$-6\left(\sin x - \frac{1}{2^3}\sin 2x + \frac{1}{3^3}\sin 3x - \cdots\right)$$

Example 4

Find the Fourier series of $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ *in the interval* $(-\pi, \pi)$ *and deduce that* $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots.$

Solution

$f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12}-\frac{x^2}{4}\right)\mathrm{d}x \\
&= \frac{1}{\pi}\left|\frac{\pi^2 x}{12}-\frac{x^3}{12}\right|_0^{\pi} \\
&= \frac{1}{\pi}\left(\frac{\pi^3}{12}-\frac{\pi^3}{12}\right) \\
&= 0 \\
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12}-\frac{x^2}{4}\right)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\left|\left(\frac{\pi^2}{12}-\frac{x^2}{4}\right)\left(\frac{\sin nx}{n}\right)-\left(-\frac{x}{2}\right)\left(-\frac{\cos nx}{n^2}\right)+\left(-\frac{1}{2}\right)\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi} \\
&= \frac{2}{\pi}\left(-\frac{\pi}{2n^2}\cos n\pi\right) \qquad [\because\ \sin n\pi = \sin 0 = 0] \\
&= \frac{-(-1)^n}{n^2} \qquad \left[\because\ \cos n\pi = (-1)^n\right]
\end{aligned}$$

Hence, $f(x) = \sum_{n=1}^{\infty} \frac{-(-1)^n}{n^2}\cos nx$

$$\frac{\pi^2}{12}-\frac{x^2}{4} = \frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \dots \qquad \dots (1)$$

Putting $x = 0$ in Eq. (1),

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots$$

Example 5

Find the Fourier series of $f(x) = \sin ax$ *in the interval* $(-\pi, \pi)$.

Solution

$$f(-x) = \sin a(-x) = -\sin ax$$
$$f(-x) = -f(x)$$

$f(x) = \sin ax$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} \sin ax \sin nx \,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} [\cos(n-a)x - \cos(n+a)x]\,\mathrm{d}x \\
&= \frac{1}{\pi}\left|\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{n+a}\right|_0^{\pi} \\
&= \frac{1}{\pi}\left[\frac{\sin(n-a)\pi}{n-a} - \frac{\sin(n+a)\pi}{n+a}\right] \qquad [\because \sin 0 = 0] \\
&= \frac{1}{\pi}\left(\frac{\sin n\pi \cos a\pi - \sin a\pi \cos n\pi}{n-a} - \frac{\sin n\pi \cos a\pi + \sin a\pi \cos n\pi}{n+a}\right) \\
&= \frac{1}{\pi}\left[\frac{-(-1)^n \sin a\pi}{n-a} - \frac{(-1)^n \sin a\pi}{n+a}\right] \qquad \left[\because \sin n\pi = 0, \cos n\pi = (-1)^n\right] \\
&= \frac{-(-1)^n \sin a\pi}{\pi}\left(\frac{1}{n-a} + \frac{1}{n+a}\right) \\
&= \frac{2n(-1)^n \sin a\pi}{\pi(a^2 - n^2)}
\end{aligned}$$

Hence, $f(x) = \dfrac{2\sin a\pi}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n}{a^2 - n^2}\sin nx$

$$\sin ax = -\frac{2\sin a\pi}{\pi}\left[\frac{1}{a^2 - 1^2}\sin x - \frac{2}{a^2 - 2^2}\sin 2x + \frac{3}{a^2 - 3^2}\sin 3x - \cdots\right]$$

Example 6

Find the Fourier series of $f(x) = x \sin x$ *in the interval* $(-\pi, \pi)$. *Hence, deduce that* $\dfrac{\pi - 1}{4} = \dfrac{1}{1\cdot 3} - \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} - \ldots$

Solution

$$f(-x) = -x \sin(-x)$$
$$= x \sin x$$
$$= f(x)$$

$f(x) = x \sin x$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x$$
$$= \frac{1}{\pi}\int_0^{\pi} x\sin x\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left| x(-\cos x) - (-\sin x)\right|_0^{\pi}$$
$$= \frac{1}{\pi}\left[\pi(-\cos\pi)\right] \quad [\because\ \sin\pi = \sin 0 = 0]$$
$$= 1 \quad [\because\ \cos\pi = -1]$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$
$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,\mathrm{d}x$$
$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left| x\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right] - \left[-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\right|_0^{\pi}, n \neq 1$$
$$= \frac{1}{\pi}\left[-\pi\frac{\cos(n+1)\pi}{n+1} + \pi\frac{\cos(n-1)\pi}{n-1}\right], n \neq 1 \quad [\because\ \sin(n+1)\pi = \sin(n-1)\pi = 0]$$
$$= \frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} \quad [\because\ \cos(n+1)\pi = \cos(n-1)\pi = -(-1)^n]$$
$$= \frac{-2(-1)^n}{n^2-1}$$
$$= \frac{2(-1)^{n+1}}{n^2-1},\ n \neq 1$$

For $n = 1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\, dx$$

$$= \frac{1}{\pi}\left| -x\frac{\cos 2x}{2} + \frac{\sin 2x}{4} \right|_0^{\pi}$$

$$= \frac{1}{\pi}\left(-\pi\frac{\cos 2\pi}{2} \right) \quad [\because \sin 2\pi = \sin 0 = 0]$$

$$= -\frac{1}{2} \quad [\because \cos 2\pi = 1]$$

Hence, $$f(x) = 1 - \frac{1}{2} + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx$$

$$x\sin x = \frac{1}{2} + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx$$

$$= \frac{1}{2} - 2\left(\frac{1}{3}\cos 2x - \frac{1}{8}\cos 3x + \frac{1}{15}\cos 4x - \cdots \right) \quad \ldots (1)$$

Putting $x = \frac{\pi}{2}$ in Eq. (1),

$$\frac{\pi}{2}\sin\frac{\pi}{2} = \frac{1}{2} + 2\left(-\frac{1}{3}\cos\pi + \frac{1}{8}\cos\frac{3\pi}{2} - \frac{1}{15}\cos 2\pi + \cdots \right)$$

$$\frac{\pi}{2} = \frac{1}{2} - \frac{2}{3}\cos\pi - \frac{2}{15}\cos 2\pi - \ldots$$

$$\frac{\pi-1}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \ldots$$

Example 7

Find the Fourier series of $f(x) = \cosh ax$ *in the interval* $(-\pi, \pi)$.

Solution

$$f(-x) = \cosh a(-x)$$
$$= \cosh ax$$
$$= f(x)$$

$f(x) = \cosh ax$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx$$
$$= \frac{1}{\pi}\int_0^\pi \cosh ax\,dx$$
$$= \frac{1}{\pi}\int_0^\pi \left(\frac{e^{ax}+e^{-ax}}{2}\right)dx$$
$$= \frac{1}{2\pi}\left|\frac{e^{ax}}{a}+\frac{e^{-ax}}{-a}\right|_0^\pi$$
$$= \frac{1}{2\pi a}\left(e^{a\pi}-e^{-a\pi}\right)$$
$$= \frac{\sinh a\pi}{\pi a}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$$
$$= \frac{2}{\pi}\int_0^\pi \cosh ax\cos nx\,dx$$
$$= \frac{2}{\pi}\int_0^\pi \left(\frac{e^{ax}+e^{-ax}}{2}\right)\cos nx\,dx$$
$$= \frac{1}{\pi}\int_0^\pi (e^{ax}\cos nx + e^{-ax}\cos nx)dx$$
$$= \frac{1}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx) + \frac{e^{-ax}}{a^2+n^2}(-a\cos nx + n\sin nx)\right|_0^\pi$$
$$= \frac{a\cos n\pi}{\pi(a^2+n^2)}(e^{a\pi}-e^{-a\pi}) \quad [\because\ \sin n\pi = \sin 0 = 0]$$
$$= \frac{a\cos n\pi}{\pi(a^2+n^2)}2\sinh a\pi$$
$$= \frac{(-1)^n 2a\sinh a\pi}{\pi(a^2+n^2)}$$

Hence, $f(x) = \dfrac{\sinh a\pi}{\pi a} + \dfrac{2a}{\pi}\sinh a\pi \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx$

Example 8

Find the Fourier series of $f(x) = e^{-|x|}$ *in the interval* $(-\pi, \pi)$.

Solution

$$f(x) = e^{-|x|}$$
$$f(-x) = e^{-|-x|}$$
$$= e^{-|x|} = f(x)$$

$f(x) = e^{-|x|}$ is an even function.

Hence, $b_n = 0$

$$f(x) = e^{x} \qquad -\pi < x < 0$$
$$= e^{-x} \qquad 0 < x < \pi$$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$
$$= \frac{1}{\pi}\int_0^{\pi} e^{-x}\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left|-e^{-x}\right|_0^{\pi}$$
$$= \frac{1}{\pi}(1 - e^{-\pi})$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$
$$= \frac{2}{\pi}\int_0^{\pi} e^{-x}\cos nx\,\mathrm{d}x$$
$$= \frac{2}{\pi}\left|\frac{e^{-x}}{n^2+1}(-\cos nx + n\sin nx)\right|_0^{\pi}$$
$$= \frac{2}{\pi(n^2+1)}\left[e^{-\pi}(-\cos n\pi) + \cos 0\right] \quad [\because\ \sin n\pi = \sin 0 = 0, \cos 0 = 1]$$
$$= \frac{2}{\pi(n^2+1)}[1 - (-1)^n e^{-\pi}]$$

Hence, $f(x) = \dfrac{1}{\pi}(1 - e^{-\pi}) + \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n e^{-\pi}}{n^2+1}\right]\cos nx$

Example 9

Find the Fourier series of $f(x) = |\cos x|$ *in the interval* $(-\pi, \pi)$.

Solution

$f(x) = |\cos x|$ is an even function.

Hence, $b_n = 0$

$$f(x) = \cos x \qquad 0 < x < \frac{\pi}{2}$$

$$= -\cos x \qquad \frac{\pi}{2} < x < \pi$$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} (-\cos x)\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\left|\sin x\right|_0^{\frac{\pi}{2}} - \left|\sin x\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\sin\frac{\pi}{2} - \left(\sin\pi - \sin\frac{\pi}{2}\right)\right]$$

$$= \frac{2}{\pi} \qquad \left[\because\ \sin\frac{\pi}{2} = 1, \sin\pi = 0\right]$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x\cos nx\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} (-\cos x)\cos nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} \left\{\cos(n+1)x + \cos(n-1)x\right\}\mathrm{d}x - \int_{-\frac{\pi}{2}}^{\pi} \{\cos(n+1)x + \cos(n-1)x\}\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right|_0^{\frac{\pi}{2}} - \left|\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right|_{\frac{\pi}{2}}^{\pi}\right], \quad n \neq 1$$

$$= \frac{2}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1}\right], \quad n \neq 1 \qquad \left[\begin{aligned} \because\ \sin\left(\frac{n\pi}{2} + \frac{\pi}{2}\right) &= \cos\frac{n\pi}{2} \\ \sin\left(\frac{n\pi}{2} - \frac{\pi}{2}\right) &= -\cos\frac{n\pi}{2} \end{aligned}\right]$$

$$= -\frac{4}{\pi(n^2-1)}\cos\frac{n\pi}{2}, \quad n \neq 1$$

For $n = 1$,

$$a_1 = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos^2 x\,dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos^2 x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2x}{2}\right)dx - \int_{\frac{\pi}{2}}^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx\right]$$

$$= \frac{1}{\pi}\left[\left|x + \frac{\sin 2x}{2}\right|_0^{\frac{\pi}{2}} - \left|x + \frac{\sin 2x}{2}\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \left(\pi - \frac{\pi}{2}\right)\right] \quad [\because \ \sin\pi = \sin 2\pi = 0]$$

$$= 0$$

Hence, $$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2}^{\infty}\frac{1}{n^2-1}\cos\frac{n\pi}{2}\cos nx$$

$$|\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\left(-\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x - \frac{1}{35}\cos 6x + \ldots\right)$$

$$= \frac{2}{\pi} + \frac{4}{\pi}\left(\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x - \ldots\right)$$

Example 10

Find the Fourier series of $f(x) = |x|$ in the interval $[-\pi, \pi]$.

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$ **[Winter 2016]**

Solution

$$f(x) = |x| \qquad -\pi < x < \pi$$

i.e., $$f(x) = -x \qquad -\pi < x \le 0$$
$$= x \qquad 0 \le x < \pi$$

$f(x) = |x|$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{2}\right)$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|x\left(\frac{\sin nx}{n}\right)+\frac{\cos nx}{n^2}\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{\cos n\pi}{n^2}-\frac{\cos 0}{n^2}\right) \quad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{2}{\pi n^2}[(-1)^n - 1] \quad \left[\because \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

Hence,
$$f(x) = \frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2}\cos nx$$

$$|x| = \frac{\pi}{2}-\frac{4}{\pi}\left(\frac{1}{1^2}\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+\dots\right) \quad \dots (1)$$

Putting $x = 0$ in Eq. (1),

$$0 = \frac{\pi}{2}-\frac{4}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots$$

Example 11

Find the Fourier series of $f(x) = -k \qquad -\pi < x < 0$

$= k \qquad 0 < x < \pi$ **[Winter 2014]**

Solution

$$
\begin{aligned}
f(-x) &= -k \qquad -\pi < -x < 0 \quad \text{or} \quad 0 < x < \pi \\
&= k \qquad 0 < -x < \pi \quad \text{or} \quad -\pi < x < 0 \\
f(-x) &= -f(x)
\end{aligned}
$$

$f(x)$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx \\
&= \frac{2}{\pi}\int_0^{\pi} k\sin nx \, dx \\
&= \frac{2k}{\pi}\left|-\frac{\cos nx}{n}\right|_0^{\pi} \\
&= \frac{2k}{n\pi}(-\cos n\pi + \cos 0) \\
&= \frac{2k}{n\pi}\left[-(-1)^n + 1\right] \qquad \left[\because \cos n\pi = (-1)^n, \cos 0 = 1\right] \\
&= \frac{2k}{n\pi}\left[1-(-1)^n\right]
\end{aligned}
$$

Hence,

$$
\begin{aligned}
f(x) &= \frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1-(-1)^n\right]\sin nx \\
&= \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)
\end{aligned}
$$

Example 12

Find the Fourier series of $f(x) = \begin{cases} \pi + x & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$ **[Summer 2015]**

Solution

$$
\begin{aligned}
f(x) &= \pi - x \qquad 0 < x < \pi \\
f(-x) &= \pi + x \qquad -\pi < x < 0
\end{aligned}
$$

$$f(-x) = f(x)$$

$f(x)$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} (\pi - x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left|\pi x - \frac{x^2}{2}\right|_0^{\pi} \\
&= \frac{1}{\pi}\left(\pi^2 - \frac{\pi^2}{2}\right) \\
&= \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{n} \\
&= \frac{2}{\pi}\left|(\pi - x)\left(\frac{\sin nx}{n}\right) - \frac{\cos nx}{n^2}\right|_0^{\pi} \\
&= \frac{2}{\pi}\left[-\frac{\cos n\pi}{n^2} + \frac{\cos 0}{n^2}\right] && [\because \sin n\pi = \sin 0 = 0] \\
&= \frac{2}{\pi}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right] && [\because \cos n\pi = (-1)^n, \cos 0 = 1] \\
&= \frac{2}{\pi}\left[\frac{1 - (-1)^n}{n^2}\right]
\end{aligned}$$

Hence, $$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx$$

$$= \frac{\pi}{2} + \frac{4}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right)$$

Example 13

Find the Fourier series of the function f(x) given by

$$f(x) = \begin{cases} 1 + \dfrac{2x}{\pi} & -\pi \le x \le 0 \\ 1 - \dfrac{2x}{\pi} & 0 \le x \le \pi \end{cases}$$

Hence, deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$. **[Winter 2016]**

Solution

$$f(-x) = 1 + \frac{2(-x)}{\pi} \quad -\pi \le -x \le 0$$

$$= 1 - \frac{2x}{\pi} \quad 0 \le x \le \pi$$

$$f(-x) = 1 - \frac{2}{\pi}(-x) \quad 0 \le x \le \pi$$

$$= 1 + \frac{2x}{\pi} \quad -\pi \le x \le 0$$

$$f(-x) = f(x)$$

$f(x)$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\mathrm{d}x$$

$$= \frac{1}{\pi}\left| x - \frac{2}{\pi}\cdot\frac{x^2}{2}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left| x - \frac{x^2}{\pi}\right|_0^{\pi}$$

$$= 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \frac{2}{\pi n^2}\cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2}{\pi n^2}\cos n\pi + \frac{2}{\pi n^2}\right] \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{4}{n^2\pi^2}[1-\cos n\pi]$$

$$= \frac{4}{n^2\pi^2}[1-(-1)^n] \qquad [\because \cos n\pi = (-1)]$$

Hence, $$f(x) = 0 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}[1-(-1)^n]\cos nx$$

$$= \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx$$

$$= \frac{4\cdot 2}{\pi^2}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right) \qquad \text{...(1)}$$

At $x = 0$,

$$f(0) = \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right]$$

$$f(0) = \frac{1}{2}[1+1] = \frac{2}{2} = 1$$

Putting $x = 0$ in Eq. (1),

$$f(0) = \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos(0) + \frac{1}{3^2}\cos(0) + \frac{1}{5^2}\cos(0) + \cdots\right]$$

$$1 = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Example 14

Find the Fourier series of $f(x) = \cos x \quad -\pi < x < 0$

$= -\cos x \quad 0 < x < \pi$

Solution

$$\begin{aligned} f(-x) &= \cos(-x) && -\pi < -x < 0 \\ &= -\cos(-x) && 0 < -x < \pi \\ f(-x) &= \cos x && 0 < x < \pi \\ &= -\cos x && -\pi < x < 0 \\ f(-x) &= -f(x) \end{aligned}$$

$f(x)$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} (-\cos x)\sin nx\,dx \\ &= -\frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x + \sin(n-1)x]dx \\ &= -\frac{1}{\pi}\left|-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right|_0^{\pi}, \; n \neq 1 \end{aligned}$$

$$= \frac{1}{\pi}\left[\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \frac{1}{n+1} - \frac{1}{n-1}\right], \ n \neq 1$$

$$= -\frac{1}{\pi}\left(\frac{1+\cos n\pi}{n+1} + \frac{1+\cos n\pi}{n-1}\right), \ n \neq 1 \ [\because \cos(n\pi+\pi) = \cos(n\pi-\pi) = -\cos n\pi]$$

$$= -\frac{2n}{\pi(n^2-1)}(1+\cos n\pi), \quad n \neq 1$$

$$= -\frac{2n}{\pi(n^2-1)}[1+(-1)^n], \ n \neq 1 \quad \left[\because \ \cos n\pi = (-1)^n\right]$$

For $n = 1$,

$$b_1 = \frac{2}{\pi}\int_0^{\pi}(-\cos x)\sin x \, dx$$

$$= -\frac{1}{\pi}\int_0^{\pi}\sin 2x \, dx$$

$$= -\frac{1}{\pi}\left|-\frac{\cos 2x}{2}\right|_0^{\pi}$$

$$= \frac{1}{2\pi}(\cos 2\pi - \cos 0)$$

$$= 0 \quad [\because \ \cos 2\pi = \cos 0 = 1]$$

Hence,
$$f(x) = -\frac{2}{\pi}\sum_{n=2}^{\infty}\frac{n}{n^2-1}[1+(-1)^n]\sin nx$$

$$= -\frac{2}{\pi}\left(\frac{2}{3}2\sin 2x + \frac{4}{15}2\sin 4x + \frac{6}{35}2\sin 6x + \cdots\right)$$

$$= -\frac{8}{\pi}\left(\frac{1}{3}\sin 2x + \frac{2}{15}\sin 4x + \frac{3}{35}\sin 6x + \cdots\right)$$

Example 15

Obtain the Fourier series of periodic function $f(x) = 2x$, where $-1 < x < 1$. **[Winter 2016]**

Solution

$$f(x) = 2x$$
$$f(-x) = -2x = -f(x)$$
$$f(-x) = -f(x)$$

$f(x)$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function $f(x)$ with period $2l = 2$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= 2\int_0^l 2x \sin n\pi x\, dx$$

$$= 2\left| 2x\left(-\frac{\cos n\pi x}{n\pi}\right) - (2)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1$$

$$= 2\left| -\frac{2x}{n\pi}\cos n\pi x + \frac{2}{n^2\pi^2}\sin n\pi x\right|_0^1$$

$$= 2\left(-\frac{2}{n\pi}\cos n\pi\right) \quad [\because \sin n\pi = \sin 0 = 0]$$

$$= -\frac{4}{n\pi}(-1)^n \qquad \left[\because \cos n\pi = (-1)^n\right]$$

$$= (-1)^{n+1}\frac{4}{n\pi}$$

Hence, $$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{4}{n\pi}\sin n\pi x$$

Example 16

Find the Fourier series of $f(x) = 1 - x^2$ in the interval (–1, 1).

Solution

$$f(-x) = 1 - (-x)^2 = 1 - x^2 = f(x)$$

$f(x) = 1 - x^2$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period $2l = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

$$
\begin{aligned}
a_0 &= \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x \\
&= \int_0^1 (1-x^2)\,\mathrm{d}x \\
&= \left| x - \frac{x^3}{3} \right|_0^1 \\
&= 1 - \frac{1}{3} \\
&= \frac{2}{3}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x \\
&= 2\int_0^l (1-x^2)\cos n\pi x\,\mathrm{d}x \\
&= 2\left| (1-x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right) \right|_0^1 \\
&= 2\left(-2\frac{\cos n\pi}{n^2\pi^2}\right) \qquad [\because \sin n\pi = \sin 0 = 0] \\
&= \frac{-4(-1)^n}{n^2\pi^2} \qquad \left[\because \cos n\pi = (-1)^n\right]
\end{aligned}
$$

Hence, $$f(x) = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$$

$$1 - x^2 = \frac{2}{3} - \frac{4}{\pi^2}\left(-\frac{1}{1^2}\cos \pi x + \frac{1}{2^2}\cos 2\pi x - \frac{1}{3^2}\cos 3\pi x + \cdots\right)$$

Example 17

Find the Fourier series of $f(x) = x^2 - 2$ *in* $-2 \le x \le 2$. **[Winter 2014]**

Solution

$f(x) = x^2 - 2$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period $2l = 4$ is given by

$$
\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \\
&= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}
\end{aligned}
$$

$$
\begin{aligned}
a_0 &= \frac{1}{l}\int_0^l f(x)\,dx \\
&= \frac{1}{2}\int_0^2 (x^2-2)\,dx \\
&= \frac{1}{2}\left|\frac{x^3}{3}-2x\right|_0^2 \\
&= \frac{1}{2}\left(\frac{8}{3}-4\right) \\
&= -\frac{2}{3}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx \\
&= \frac{2}{2}\int_0^2 (x^2-2)\cos\frac{n\pi x}{2}\,dx \\
&= \left|(x^2-2)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right)-(2x)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)+(2)\left(-\frac{8}{n^3\pi^3}\sin\frac{n\pi x}{2}\right)\right|_0^2 \\
&= -(4)\left(-\frac{4}{n^2\pi^2}\cos n\pi\right) \\
&= \frac{16}{n^2\pi^2}\cos n\pi \qquad [\because\ \sin n\pi = \sin 0 = 0] \\
&= \frac{16}{n^2\pi^2}(-1)^n \qquad \left[\because\ \cos n\pi = (-1)^n\right]
\end{aligned}
$$

Hence,
$$f(x) = -\frac{2}{3}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n\cos\frac{n\pi x}{2}$$

$$x^2-2 = -\frac{2}{3}-\frac{16}{\pi^2}\left(\cos\frac{\pi x}{2}-\frac{1}{4}\cos\pi x+\frac{1}{9}\cos\frac{3\pi x}{2}-\cdots\right)$$

Example 18

Find the Fourier series of $f(x) = x|x|$ *in the interval* $(-1, 1)$.

Solution

$$
\begin{aligned}
f(x) &= x|x| \\
f(-x) &= -x|-x| \\
&= -x|x| = -f(x)
\end{aligned}
$$

$f(x) = x|x|$ is an odd function.
Hence, $a_0 = 0$ and $a_n = 0$

$$f(x) = -x^2 \qquad -1 < x < 0$$
$$= x^2 \qquad 0 < x < 1$$

The Fourier series of an odd function with period $2l = 2$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$
$$= \sum_{n=1}^{\infty} b_n \sin n\pi x$$
$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$
$$= 2\int_0^1 x^2 \sin n\pi x\,dx$$
$$= 2\left| x^2\left(-\frac{\cos n\pi x}{n\pi}\right) - 2x\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + 2\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right|_0^1$$
$$= 2\left[-\frac{\cos n\pi}{n\pi} + \frac{2\cos n\pi}{n^3\pi^3} - \frac{2\cos 0}{n^3\pi^3}\right] \qquad [\because \sin n\pi = \sin 0 = 0]$$
$$= 2\left[-\frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right] \qquad \left[\because \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

Hence, $$f(x) = 2\sum_{n=1}^{\infty}\left[-\frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right]\sin n\pi x$$

Example 19

Find the Fourier series of $f(x) = x - x^3$ in $-1 < x < 1$. **[Winter 2013]**

Solution

$$f(-x) = -x + x^3 \qquad -1 < x < 1$$
$$= -(x - x^3)$$
$$= -f(x)$$

$f(x) = x - x^3$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period $2l = 2$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$
$$= \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= 2\int_0^1 (x - x^3)\sin n\pi x\,dx$$

$$= 2\left|(x-x^3)\left(-\cos\frac{n\pi x}{n\pi}\right) - (1-3x^2)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + (-6x)\left(\frac{\cos n\pi x}{n^3\pi^3}\right) - (-6)\left(\frac{\sin n\pi x}{n^4\pi^4}\right)\right|_0^1$$

$$= 2\left[-6\frac{\cos n\pi}{n^3\pi^3}\right] \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= -12\frac{(-1)^n}{n^3\pi^3} \quad \left[\because\ \cos n\pi = (-1)^n\right]$$

Hence, $$f(x) = -\frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n\pi x$$

$$x - x^3 = \frac{12}{\pi^3}\left(\sin\pi x - \frac{\sin 2\pi x}{2^3} + \frac{\sin 3\pi x}{3^3} - \cdots\right)$$

Example 20

Find the Fourier series of $f(x) = \frac{1}{2} + x \quad -\frac{1}{2} < x < 0$

$$= \frac{1}{2} - x \qquad 0 < x < \frac{1}{2}$$

Solution

$$f(-x) = \frac{1}{2} - x \qquad -\frac{1}{2} < -x < 0 \ \text{ or } \ 0 < x < \frac{1}{2}$$

$$= \frac{1}{2} + x \qquad 0 < -x < \frac{1}{2} \ \text{ or } \ -\frac{1}{2} < x < 0$$

$$f(-x) = f(x)$$

$f(x)$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period $2l = 1$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\pi x$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\mathrm{d}x$$

$$= \frac{1}{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(\frac{1}{2} - x\right)\mathrm{d}x$$

$$= 2\left|\frac{x}{2} - \frac{x^2}{2}\right|_0^{\frac{1}{2}}$$

$$= 2\left(\frac{1}{4} - \frac{1}{8}\right)$$

$$= \frac{1}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$= \frac{2}{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(\frac{1}{2} - x\right)\cos 2n\pi x\ \mathrm{d}x$$

$$= 4\left|\left(\frac{1}{2} - x\right)\left(\frac{\sin 2n\pi x}{2n\pi}\right) - (-1)\left(-\frac{\cos 2n\pi x}{4n^2\pi^2}\right)\right|_0^{\frac{1}{2}}$$

$$= 4\left[\left(-\frac{\cos\ n\pi}{4n^2\pi^2} + \frac{\cos 0}{4n^2\pi^2}\right)\right] \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{n^2\pi^2}\left[1-(-1)^n\right] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

Hence,
$$f(x) = \frac{1}{4} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos\ 2n\pi x$$

EXERCISE 7.2

Find the Fourier series of the following functions:

1. $f(x) = \dfrac{x(\pi^2 - x^2)}{12} \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.:} \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx\right]$$

2. $f\ (x) = \cos\ ax \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.:} \frac{\sin\ a\pi}{\pi} + \frac{2a\ \sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2 - n^2}\cos\ nx\right]$$

3. $f(x) = x\cos x \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.:} -\frac{1}{2}\sin x + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\sin nx\right]$$

4. $f(x) = |\sin x| \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.:} \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx\right]$$

5. $f(x) = \sqrt{1-\cos x} \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.:} \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos nx\right]$$

6. $f(x) = \sinh ax \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.:} \frac{2}{\pi}\sinh a\pi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{a^2+n^2}\sin nx\right]$$

7. $f(x) = \frac{-(\pi+x)}{2} \qquad -\pi < x < 0$

$= \frac{\pi - x}{2} \qquad 0 < x < \pi$

$$\left[\textbf{Ans.:} \sum_{n=1}^{\infty}\frac{1}{n}\sin nx\right]$$

8. $f(x) = x + \frac{\pi}{2} \qquad -\pi < x < 0$

$= \frac{\pi}{2} - x \qquad 0 < x < \pi$

$$\left[\textbf{Ans.:} \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx\right]$$

9. $f(x) = |x| \qquad -2 < x < 2$

$$\left[\textbf{Ans.:} 1 - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\cos n\pi x\right]$$

10. $f(x) = a^2 - x^2 \qquad -a < x < a$

$$\left[\textbf{Ans.:} \frac{2a^2}{3} - \frac{4a^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{a}\right]$$

11. $f(x) = \sin ax \qquad -l < x < l$

$$\left[\textbf{Ans.:} \sin al\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2\pi^2 - a^2l^2}\sin\frac{n\pi x}{l}\right]$$

7.8 HALF-RANGE FOURIER SERIES

Any arbitrary function $f(x)$ with period $2l$ which is defined in half of the interval $(0, l)$ can also be represented in terms of sine and cosine functions. A half-range expansion containing only cosine terms is known as a *half-range cosine series*. Similarly, a half-range expansion containing only sine terms is known as a *half-range sine series*.

To represent any function $f(x)$ in the half-range cosine series in the interval $(0, l)$, we extend the function by reflecting it in the vertical axis (i.e., y axis) so that $f(-x) = f(x)$. The extended function is an even function in $(-l, l)$ and is periodic with period $2l$. The half-range cosine series of such a function is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where
$$a_0 = \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

Corollary If any function with period 2π is defined in the interval $(0, \pi)$ then the half-range cosine series of such a function is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where
$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

Similarly, to represent any function $f(x)$ in the half-range sine series in the interval $(0, l)$, we extend the function by reflecting it in the origin so that $f(-x) = -f(x)$. The extended function is an odd function in $(-l, l)$ and is periodic with period $2l$. The half-range sine series of such a function is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where
$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

Corollary If any function with period 2π is defined in the interval $(0, \pi)$ then the half-range sine series of such a function is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x$$

Example 1

Find the half-range cosine series of $f(x) = x$ in the interval $(0, \pi)$.

Solution

The half-range cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} x\,\mathrm{d}x \\
&= \frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{\pi} \\
&= \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\left|x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi} \\
&= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] \qquad [\because\ \sin n\pi = \sin 0 = 0] \\
&= \frac{2}{\pi n^2}[(-1)^n - 1] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]
\end{aligned}$$

Hence,

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos nx$$

$$x = \frac{\pi}{2} + \frac{2}{\pi}\left(-\frac{2}{1^2}\cos x - \frac{2}{3^2}\cos 3x - \cdots\right)$$

Example 2

Find the Fourier cosine series of $f(x) = x^2 \quad 0 < x < \pi$. **[Summer 2015]**

Solution

The Fourier cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x^2 dx$$

$$a_0 = \frac{1}{\pi}\left|\frac{x^3}{3}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^3}{3}\right)$$

$$= \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left|x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left|x^2\left(\frac{\sin nx}{n}\right) + 2x\left(\frac{\cos nx}{n^2}\right) - \left(\frac{2\sin nx}{n^3}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[2\pi\frac{\cos n\pi}{n^2}\right] \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{4\cos n\pi}{n^2}$$

$$= \frac{4(-1)^n}{n^2} \qquad [\because \cos n\pi = (-)^n]$$

Hence, $$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

Example 3

Find the half-range sine series of $f(x) = x^2$ *in the interval* $(0, \pi)$.

Solution

The half-range sine series of $f(x)$ with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\,dx \\ &= \frac{2}{\pi}\left| x^2\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right|_0^{\pi} \\ &= \frac{2}{\pi}\left[-\pi^2\left(\frac{\cos n\pi}{n}\right) + 2\left(\frac{\cos n\pi}{n^3}\right) - 2\left(\frac{\cos 0}{n^3}\right)\right] \quad [\because\ \sin n\pi = \sin 0 = 0] \\ &= \frac{2}{\pi}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right] \quad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right] \end{aligned}$$

Hence, $f(x) = \dfrac{2}{\pi}\sum_{n=1}^{\infty}\left[\dfrac{-\pi^2(-1)^n}{n} + \dfrac{2(-1)^n}{n^3} - \dfrac{2}{n^3}\right]\sin nx$

Example 4

Find the half-range sine series of $f(x) = x^3$ *in* $0 \le x \le \pi$.

[Summer 2017]

Solution

The half-range sine series of $f(x)$ with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin x\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx \, dx$$

$$= \frac{2}{\pi}\left| x^3\left(-\frac{\cos nx}{x}\right) - (3x^2)\left(-\frac{\sin nx}{n^2}\right) + (6x)\left(\frac{\cos nx}{n^3}\right) - (6)\left(\frac{\sin nx}{n^4}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left| -x^3\left(\frac{\cos nx}{n}\right) + 3x^2\left(\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi^3\frac{(-1)^n}{n} + 6\pi\frac{(-1)^n}{n^3}\right] \qquad \left[\begin{array}{l}\because \sin n\pi = \sin = 0\\ \cos n\pi = (-1)\end{array}\right]$$

$$= \frac{2}{\pi}\cdot\pi\left[\frac{6}{n^3} - \frac{\pi^2}{n}\right](-1)^n$$

$$= \frac{2(-1)^n}{n^3}(6 - n^2\pi^2)$$

Hence, $f(x) = 2\sum_{n=1}^{\infty}\frac{1}{n^3}(-1)^n\,(6 - n^2\pi^2)\sin nx$

Example 5

Find the cosine series for $f(x) = \pi - x$ in the interval $0 < x < \pi$.

[Winter 2017]

Solution

The cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(\pi - x)\,dx$$

$$= \frac{1}{\pi}\left|\pi x - \frac{x^2}{2}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\pi^2 - \frac{\pi^2}{2}\right)$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\cos n\pi}{n^2} + \frac{1}{n^2}\right] \quad \left[\because \begin{array}{l}\sin n\pi = \sin 0 = 0\\ \cos 0 = 1\end{array}\right]$$

$$= \frac{2}{\pi}\left[\frac{1-(-1)^n}{n^2}\right] \quad \left[\because \cos n\pi = (-1)^n\right]$$

Hence, $f(x) = \dfrac{\pi}{2} + \dfrac{2}{\pi}\sum_{n=1}^{\infty}\left[\dfrac{1-(-1)^n}{n^2}\right]\cos nx$

Example 6

Find the half-range cosine series of f(x) = x(π – x) in the interval (0, π) *and, hence, deduce that*

(i) $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$ (ii) $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \ldots$

Solution

The half-range cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} x(\pi - x)\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\pi\frac{x^2}{2} - \frac{x^3}{3}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\pi\frac{\pi^2}{2} - \frac{\pi^3}{3}\right)$$

$$= \frac{\pi^2}{6}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|(\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[(\pi - 2\pi)\frac{\cos n\pi}{n^2} - \frac{\pi\cos 0}{n^2}\right] \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= -\frac{2}{n^2}[1 + (-1)^n] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

Hence, $$f(x) = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty}\left[\frac{1+(-1)^n}{n^2}\right]\cos nx$$

$$x(\pi - x) = \frac{\pi^2}{6} - 2\left(\frac{2}{2^2}\cos 2x + \frac{2}{4^2}\cos 4x + \frac{2}{6^2}\cos 6x + \cdots\right) \qquad \dots (1)$$

Putting $x = 0$ in Eq. (1),

$$0 = \frac{\pi^2}{6} - 4\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Putting $x = \frac{\pi}{2}$ in Eq. (1),

$$\frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{\pi^2}{6} - 2\left(\frac{2}{2^2}\cos\pi + \frac{2}{4^2}\cos 2\pi + \frac{2}{6^2}\cos 3\pi + \cdots\right)$$

$$\frac{\pi^2}{4} = \frac{\pi^2}{6} - 4\left(-\frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{6^2} + \dots\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots$$

Example 7

Find the Fourier sine series of $f(x) = e^x$ *in* $0 < x < \pi$. **[Summer 2015]**

Solution

The half-range sine series of $f(x)$ with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} e^x \sin nx \, dx$$

$$= \frac{2}{\pi}\left|\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^{\pi}}{1+n^2}(-n\cos n\pi) - \frac{e^0}{1+n^2}(-n\cos 0)\right] \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{2}{\pi}\left[\frac{e^{\pi}}{1+n^2}(-1)^n(-n) + n\cdot\frac{1}{1+n^2}\right] \qquad [\because \cos n\pi = (-1)^n, \cos 0 = 1]$$

$$= \frac{2}{\pi}\cdot\frac{n}{(1+n^2)}[e^{\pi}(-1)^{n+1} + 1]$$

Hence, $$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n}{(1+n^2)}[e^{\pi}(-1)^{n+1} + 1]\sin nx$$

$$e^x = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n}{1+n^2}[e^{\pi}(-1)^{n+1} + 1]\sin nx$$

Example 8

Find the Fourier cosine series of $f(x) = e^{-x}$, *where* $0 \le x \le \pi$.

[Winter 2015]

Solution

The Fourier cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} e^{-x}\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{e^{-x}}{-1}\right|_0^{\pi}$$

$$= -\frac{1}{\pi}\left[e^{-\pi} - e^{-0}\right]$$

$$= -\frac{1}{\pi}\left[e^{-\pi} - 1\right]$$

$$= \frac{1}{\pi}(1 - e^{-\pi})$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi} e^{-x}\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|\frac{e^{-x}}{1+n^2}\{(-1)\cos nx + n\sin x\}\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left|\frac{e^{-x}}{1+n^2}(-\cos nx + n\sin nx)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^{-\pi}}{1+n^2}(-\cos n\pi) - \frac{e^0}{1+n^2}(-1)\right] \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{2}{\pi}\left[\frac{-e^{-\pi}(-1)^n}{1+n^2} + \frac{1}{1+n^2}\right] \qquad [\because\ \cos n\pi = (-1)^n]$$

$$= \frac{2}{\pi}\cdot\frac{1}{1+n^2}\left[1 - (-1)^n e^{-\pi}\right]$$

Hence, $$f(x) = \frac{1}{\pi}(1 - e^{-\pi}) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n e^{\pi}]}{1+n^2}\cos nx$$

Example 9

Find the half-range cosine series of $f(x) = \sin x$ *in the interval* $(0, \pi)$ *and hence, deduce that* $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$

[Winter 2014; Summer 2018]

Solution

The half-range cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} \sin x\,dx \\
&= \frac{1}{\pi}\left| -\cos x \right|_0^{\pi} \\
&= \frac{1}{\pi}(-\cos \pi + \cos 0) \\
&= \frac{2}{\pi} \qquad [\because\ \cos \pi = -1, \cos 0 = 1]
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\,dx \\
&= \frac{1}{\pi}\left| -\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right|_0^{\pi}, \ n \ne 1 \\
&= \frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{\cos 0}{n+1} - \frac{\cos 0}{n-1} \right], \ n \ne 1 \\
&= -\frac{2}{\pi(n^2-1)}[1 + (-1)^n], \ n \ne 1
\end{aligned}$$

$$\left[\because \cos(n\pi + \pi) = \cos(n\pi - \pi) = -\cos n\pi = -(-1)^n, \cos 0 = 1\right]$$

For $n = 1$,

$$\begin{aligned}a_1 &= \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx \\ &= \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx \\ &= \frac{1}{\pi}\left| -\frac{\cos 2x}{2} \right|_0^{\pi} \\ &= \frac{1}{2\pi}(-\cos 2\pi + \cos 0) \\ &= 0 \quad [\because \cos 2\pi = \cos 0 = 1]\end{aligned}$$

Hence, $$f(x) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\left[\frac{1+(-1)^n}{n^2-1}\right]\cos nx$$

$$\sin x = \frac{2}{\pi} - \frac{2}{\pi}\left(\frac{2}{3}\cos 2x + \frac{2}{15}\cos 4x + \cdots\right) \qquad \ldots (1)$$

Putting $x = \frac{\pi}{2}$ in Eq. (1),

$$\begin{aligned}\sin \frac{\pi}{2} &= \frac{2}{\pi} - \frac{2}{\pi}\left(\frac{2}{3}\cos \pi + \frac{2}{15}\cos 2\pi + \cdots\right) \\ 1 &= \frac{2}{\pi} - \frac{2}{\pi}\left(-\frac{2}{3} + \frac{2}{15} - \ldots\right) \\ 1 &= \frac{2}{\pi} + \frac{2}{\pi}\left[\left(1 - \frac{1}{3}\right) - \left(\frac{1}{3} - \frac{1}{5}\right) + \ldots\right] \\ 1 &= \frac{2}{\pi}\left(2 - \frac{2}{3} + \frac{2}{5} - \ldots\right) \\ \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \ldots\end{aligned}$$

Example 10

For the function $f(x) = \cos 2x$, find the Fourier sine series over $(0, \pi)$.

[Winter 2015]

Solution

The Fourier sine series of $f(x)$ with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos 2x\,\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\cos 2x\,\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} \left[\sin(2+n)x - \sin(2-n)x\right]\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\left(-\frac{\cos(n+2)x}{n+2}\right) - \left(-\frac{\cos(2-n)x}{2-n}\right)\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\cos(2\pi+n\pi)}{n+2} - \frac{\cos(2\pi-n\pi)}{n-2} + \frac{1}{n+2} + \frac{1}{n-2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\cos n\pi}{n+2} - \frac{\cos n\pi}{n-2} + \frac{1}{n+2} + \frac{1}{n-2}\right]$$

$$= \frac{1}{\pi}\left[(-1)^{n+1}\left\{\frac{1}{n+2} + \frac{1}{n-2}\right\} + \left\{\frac{1}{n+2} + \frac{1}{n-2}\right\}\right]$$

$$[\because \cos n\pi = (-1)^n]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^{n+1} + 1\{(n-2+n+2)\}}{(n+2)(n-2)}\right]$$

$$= \frac{2n\left[(-1)^{n+1} + 1\right]}{n^2 - 4}, \quad \text{if} \quad n \neq 2$$

$$b_2 = \frac{1}{\pi}\int_0^{\pi} 2\cos 2x\ \ \sin 2x\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 4x\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{\cos 4x}{4}\right|_0^{\pi}$$

$$= \frac{1}{4\pi}(-1+1)$$
$$= 0$$

Hence, $$f(x) = 2\sum_{\substack{n=1 \\ n\neq 2}}^{\infty}\left[\frac{n\left\{(-1)^{n+1}+1\right\}}{n^2-4}\right]\sin nx$$

Example 11

Find the half-range cosine series of $f(x)$, *where*

$$f(x) = x \qquad 0 < x < \frac{\pi}{2}$$
$$= \pi - x \qquad \frac{\pi}{2} < x < \pi$$

Solution

The half-range cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$
$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} x\,dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)\,dx\right]$$
$$= \frac{1}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\frac{\pi}{2}} + \left|\pi x - \frac{x^2}{2}\right|_{\frac{\pi}{2}}^{\pi}\right]$$
$$= \frac{1}{\pi}\left[\frac{\pi^2}{8} + \left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right]$$
$$= \frac{\pi}{4}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\cos nx\,dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left|x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\frac{\pi}{2}} + \left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2}\cdot\frac{\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2} - \frac{\cos 0}{n^2}\right) + \left(-\frac{\cos n\pi}{n^2} - \frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2}\right)\right]$$

$$[\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right] \qquad \left[\because \cos 0 = 1, \cos n\pi = (-1)^n\right]$$

Hence, $$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]\cos nx$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{1}{2^2}(-4)\cos 2x + \frac{1}{6^2}(-4)\cos 6x + \frac{1}{10^2}(-4)\cos 10x + \ldots\right]$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2}\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + \ldots\right)$$

Example 12

Find the half-range sine series of $f(x)$, where

$$f(x) = \frac{\pi}{3} \qquad 0 \le x < \frac{\pi}{3}$$

$$= 0 \qquad \frac{\pi}{3} \le x < \frac{2\pi}{3}$$

$$= -\frac{\pi}{3} \qquad \frac{2\pi}{3} \le x \le \pi$$

Solution

The half-range sine series of $f(x)$ with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{3}} \frac{\pi}{3}\sin nx\,\mathrm{d}x + \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 0\cdot \sin nx\,\mathrm{d}x + \int_{\frac{2\pi}{3}}^{\pi}\left(-\frac{\pi}{3}\right)\sin nx\,\mathrm{d}x\right]$$

$$= \frac{2}{3}\left[\left|-\frac{\cos nx}{n}\right|_0^{\frac{\pi}{3}} - \left|\frac{-\cos nx}{n}\right|_{\frac{2\pi}{3}}^{\pi}\right]$$

$$= \frac{2}{3n}\left[-\cos\frac{n\pi}{3} + \cos 0 + \cos n\pi - \cos\frac{2n\pi}{3}\right]$$

$$= \frac{2}{3n}\left[1 + (-1)^n - 2\cos\frac{n\pi}{2}\cos\frac{n\pi}{6}\right] \quad \left[\begin{array}{l}\because\ \cos 0 = 1, \cos n\pi = (-1)^n \\ \cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}\end{array}\right]$$

Hence, $$f(x) = \frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{n}\left[1 + (-1)^n - 2\cos\frac{n\pi}{2}\cos\frac{n\pi}{6}\right]\sin nx$$

Example 13

Find the half-range sine series of $f(x) = lx - x^2$ in the interval $(0, l)$ and, hence, deduce that

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \ldots$$

Solution

The half-range sine series of $f(x)$ with period $2l$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{l}\int_0^{l} (lx - x^2)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{l}\left|(lx - x^2)\frac{l}{n\pi}\left[-\cos\frac{n\pi x}{l}\right] - (l - 2x)\frac{l^2}{n^2\pi^2}\left[-\sin\frac{n\pi x}{l}\right] + \left[(-2)\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]\right|_0^l$$

$$= \frac{2}{l}\left[-\frac{2l^3}{n^3\pi^3}(\cos n\pi - \cos 0)\right] \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{4l^2}{n^3\pi^3}[1 - (-1)^n] \quad [\because\ \cos n\pi = (-1)^n]$$

Hence, $$f(x) = \frac{4l^2}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^3}\right]\sin\frac{n\pi x}{l}$$

$$lx - x^2 = \frac{8l}{\pi^3}\left[\frac{1}{1^3}\sin\frac{\pi x}{l} + \frac{1}{3^3}\sin\frac{3\pi x}{l} + \frac{1}{5^3}\sin\frac{5\pi x}{l} + \cdots\right] \quad \ldots (1)$$

Putting $x = \frac{l}{2}$ in Eq. (1),

$$\frac{l^2}{2} - \frac{l^2}{4} = \frac{8l^2}{\pi^3}\left(\frac{1}{1^3}\sin\frac{\pi}{2} + \frac{1}{3^3}\sin\frac{3\pi}{2} + \frac{1}{5^3}\sin\frac{5\pi}{2} + \cdots\right)$$

$$\frac{l^2}{4} = \frac{8l^2}{\pi^3}\left(\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \ldots\right)$$

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \ldots$$

Example 14

Find the Fourier cosine series of f(x) = x in $0 < x < l$. **[Winter 2013]**

Solution

The half-range cosine series of $f(x)$ with period $2l$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx$$

$$= \frac{1}{l}\int_0^l x\,dx$$

$$= \frac{1}{l}\left|\frac{x^2}{2}\right|_0^l$$

$$= \frac{1}{l}\left(\frac{l^2}{2}\right)$$

$$= \frac{l}{2}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left| x\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - (1)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{2}{l}\left(\frac{l^2}{n^2\pi^2}\cos n\pi - \frac{l^2}{n^2\pi^2}\cos 0\right) \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{2l}{n^2\pi^2}[(-1)^n - 1] \quad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

Hence,
$$f(x) = \frac{l}{2} + \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos\frac{n\pi x}{l}$$

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left[\frac{1}{1^2}\cos\left(\frac{\pi x}{l}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{l}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{l}\right) + \cdots\right]$$

Example 15

Find the Fourier cosine series of $f(x) = x^2$ *in* $0 < x < l$. **[Summer 2013]**

Solution

The half-range cosine series of $f(x)$ with period $2l$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx$$

$$= \frac{1}{l}\int_0^l x^2 dx$$

$$= \frac{1}{l}\left|\frac{x^3}{3}\right|_0^l$$

$$= \frac{1}{l}\left(\frac{l^3}{3}\right)$$

$$= \frac{1}{3}l^2$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l x^2\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left| x^2\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - (2x)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right) + 2\left(\frac{l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{2}{l}\left[\frac{2l^3}{n^2\pi^2}\cos n\pi\right] \quad [\because \ \sin n\pi = \sin 0 = 0]$$

$$= \frac{4l^2}{n^2\pi^2}(-1)^n \quad \left[\because \ \cos n\pi = (-1)^n\right]$$

Hence, $$f(x) = \frac{1}{3}l^2 + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{l}$$

$$x^2 = \frac{1}{3}l^2 - \frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos\left(\frac{\pi x}{l}\right) - \frac{1}{2^2}\cos\left(\frac{2\pi x}{l}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{l}\right) - \cdots\right]$$

Example 16

Obtain the Fourier cosine series of the function $f(x) = e^x$ in the range $(0, l)$. **[Winter 2014]**

Solution

The half-range cosine series of $f(x)$ with period $2l$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx$$

$$= \frac{1}{l}\int_0^l e^x\,dx$$

$$= \frac{1}{l}\left|e^x\right|_0^l$$

$$= \frac{1}{l}(e^l - 1)$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l e^x\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left|\frac{e^x}{1+\frac{n^2\pi^2}{l^2}}\left\{\cos\left(\frac{n\pi x}{l}\right) + \frac{n\pi}{l}\sin\left(\frac{n\pi x}{l}\right)\right\}\right|_0^l$$

$$= \frac{2l}{l^2 + n^2\pi^2}(e^l\cos n\pi - e^0\cos 0) \quad [\because \ \sin n\pi = \sin 0 = 0]$$

$$= \frac{2l}{l^2 + n^2\pi^2}[e^l(-1)^n - 1] \quad [\because \cos n\pi = (-1)^n, \cos 0 = 1]$$

Hence, $f(x) = \frac{1}{l}(e^l - 1) + \sum_{n=1}^{\infty} \frac{2l}{l^2 + n^2\pi^2}\left[e^l(-1)^n - 1\right]\cos\frac{n\pi x}{l}$

Example 17

Find the half-range cosine series of $f(x)$, where

$$f(x) = kx \qquad 0 \le x \le \frac{l}{2}$$
$$= k(l - x) \qquad \frac{l}{2} \le x \le l$$

Hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.$ **[Summer 2016]**

Solution

The half-range cosine series of $f(x)$ with period $2l$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x$$
$$= \frac{1}{l}\left[\int_0^{\frac{l}{2}} kx\,\mathrm{d}x + \int_{\frac{l}{2}}^{l} k(l - x)\,\mathrm{d}x\right]$$
$$= \frac{1}{l}\left[k\left|\frac{x^2}{2}\right|_0^{\frac{l}{2}} + k\left|lx - \frac{x^2}{2}\right|_{\frac{l}{2}}^{l}\right]$$
$$= \frac{k}{l}\left[\frac{l^2}{8} + \left(l^2 - \frac{l^2}{2}\right) - \left(\frac{l^2}{2} - \frac{l^2}{8}\right)\right]$$
$$= \frac{kl}{4}$$

$$
\begin{aligned}
a_n &= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\mathrm{d}x \\
&= \frac{2}{l}\left[\int_0^{\frac{l}{2}} kx\cos\frac{n\pi x}{l}\mathrm{d}x + \int_{\frac{l}{2}}^{l} k(l-x)\cos\frac{n\pi x}{l}\mathrm{d}x\right] \\
&= \frac{2k}{l}\left[\left| x\left(\sin\frac{n\pi x}{l}\right)\cdot\left(\frac{l}{n\pi}\right) - \left(-\cos\frac{n\pi x}{l}\right)\cdot\left(\frac{l^2}{n^2\pi^2}\right)\right|_0^{\frac{l}{2}}\right. \\
&\quad \left. + \left|(l-x)\left(\sin\frac{n\pi x}{l}\right)\cdot\left(\frac{l}{n\pi}\right) - (-1)\left(-\cos\frac{n\pi x}{l}\right)\cdot\left(\frac{l^2}{n^2\pi^2}\right)_{\frac{l}{2}}^{l}\right|\right] \\
&= \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - \left\{1+(-1)\right\}^n\right] \quad \left[\because\ \sin n\pi = 0, \cos n\pi = (-1)^n,\ \cos 0 = 1\right]
\end{aligned}
$$

Hence,
$$
f(x) = \frac{kl}{4} + \frac{2kl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[2\cos\frac{n\pi}{2} - \left\{1+(-1)^n\right\}\right]\cos\frac{n\pi x}{l} \qquad \dots (1)
$$

Putting $x = 0$ in Eq. (1),

$$
\begin{aligned}
0 &= \frac{kl}{4} + \frac{2kl}{\pi^2}\left(-\frac{4}{2^2} - \frac{4}{6^2} - \frac{4}{10^2} - \dots\right) \\
0 &= \frac{kl}{4} - \frac{2kl}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) \qquad \dots (2) \\
\frac{\pi^2}{8} &= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots
\end{aligned}
$$

Example 18

Find the half-range sine series of
$$
\begin{aligned}
f(x) &= \frac{2x}{l} & 0 \le x \le \frac{l}{2} \\
&= \frac{2(l-x)}{l} & \frac{l}{2} \le x \le l
\end{aligned}
$$

Solution

The half-range sine series of $f(x)$ with period $2l$ is given by

$$
f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}
$$

$$
b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\mathrm{d}x
$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{2}} \frac{2x}{l}\sin\frac{n\pi x}{l}\,\mathrm{d}x + \int_{\frac{l}{2}}^{l} \frac{2(l-x)}{l}\sin\frac{n\pi x}{l}\,\mathrm{d}x\right]$$

$$= \frac{4}{l^2}\left[\left| x\left(-\cos\frac{n\pi x}{l}\right)\frac{l}{n\pi} - \left(-\sin\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right|_0^{\frac{l}{2}}\right.$$

$$\left. + \left|(l-x)\left(-\cos\frac{n\pi x}{l}\right)\frac{l}{n\pi} - (-1)\left(-\sin\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right|_{\frac{l}{2}}^{l}\right]$$

$$= \frac{4}{l^2}\frac{l^2}{n^2\pi^2}\left(2\sin\frac{n\pi}{2}\right)$$

$$= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Hence, $f(x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}$

Example 19

Express $f(x) = x$ as a

(i) half-range sine series in $0 < x < 2$

(ii) half-range cosine series in $0 < x < 2$ **[Summer 2014]**

Solution

(i) The half-range sine series of $f(x)$ with period $2l = 4$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,\mathrm{d}x$$

$$= \left| x\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right|_0^2$$

$$= 2\left(-\frac{2}{n\pi}\right)\cos n\pi \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= -\frac{4(-1)^n}{n\pi} \quad [\because \cos n\pi = (-1)^n]$$

Hence, $$f(x) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{2}$$

$$x = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin \pi x + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right)$$

(ii) The half-range cosine series of $f(x)$ with period $2l = 4$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx$$

$$= \frac{1}{2}\int_0^2 x\,dx$$

$$= \frac{1}{2}\left|\frac{x^2}{2}\right|_0^2$$

$$= \frac{1}{2}(2)$$

$$= 1$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx$$

$$= \left| x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right|_0^2$$

$$= \frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}\cos 0 \quad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right] \quad [\because \cos n\pi = (-1)^n, \cos 0 = 1]$$

Hence, $$f(x) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos\frac{n\pi x}{2}$$

$$x = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right]$$

Example 20

Find the Fourier sine series of $f(x) = 2x$ in $0 < x < 1$. **[Summer 2015]**

Solution

The Fourier sine series of $f(x)$ with period $2l = 2$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= 2\int_0^1 (2x) \sin n\pi x \, dx$$

$$= 4\left| (x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1$$

$$= 4\left| -x\left(\frac{\cos n\pi x}{n\pi}\right) + \frac{\sin n\pi x}{n^2\pi^2}\right|_0^1$$

$$= 4\left[-\frac{\cos n\pi}{n\pi}\right] \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= -\frac{4(-1)^n}{n\pi} \qquad [\because \cos n\pi = (-1)^n]$$

$$= \frac{4}{n\pi}(-1)^{n+1}$$

Hence, $$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi x$$

Example 21

Find the half-range cosine series of $f(x) = (x - 1)^2$ in $0 < x < 1$.

[Summer 2015]

Solution

The half-range cosine series of $f(x)$ with period $2l = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x$$

$$= \int_0^1 (x-1)^2\,\mathrm{d}x$$

$$= \left|\frac{(x-1)^3}{3}\right|_0^1$$

$$= \left[0 - \frac{(-1)}{3}\right]$$

$$= \frac{1}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\frac{\cos n\pi x}{l}\,\mathrm{d}x$$

$$= 2\int_0^1 (x-1)^2 \cos n\pi x\,\mathrm{d}x$$

$$= 2\left|(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2(x-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right|_0^1$$

$$= 2\left|(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) + 2(x-1)\left(\frac{\cos n\pi x}{n^2\pi^2}\right) - 2\left(\frac{\sin n\pi x}{n^3\pi^3}\right)\right|_0^1$$

$$= \frac{4\cos 0}{n^2\pi^2} \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{4}{n^2\pi^2} \qquad [\because \cos 0 = 1]$$

Hence, $$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$$

$$(x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$$

Example 22

Find the half-range sine series of $f(x) = x \qquad 0 < x < 1$

$= 2 - x \qquad 1 < x < 2$

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$

Solution

The half-range sine series of $f(x)$ with period $2l = 4$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \int_0^1 x\sin\frac{n\pi x}{2}\,dx + \int_1^2 (2-x)\sin\frac{n\pi x}{2}\,dx$$

$$= \left| x\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right|_0^1$$

$$+ \left|(2-x)\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - (-1)\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right|_1^2$$

$$= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Hence, $f(x) = \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2}$

$$= \frac{8}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi}{2}\sin\frac{\pi x}{2} + \frac{1}{3^2}\sin\frac{3\pi}{2}\sin\frac{3\pi x}{2} + \frac{1}{5^2}\sin\frac{5\pi}{2}\sin\frac{5\pi x}{2} + \cdots\right]$$

$$= \frac{8}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{2} - \frac{1}{3^2}\sin\frac{3\pi x}{2} + \frac{1}{5^2}\sin\frac{5\pi x}{2} - \cdots\right] \qquad \ldots(1)$$

At $x = 1$,

$$f(1) = \frac{1}{2}\left[\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)\right] = \frac{1+(2-1)}{2} = 1$$

Putting $x = 1$ in Eq. (1),

$$f(1) = \frac{8}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{2} - \frac{1}{3^2}\sin\frac{3\pi}{2} + \frac{1}{5^2}\sin\frac{5\pi}{2} - \cdots\right)$$

$$1 = \frac{8}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 23

Find the half-range cosine series of $f(x) = 1 \qquad 0 \le x \le 1$

$$= x \qquad 1 \le x \le 2$$

Solution

The half-range cosine series of $f(x)$ with period $2l = 4$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\mathrm{d}x$$

$$= \frac{1}{2}\left[\int_0^1 1\mathrm{d}x + \int_1^2 x\mathrm{d}x\right]$$

$$= \frac{1}{2}\left[\left|x\right|_0^1 + \left|\frac{x^2}{2}\right|_1^2\right]$$

$$= \frac{5}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$= \int_0^1 1\cdot\cos\frac{n\pi x}{2}\mathrm{d}x + \int_1^2 x\cos\frac{n\pi x}{2}\mathrm{d}x$$

$$= \left|\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right|_0^1 + \left[\left|x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right|_1^2\right]$$

$$= \left(\frac{2}{n\pi}\sin\frac{n\pi}{2}\right) + \left(\frac{4}{n^2\pi^2}\cos n\pi - \frac{2}{n\pi}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right)$$

$$= \frac{4}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right] \quad [\because \cos n\pi = (-1)^n]$$

Hence, $$f(x) = \frac{5}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right]\cos\frac{n\pi x}{2}$$

EXERCISE 7.3

1. Find the half-range cosine series of $f(x) = x \sin x$ in $0 < x < \pi$.

$$\left[\textbf{Ans.:}\ 1 - \frac{1}{2}\cos x + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-1}\cos nx\right]$$

2. Find the half-range cosine series of $f(x) = (x-1)^2$ in $0 < x < 1$.

$$\left[\textbf{Ans.:}\ \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x\right]$$

3. Find the half-range cosine series of $f(x) = e^x$ in $0 < x < 1$.

$$\left[\textbf{Ans.:}\ (e-1) + 2\sum_{n=1}^{\infty}\frac{1}{1+n^2\pi^2}[e(-1)^n - 1]\cos n\pi x\right]$$

4. Find the half-range sine series of

$$f(x) = \begin{cases} x & 0 \le x \le 2 \\ 4 - x & 2 \le x \le 4 \end{cases}$$

$$\left[\textbf{Ans.:}\ \frac{16}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\sin\frac{n\pi}{2}\right)\sin\frac{n\pi x}{4}\right]$$

5. Find the half-range sine and cosine series of $f(x) = x - x^2$ in $0 < x < 1$.

$$\left[\textbf{Ans.:}\ \frac{8}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sin(2n+1)\pi x,\quad \frac{1}{6} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n)^2}\cos 2n\pi x\right]$$

6. Find the half-range sine and cosine series of $f(x) = a\left(1 - \frac{x}{l}\right)$ in $0 < x < l$.

$$\left[\textbf{Ans.:}\ \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l},\quad \frac{a}{2} + \frac{4a}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{(2n+1)\pi x}{l}\right]$$

7. Find the half-range sine series of $f(x) = \sin^2 x$ in $0 < x < \pi$.

$$\left[\textbf{Ans.:} -\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)(2n+1)(2n+3)}\right]$$

8. Find the half-range sine series of

$$f(x) = \frac{2x}{3} \qquad 0 \le x \le \frac{\pi}{3}$$
$$= \frac{\pi - x}{3} \qquad \frac{\pi}{3} \le x \le \pi$$

$$\left[\textbf{Ans.:} \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin nx\right]$$

9. Find the half-range sine series of

$$f(x) = x \qquad 0 \le x < 1$$
$$= 1 \qquad 1 \le x < 2$$
$$= 3 - x \qquad 2 \le x \le 3$$

$$\left[\textbf{Ans.:} \frac{6}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\left(\sin\frac{n\pi}{3} + \sin\frac{2n\pi}{3}\right)\right]\sin\frac{n\pi x}{3}\right]$$

Points to Remember

Fourier Series in the Interval (0, 2π)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Fourier Series in the Interval (c, c + 2l)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

where $2l$ is the length of the interval.

Fourier Series of Even Function in the Interval $(-\pi, \pi)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = 0$$

Fourier Series of Even Function in the interval $(-l, l)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^{l} f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = 0$$

Fourier Series of Odd Function in the Interval $(-\pi, \pi)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Fourier Series of Odd Function in the Interval $(-l, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Half-Range Cosine Series in the Interval (0, π)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Half-Range Cosine Series in the Interval (0, l)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^{l} f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

Half-Range Sine Series in the Interval (0, π)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Half-Range Sine Series in the Interval (0, l)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. If $f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$

then $f(x)$ is a/an ________ function in (–1 , 1).

(a) even (b) odd (c) constant (d) none of these

2. If $f(x) = \begin{cases} -x & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$

then $f(x)$ is a/an _____ function in $(-\pi, \pi)$.

(a) even (b) odd (c) constant (d) none of these

3. If $f(x) = \begin{cases} 1+\dfrac{2x}{\pi} & -\pi \le x \le 0 \\ 1-\dfrac{2x}{\pi} & 0 \le x \le \pi \end{cases}$

then $f(x)$ is a/an _____ function in $(-\pi, \pi)$.

(a) even (b) odd (c) constant (d) none of these

4. The Fourier series expansion of $f(x) = \begin{cases} -x^2 & -\pi < x \le 0 \\ x^2 & 0 \le x \le \pi \end{cases}$ contains no ______ terms.

(a) sine (b) cosine (c) constant (d) none of these

5. The Fourier series expansion of $f(x) = \begin{cases} -x & -4 \le x \le 0 \\ x & 0 \le x \le 4 \end{cases}$ contains no ______ terms.

(a) sine (b) cosine (c) constant (d) none of these

6. If $f(x)$ is an even function in $(-\pi, \pi)$, then the graph of $f(x)$ is symmetrical about the ______.

(a) x-axis (b) y-axis (c) origin (d) none of these

7. If $f(x)$ is an odd function in $(-l, l)$, then the graph of $f(x)$ is symmetrical about the ______

(a) x-axis (b) y-axis (c) origin (d) none of these

8. If $f(x)$ is an even function in the interval $(-l, l)$, then the value of b_n is

(a) $\dfrac{\pi}{2}$ (b) π (c) 1 (d) 0

9. If $f(x)$ is an odd function in $(-l, l)$, then the values of a_0 and a_1 are

(a) 0, 0 (b) π, π (c) $\dfrac{\pi}{2}, \pi$ (d) 1, 1

10. If $f(x) = x$ in $(-\pi, \pi)$, then the Fourier coefficient a_2 is

(a) π (b) 0 (c) 1 (d) –1

11. If $f(x) = \cos x$ in $(-\pi, \pi)$, then the Fourier coefficient b_n is

(a) 0 (b) π (c) 1 (d) none of these

12. In the Fourier series expansion of $f(x) = x \sin x$ in $(-\pi, \pi)$, the _____ terms are absent.

(a) sine (b) cosine (c) constant (d) none of these

13. If $f(x) = x \cos x$ in $(-\pi, \pi)$, then b_1 is

(a) 0 (b) π (c) 1 (d) none of these

14. Which of the following is neither an even function nor an odd function?

(a) $x \sin x$ (b) x^2 (c) e^{-x} (d) $x \cos x$

15. Fundamental period of $\sin 2x$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) 2π (d) π

16. Fundamental period of $\tan 3x$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) π (d) $\frac{\pi}{4}$

17. If $f(x + nT) = f(x)$ where n is any integer, then the fundamental period of $f(x)$ is

(a) $2T$ (b) $\frac{T}{2}$ (c) T (d) $3T$

18. For half-range sine series of $f(x) = \cos x$, $0 \le x \le \pi$ and period 2π, Fourier series is represented by $\sum_{n=1}^{\infty} b_n \sin nx$, then Fourier coefficient b_1 is

(a) $\frac{1}{\pi}$ (b) 0 (c) $\frac{2}{\pi}$ (d) $-\frac{2}{\pi}$

19. A function $f(x)$ is said to be periodic of period T if

(a) $f(x + T) = f(x)$ for all x (b) $f(x + T) = f(T)$ for all x

(c) $f(-x) = f(x)$ for all x (d) $f(-x) = -f(x)$ for all x

20. Fourier series representation of a periodic function $f(x)$ with period 2π which satisfies Dirichlet's conditions is

(a) $a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

(b) $a_0 + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x)$

(c) $a_0 + \sum_{n=1}^{\infty} (a_n \cos nx)(b_n \sin nx)$

(d) $a_0 + a_n \cos nx + b_n \sin nx$

21. A function $f(x)$ is said to be even if

(a) $f(-x) = f(x)$ (b) $f(-x) = -f(x)$

(c) $f(x + 2\pi) = f(x)$ (d) $f(-x) = [f(x)]^2$

22. A function $f(x)$ is said to be odd if

(a) $f(-x) = f(x)$ (b) $f(-x) = -f(x)$

(c) $f(x + 2\pi) = f(x)$ (d) $f(-x) = [f(x)]^2$

23. Which of the following is an odd function?

(a) $\sin x$ (b) $e^x + e^{-x}$ (c) $e^{|x|}$ (d) $\pi^2 - x^2$

24. Which of the following is an even function?

(a) $\sin x$ (b) $e^x - e^{-x}$ (c) $x\cos x$ (d) $\cos x$

25. For an even function $f(x)$ defined in the interval $-\pi \le x \le \pi$ and $f(x+2\pi) = f(x)$, the Fourier series is

(a) $\sum_{n=1}^{\infty} b_n \sin x$ (b) $a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

(c) $a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ (d) $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

26. For an odd function $f(x)$ defined in the interval $-\pi \le x \le \pi$ and $f(x + 2\pi) = f(x)$, the Fourier series is

(a) $\sum_{n=1}^{\infty} b_n \sin nx$ (b) $a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

(c) $a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ (d) $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

27. Half-range Fourier cosine series for $f(x)$ defined in the interval $(0, \pi)$ is

(a) $a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ (b) $a_n + \sum_{n=1}^{\infty} a_n \cos\frac{nx}{l}$

(c) $\sum_{n=1}^{\infty} a_n \cos nx$ (d) $a_n + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

28. Half-range Fourier sine series for $f(x)$ defined in the interval $(0, \pi)$ is

(a) $\sum_{n=1}^{\infty} b_n \sin nx$ (b) $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

(b) $a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$ (d) $a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

29. The Fourier series of an odd periodic function contains only

(a) odd harmonics (b) even harmonics

(c) cosine terms (d) sine terms

30. The trigonometric Fourier series of an even function does not have

(a) constant (b) cosine terms (c) sine terms (d) odd harmonic terms

31. For the function $f(x)=\begin{cases}-k & -\pi<x<0\\ k & 0<x<\pi\end{cases}$, the value of a_0 in the Fourier series expansion will be

(a) k (b) $2k$ (c) 0 (d) $-k$

32. The value of a_0 in Fourier series expansion of $f(x) = x^2$, $-1 < x < 1$ is

(a) $\frac{1}{3}$ (b) 3 (c) $\frac{1}{2}$ (d) 1

Answers

1. (b)	2. (a)	3. (a)	4. (b)	5. (a)	6. (b)	7. (c)	8. (d)
9. (a)	10. (b)	11. (a)	12. (a)	13. (a)	14. (c)	15. (d)	16. (b)
17. (c)	18. (b)	19. (a)	20. (a)	21. (a)	22. (b)	23. (a)	24. (d)
25. (c)	26. (a)	27. (a)	28. (a)	29. (d)	30. (c)	31. (c)	32. (a)

UNIT-4

Chapter 8. Partial Derivatives

CHAPTER 8

Partial Derivatives

Chapter Outline

8.1 INTRODUCTION

We often come across functions which depend on two or more variables. For example, area of a triangle depends on its base and height, hence we can say that area is the function of two variables, i.e., its base and height. u is called a function of two variables x and y, if u has a definite value for every pair of x and y. It is written as $u = f(x, y)$. The variables x and y are independent variables while u is dependent variable. The set of all the pairs (x, y) for which u is defined is called the domain of the function. Similarly, we can define function of more than two variables.

8.2 FUNCTIONS OF TWO OR MORE VARIABLES

The function $f(x, y)$ is called a real-valued function of two or more variables if there are two or more independent variables, e.g., total surface area of a rectangular parallelepiped is $2(xy + yz + zx)$ which is a function of three variables.

8.3 LIMIT AND CONTINUITY OF FUNCTIONS OF SEVERAL VARIABLES

8.3.1 Limits

If $f(x, y)$ is a function of two variables x, y then

$$\lim_{(x, y)\to(a, b)} f(x, y) = l$$

if and only if for any chosen number $\varepsilon > 0$ however small, there exists a number $\delta > 0$ such that

$$|f(x, y) - l| < \varepsilon$$

for all values of (x, y) for which, $|x - a| < \delta$ and $|y - b| < \delta$

8.3.2 Test for Non-existence of a Limit

1. Evaluate limits

(i) $\lim_{x\to a}\left\{\lim_{y\to b} f(x, y)\right\}$ and (ii) $\lim_{y\to b}\left\{\lim_{x\to a} f(x, y)\right\}$

If both the limit values are equal, then $\lim_{(x, y)\to(a, b)} f(x, y)$ exists.

2. If $a = 0$, $b = 0$, evaluate limit along different paths say $y = mx$ or $y = mx^n$, etc. If all limit values are equal, then $\lim_{(x, y)\to(a, b)} f(x, y)$ exists.

8.3.3 Theorems on Limit

If $\lim_{(x, y)\to(a, b)} f(x, y) = l$ and $\lim_{(x, y)\to(a, b)} g(x, y) = m$,

(i) $$\lim_{(x, y)\to(a, b)} [f(x, y) + g(x, y)] = \lim_{(x, y)\to(a, b)} f(x, y) + \lim_{(x, y)\to(a, b)} g(x, y) = l + m$$

(ii) $$\lim_{(x, y)\to(a, b)} [f(x, y) - g(x, y)] = \lim_{(x, y)\to(a, b)} f(x, y) - \lim_{(x, y)\to(a, b)} g(x, y) = l - m$$

(iii) $$\lim_{(x, y)\to(a, b)} [f(x, y) \cdot g(x, y)] = \lim_{(x, y)\to(a, b)} f(x, y) \cdot \lim_{(x, y)\to(a, b)} g(x, y) = lm$$

(iv) $$\lim_{(x, y)\to(a, b)} \left[\frac{f(x, y)}{g(x, y)}\right] = \frac{\lim_{(x, y)\to(a, b)} f(x, y)}{\lim_{(x, y)\to(a, b)} g(x, y)} = \frac{l}{m}, \text{ provided } m \neq 0$$

8.3.4 Continuity

Let $f(x, y)$ be a function of x and y defined at (a, b) as well as in the neighbourhood of it. The function $f(x, y)$ is continuous at (a, b) if the following three conditions are satisfied:

(i) $f(a, b)$ exists, i.e., $f(x, y)$ is defined at (a, b).

(ii) $\lim_{(x, y)\to(a, b)} f(x, y)$ exists.

(iii) $\lim_{(x, y)\to(a, b)} f(x, y) = f(a, b)$.

A function $f(x, y)$ is continuous in a domain if it is continuous at each point of that domain.

Note: If $f(x, y)$ and $g(x, y)$ are continuous at (a, b), then $f+g, f-g, fg, \frac{f}{g}$ (provided $g \neq 0$) are continuous at (a, b).

Example 1

Find $\lim_{(x, y)\to(1, 2)} \frac{x^2+y}{3x+y^2}$.

Solution

$$\lim_{(x, y)\to(1, 2)} \frac{x^2+y}{3x+y^2} = \frac{1^2+2}{3(1)+2^2}$$
$$= \frac{3}{7}$$

Example 2

Find $\lim_{(x, y)\to(0, 0)} \frac{x-y}{x+y}$.

Solution

$$\lim_{x\to 0}\left(\lim_{y\to 0} \frac{x-y}{x+y}\right) = \lim_{x\to 0} 1 = 1$$
$$\lim_{y\to 0}\left(\lim_{x\to 0} \frac{x-y}{x+y}\right) = \lim_{y\to 0}(-1) = -1$$

Since both the limits are different, the limit does not exist.

Example 3

By considering different paths of approach, show that the function $f(x, y) = \frac{x^4-y^2}{x^4+y^2}$ *has no limit as* $(x, y)\to(0, 0)$. **[Winter 2015]**

Solution

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^4-y^2}{x^4+y^2}\right)=\lim_{x\to 0}\frac{x^4}{x^4}=\lim_{x\to 0}1=1$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^4-y^2}{x^4+y^2}\right)=\lim_{y\to 0}\left(\frac{-y^2}{y^2}\right)=\lim_{y\to 0}(-1)=-1$$

Since both the limits are different, $f(x, y)$ has no limit as $(x, y)\to(0, 0)$.

Example 4

Find $\displaystyle\lim_{(x,\,y)\to(0,\,0)}\frac{xy}{y^2-x^2}$.

Solution

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{xy}{y^2-x^2}\right)=\lim_{x\to 0}0=0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{xy}{y^2-x^2}\right)=\lim_{y\to 0}0=0$$

Putting $y = mx$ and taking limit $x\to 0$,

$$\lim_{x\to 0}\frac{x(mx)}{(mx)^2-x^2}=\lim_{x\to 0}\frac{m}{m^2-1}=\frac{m}{m^2-1}$$

Since the last limit depends on m and m is not fixed, the limit does not exist.

Example 5

Find $\displaystyle\lim_{(x,\,y)\to(0,\,0)}\frac{x^2y}{x^4+y^2}$.

Solution

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^2y}{x^4+y^2}\right)=\lim_{x\to 0}0=0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^2y}{x^4+y^2}\right)=\lim_{y\to 0}0=0$$

Putting $y = mx$ and taking limit $x\to 0$,

$$\lim_{x\to 0}\frac{x^2(mx)}{x^4+(mx)^2}=\lim_{x\to 0}\frac{mx}{x^2+m^2}=0$$

Putting $y = mx^2$ and taking limit $x\to 0$,

$$\lim_{x\to 0}\frac{x^2(mx^2)}{x^4+(mx^2)^2}=\lim_{x\to 0}\frac{m}{1+m^2}=\frac{m}{1+m^2}$$

Since the last limit depends on m and m is not fixed, the limit does not exist.

Example 6

Show that $f(x, y) = x^2 + 2y$ *is continuous at* (1, 2).

Solution

$$\lim_{(x,y)\to(1,2)} f(x, y) = \lim_{(x,y)\to(1,2)} (x^2 + 2y) = 1^2 + 2(2) = 5$$

$$f(1, 2) = 1^2 + 2(2) = 5$$

$$\lim_{(x,y)\to(1,2)} f(x, y) = f(1, 2)$$

Hence, $f(x, y)$ is continuous at (1, 2).

Example 7

Show that $f(x, y) = 2x^2 + y, \quad (x, y) \neq (1, 2)$

$= 0, \quad (x, y) = (1, 2)$

is discontinuous at (1, 2).

Solution

$$\lim_{(x,y)\to(1,2)} f(x, y) = \lim_{(x,y)\to(1,2)} (2x^2 + y) = 2(1^2) + 2 = 4$$

$$f(1, 2) = 0$$

$$\lim_{(x,y)\to(1,2)} f(x, y) \neq f(1, 2)$$

Hence, $f(x, y)$ is discontinuous at (1, 2).

Example 8

Discuss the continuity of $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}, \quad (x, y) \neq (0, 0)$

$= 0, \quad (x, y) = (0, 0)$

at (0, 0).

Solution

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x, y)\right] = \lim_{x\to 0}\left(\lim_{y\to 0} \frac{x^2 - y^2}{x^2 + y^2}\right)$$

$$= \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1$$

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x, y)\right] = \lim_{y\to 0}\left(\lim_{x\to 0} \frac{x^2 - y^2}{x^2 + y^2}\right)$$

$$= \lim_{y\to 0}\left(-\frac{y^2}{y^2}\right) = \lim_{y\to 0}(-1) = -1$$

Since both the limits are not equal, $f(x, y)$ is discontinuous at (0, 0).

Example 9

Determine the set of points at which the given function is continuous:

$$f(x, y) = \begin{cases} \dfrac{3x^2 y}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

[Winter 2013]

Solution

For $(x, y) \neq (0, 0)$, the function $f(x, y) = \dfrac{3x^2 y}{x^2 + y^2}$ is a rational function and hence, it is continuous.

For $(x, y) = (0, 0)$, $f(x, y) = 0$

$$\lim_{x \to 0}\left[\lim_{y \to 0} f(x, y)\right] = \lim_{x \to 0}\left(\lim_{y \to 0} \frac{3x^2 y}{x^2 + y^2}\right)$$
$$= \lim_{x \to 0} 0 = 0$$
$$\lim_{y \to 0}\left[\lim_{x \to 0} f(x, y)\right] = \lim_{y \to 0}\left(\lim_{x \to 0} \frac{3x^2 y}{x^2 + y^2}\right)$$
$$= \lim_{y \to 0} 0 = 0$$

Putting $y = mx$ and taking limit $x \to 0$,

$$\lim_{x \to 0} \frac{3x^2 (mx)}{x^2 + m^2 x^2} = \lim_{x \to 0} \frac{3x^3 m}{x^2 (1 + m^2)} = \lim_{x \to 0} \frac{3mx}{1 + m^2} = 0$$

Hence, the limit exists at (0, 0).

$$\lim_{(x, y) \to (0,0)} f(x, y) = f(0, 0)$$

Hence, $f(x, y)$ is continuous at (0, 0).

Example 10

Show that $f(x, y) = \dfrac{2xy}{x^2 + y^2}, \quad (x, y) \neq (0, 0)$

$$= 0, \qquad (x, y) = (0, 0)$$

is continuous at every point except at the origin. **[Summer 2017]**

Solution

$$\lim_{x \to 0}\left[\lim_{y \to 0} f(x, y)\right] = \lim_{x \to 0}\left(\lim_{y \to 0} \frac{2xy}{x^2 + y^2}\right)$$
$$= \lim_{x \to 0} 0 = 0$$

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x, y)\right] = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{2xy}{x^2+y^2}\right)$$
$$= \lim_{y\to 0} 0 = 0$$

Putting $y = mx$ and taking limit $x \to 0$,

$$\lim_{x\to 0}\frac{2x(mx)}{x^2+(mx)^2} = \lim_{x\to 0}\frac{2m}{1+m^2} = \frac{2m}{1+m^2}$$

Since the last limit depends on m and m is not fixed, the limit does not exist.
Hence, $f(x)$ is discontinuous at the origin, i.e., (0, 0).
Let $(x, y) = (a, b) \neq (0, 0)$ be an arbitrary point in xy-plane, where a and b are real numbers.

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{(x,y)\to(a,b)}\frac{2xy}{x^2+y^2}$$
$$= \frac{2ab}{a^2+b^2}$$
$$f(a, b) = \frac{2ab}{a^2+b^2}$$

which is finite for real values of a and b.

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

This shows that $f(x, y)$ is continuous at (a, b).
Hence, $f(x, y)$ is continuous at every point except at the origin.

Example 11

Show that $f(x, y) = \begin{cases} \dfrac{2x^2y}{x^3+y^3} & (x, y) \neq 0 \\ 0 & (x, y) = 0 \end{cases}$

is not continuous at the origin. **[Winter 2016]**

Solution

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x, y)\right] = \lim_{x\to 0}\left(\lim_{y\to 0}\frac{2x^2y}{x^3+y^3}\right)$$
$$= \lim_{x\to 0}(0) = 0$$

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x, y)\right] = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{2x^2y}{x^3+y^3}\right)$$
$$= \lim_{y\to 0}(0) = 0$$

Putting $y = mx$ and taking limit $x \to 0$,

$$\lim_{x\to 0} \frac{2x^2(mx)}{x^3 + (mx)^3} = \lim_{x\to 0} \frac{2mx^3}{x^3(1+m^3)}$$

$$= \frac{2m}{1+m^3}$$

Since the last limit depends on m and m is not fixed, the limit does not exist. Hence, $f(x)$ is discontinuous at the origin.

Example 12

Determine the continuity of the function

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{x^2+y^2}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

at origin. **[Summer 2016]**

Solution

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = \lim_{x\to 0}\left[\lim_{y\to 0}(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)\right]$$

$$= \lim_{x\to 0}\left[x^2 \sin\left(\frac{1}{x^2}\right)\right]$$

$$= \lim_{x\to 0}\left[\frac{\sin\left(\frac{1}{x^2}\right)}{\frac{1}{x^2}}\right]$$

$$= 1$$

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = \lim_{y\to 0}\left[\lim_{x\to 0}(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)\right]$$

$$= \lim_{y\to 0}\left[y^2 \sin\left(\frac{1}{y^2}\right)\right]$$

$$= \lim_{y\to 0}\left[\frac{\sin\left(\frac{1}{y^2}\right)}{\frac{1}{y^2}}\right]$$

$$= 1$$

Putting $y = mx$ and taking limit $x \to 0$,

$$\lim_{x \to 0} (x^2 + m^2x^2) \sin\left(\frac{1}{x^2 + m^2x^2}\right) = (1 + m^2) \sin\left(\frac{1}{1 + m^2}\right)$$

Since the last limit depends on m and m is not fixed, the limit does not exist. Hence, $f(x)$ is discontinuous at origin (0, 0).

Let $(x, y) = (a, b)$ be an arbitrary point in xy-plane, where a and b are real numbers.

$$\lim_{(x,y) \to (a,b)} f(x, y) = \lim_{(x,y) \to (a,b)} (x^2 + y^2) \sin\left(\frac{1}{x^2 + y^2}\right)$$

$$= (a^2 + b^2) \sin\left(\frac{1}{a^2 + b^2}\right)$$

$$f(a, b) = (a^2 + b^2) \sin\left(\frac{1}{a^2 + b^2}\right)$$

which is finite for real values of a and b.

$$\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)$$

This shows that $f(x, y)$ is continuous at (a, b).

Hence, $f(x, y)$ is continuous at every point except at the origin.

Example 13

Show that $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}, \qquad (x, y) \neq (0, 0)$

$= 0, \qquad (x, y) = (0, 0)$

is continuous at the origin. **[Summer 2014]**

Solution

For $(x, y) \neq (0, 0)$, the function $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$ is a rational function and hence, it is continuous.

For $(x, y) = (0, 0)$, $f(x, y) = 0$

$$\lim_{x \to 0}\left[\lim_{y \to 0} f(x, y)\right] = \lim_{x \to 0}\left(\lim_{y \to 0} \frac{xy}{\sqrt{x^2 + y^2}}\right)$$

$$= \lim_{x \to 0} 0 = 0$$

$$\lim_{y \to 0}\left[\lim_{x \to 0} f(x, y)\right] = \lim_{y \to 0}\left(\lim_{x \to 0} \frac{xy}{\sqrt{x^2 + y^2}}\right)$$

$$= \lim_{y \to 0} 0 = 0$$

Putting $y = mx$ and taking limit $x \to 0$,

$$\lim_{x\to 0} \frac{x(mx)}{\sqrt{x^2 + m^2x^2}} = \lim_{x\to 0} \frac{mx^2}{x\sqrt{1+m^2}} = \lim_{x\to 0}\left(\frac{m}{\sqrt{1+m^2}}\right)x = 0$$

The limit exists at the origin.
Hence, $f(x, y)$ is continuous at the origin.

EXERCISE 8.1

1. Evaluate the following limits:

(i) $\lim\limits_{(x, y)\to(1, 2)} \frac{3x^2y}{x^2 + y^2 + 5}$ (ii) $\lim\limits_{(x, y)\to(-2, 2)} \frac{xy + 4}{x^3 + 2y^2}$ (iii) $\lim\limits_{(x, y)\to(0, 0)} \frac{x + y}{x + 2y}$

(iv) $\lim\limits_{(x, y)\to(0, 1)} e^{\frac{1}{x^2(y-1)^2}}$ (v) $\lim\limits_{(x, y)\to(0, 0)} \frac{2x - y}{x^2 + y^2}$

[Ans.: (i) $\frac{3}{5}$ (ii) 0 (iii) does not exist
(iv) 0 (v) does not exist]

2. Show that for $f(x, y) = \frac{2x - y}{2x + y}$, $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x, y)\right] \neq \lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} f(x, y)\right]$

3. Check the continuity of the following functions:

(i) $f(x,y) = \frac{x}{3x + 5y}$ at $(0, 0)$

(ii) $f(x, y) = \frac{xy}{x^2 + y^2}$, $(x, y) \neq (0, 0)$
$= 0$, $(x, y) = (0, 0)$

at origin.

(iii) $f(x, y) = \frac{x^2y^2}{x^4 + y^4}$ at $(0, 0)$.

[Ans.: (i) Discontinuous (ii) Discontinuous
(iii) Discontinuous]

8.4 PARTIAL DERIVATIVES

A partial derivative of a function of several variables is the ordinary derivative w.r.t. one of the variables, when all the remaining variables are kept constant. Consider a function $u = f(x, y)$. Here, u is the dependent variable and x and y are independent variables. The partial derivative of $u = f(x, y)$ w.r.t. x is the ordinary derivative of u w.r.t. x, keeping y constant. It is denoted by $\frac{\partial u}{\partial x}$ or $\frac{\partial f}{\partial x}$ or u_x or f_x and is known as first-order partial derivative of u w.r.t. x.

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}\right]$$

Similarly, the partial derivative of $u = f(x, y)$ w.r.t. y is the ordinary derivative of u w.r.t. y treating x as constant. It is denoted by $\frac{\partial u}{\partial y}$ or $\frac{\partial f}{\partial y}$ or u_y or f_y and is known as *first-order partial derivative* of u w.r.t. y.

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0}\left[\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\right]$$

8.5 HIGHER-ORDER PARTIAL DERIVATIVES

Partial derivatives of higher order, of a function $u = f(x, y)$, are obtained by partial differentiation of first-order partial derivative. Thus, if $u = f(x, y)$ then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

are called second-order partial derivatives. Similarly, other higher-order derivatives can also be obtained.

Mixed Derivative Theorem

If $u = f(x, y)$ possesses continuous second-order partial derivatives $\frac{\partial^2 u}{\partial x\,\partial y}$ and $\frac{\partial^2 u}{\partial y\,\partial x}$ then $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$. This is also called *commutative property*.

Note: Standard rules for derivatives of sum, difference, product and quotient are also applicable for partial derivatives.

Example 1

If $f(x, y) = x^2y + xy^2$ then find $f_x(1, 2)$ and $f_y(1, 2)$ by definition.

[Summer 2017]

Solution

$$f(x, y) = x^2y + xy^2$$

$$f_x = \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \frac{[(x+\Delta x)^2 y + (x+\Delta x)y^2] - x^2 y - xy^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 y + 2x\,\Delta xy + (\Delta x)^2 y + xy^2 + \Delta xy^2 - x^2 y - xy^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\,\Delta xy + (\Delta x)^2 y + \Delta xy^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2xy + y^2) + \Delta xy$$

$$f_x = \frac{\partial f}{\partial x} = 2xy + y^2$$

$$f_x(1, 2) = 2(1)(2) + (2)^2 = 4 + 4 = 8$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0}\left[\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\right]$$

$$= \lim_{\Delta y \to 0} \frac{[x^2(y+\Delta y) + x(y+\Delta y)^2] - x^2 y - xy^2}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x^2 y + x^2\Delta y + xy^2 + 2xy\Delta y + x(\Delta y)^2 - x^2 y - xy^2}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{2xy(\Delta y) + x^2(\Delta y) + x(\Delta y)^2}{\Delta y}$$

$$= \lim_{\Delta y \to 0} (2xy + x^2) + x(\Delta y)$$

$$f_y = \frac{\partial f}{\partial y} = 2xy + x^2$$

$$f_y(1, 2) = 2(1)(2) + (1)^2 = 4 + 1 = 5$$

Example 2

If $u=(1-2xy+y^2)^{-\frac{1}{2}}$ *then show that* $x\dfrac{\partial u}{\partial x}-y\dfrac{\partial u}{\partial y}=u^3y^2$.

Solution

$$u=(1-2xy+y^2)^{-\frac{1}{2}}$$

Differentiating u partially w.r.t. x and y,

$$\frac{\partial u}{\partial x}=-\frac{1}{2}(1-2xy+y^2)^{-\frac{3}{2}}(-2y)$$

$$\frac{\partial u}{\partial y}=-\frac{1}{2}(1-2xy+y^2)^{-\frac{3}{2}}(-2x+2y)$$

Hence, $$x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=(1-2xy+y^2)^{-\frac{3}{2}}(xy-xy+y^2)$$

$$=\left[(1-2xy+y^2)^{-\frac{1}{2}}\right]^3 y^2$$

$$=u^3y^2$$

Example 3

If u = log (tan x + tan y + tan z) *then show that*

$$\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z}=2.$$

Solution

$$u=\log(\tan x+\tan y+\tan z)$$

Differentiating u partially w.r.t. x, y and z,

$$\frac{\partial u}{\partial x}=\frac{1}{\tan x+\tan y+\tan z}\cdot\sec^2 x$$

$$\frac{\partial u}{\partial y}=\frac{1}{\tan x+\tan y+\tan z}\cdot\sec^2 y$$

$$\frac{\partial u}{\partial z}=\frac{1}{\tan x+\tan y+\tan z}\cdot\sec^2 z$$

Hence,

$$\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z}$$
$$=\frac{2\sin x\cos x\sec^2 x+2\sin y\cos y\sec^2 y+2\sin z\cos z\sec^2 z}{\tan x+\tan y+\tan z}$$
$$=\frac{2(\tan x+\tan y+\tan z)}{\tan x+\tan y+\tan z}$$
$$=2$$

Example 4

If $u=\dfrac{e^{x+y+z}}{e^x+e^y+e^z}$, *show that* $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=2u.$ **[Summer 2017]**

Solution

$$u=\frac{e^{x+y+z}}{e^x+e^y+e^z}$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x}=\frac{e^{x+y+z}}{e^x+e^y+e^z}-\frac{e^{x+y+z}}{(e^x+e^y+e^z)^2}\cdot e^x$$
$$=\frac{e^{x+y+z}}{e^x+e^y+e^z}\left(1-\frac{e^x}{e^x+e^y+e^z}\right) \qquad \ldots(1)$$

Similarly,

$$\frac{\partial u}{\partial y}=\frac{e^{x+y+z}}{e^x+e^y+e^z}\left(1-\frac{e^y}{e^x+e^y+e^z}\right) \qquad \ldots(2)$$

$$\frac{\partial u}{\partial z}=\frac{e^{x+y+z}}{e^x+e^y+e^z}\left(1-\frac{e^z}{e^x+e^y+e^z}\right) \qquad \ldots(3)$$

Adding Eqs (1), (2) and (3),

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{e^{x+y+z}}{e^x+e^y+e^z}\left(3-\frac{e^x+e^y+e^z}{e^x+e^y+e^z}\right)$$
$$=\frac{e^{x+y+z}}{e^x+e^y+e^z}(3-1)$$
$$=2u$$

Example 5

If $u = \frac{x^2+y^2}{x+y}$, *show that* $\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$.

Solution

$$u = \frac{x^2+y^2}{x+y}$$

$$u(x+y) = x^2 + y^2 \qquad \ldots(1)$$

Differentiating Eq. (1) partially w.r.t. *x*,

$$u + (x+y)\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial u}{\partial x} = \frac{2x-u}{x+y}$$

Differentiating Eq. (1) partially w.r.t. *y*,

$$u + (x+y)\frac{\partial u}{\partial y} = 2y$$

$$\frac{\partial u}{\partial y} = \frac{2y-u}{x+y}$$

$$\text{LHS} = \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = \left(\frac{2x-u}{x+y} - \frac{2y-u}{x+y}\right)^2$$

$$= \left[\frac{2(x-y)}{(x+y)}\right]^2 \qquad \ldots(2)$$

$$\text{RHS} = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = 4\left(1 - \frac{2x-u}{x+y} - \frac{2y-u}{x+y}\right)$$

$$= 4\left(1 - \frac{2x-u+2y-u}{x+y}\right)$$

$$= 4\left[1 - \frac{2(x+y)}{(x+y)} + \frac{2u}{(x+y)}\right]$$

$$= 4\left[1 - 2 + 2\left\{\frac{x^2+y^2}{(x+y)^2}\right\}\right]$$

$$= 4\left[\frac{-(x+y)^2 + 2x^2 + 2y^2}{(x+y)^2}\right]$$

$$= \frac{4(x^2+y^2-2xy)}{(x+y)^2}$$

$$= \left[\frac{2(x-y)}{(x+y)}\right]^2 \qquad \ldots(3)$$

From Eqs (1) and (2),

$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Example 6

Find the value of n for which $u = kt^{-\frac{1}{2}} e^{-\left(\frac{x^2}{na^2 t}\right)}$ *satisfies the equation* $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$.

Solution

$$u = kt^{-\frac{1}{2}} e^{-\left(\frac{x^2}{na^2 t}\right)}$$

Differentiating u partially w.r.t. t,

$$\frac{\partial u}{\partial t} = -\frac{1}{2} kt^{-\frac{3}{2}} e^{-\left(\frac{x^2}{na^2 t}\right)} + kt^{-\frac{1}{2}} e^{-\left(\frac{x^2}{na^2 t}\right)} \left(\frac{x^2}{na^2 t^2}\right)$$

$$= -\frac{1}{2} t^{-1} u + u \frac{x^2}{na^2 t^2}$$

$$= u\left(\frac{x^2}{na^2 t^2} - \frac{1}{2t}\right) \qquad \ldots(1)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = kt^{-\frac{1}{2}} e^{-\left(\frac{x^2}{na^2 t}\right)} \left(-\frac{2x}{na^2 t}\right)$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{kt^{-\frac{1}{2}}}{na^2 t}\left[2x \cdot e^{-\left(\frac{x^2}{na^2 t}\right)} \left(-\frac{2x}{na^2 t}\right) + e^{-\left(\frac{x^2}{na^2 t}\right)} (2)\right]$$

$$= -\frac{2}{na^2 t}\left(-\frac{2x^2}{na^2 t} u + u\right)$$

$$= u\left(\frac{4x^2}{n^2 a^4 t^2} - \frac{2}{na^2 t}\right)$$

$$a^2 \frac{\partial^2 u}{\partial x^2} = u\left(\frac{4x^2}{n^2 a^2 t^2} - \frac{2}{nt}\right) \qquad \ldots(2)$$

From Eqs (1) and (2),

$$n = 4$$

Example 7

Find the value of n for which $v = Ae^{-gx}\sin(nt - gx)$ *satisfies the partial differential equation* $\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$ *where g, A are constants.*

Solution

$$v = Ae^{-gx}\sin(nt - gx)$$

Differentiating v partially w.r.t. t,

$$\frac{\partial v}{\partial t} = Ae^{-gx}[\cos(nt - gx)]\cdot n$$

Differentiating v partially w.r.t. x,

$$\begin{aligned}\frac{\partial v}{\partial x} &= -Age^{-gx}\sin(nt - gx) + [Ae^{-gx}\cos(nt - gx)](-g) \\ &= -Age^{-gx}[\sin(nt - gx) + \cos(nt - gx)]\end{aligned}$$

Differentiating $\frac{\partial v}{\partial x}$ partially w.r.t. x,

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2} &= Ag^2 e^{-gx}[\sin(nt - gx) + \cos(nt - gx)] \\ &\quad - Age^{-gx}[-g\cos(nt - gx) + g\sin(nt - gx)] \\ &= Ag^2 e^{-gx}\cdot 2\cos(nt - gx)\end{aligned}$$

Also,

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$$

$$n\cdot A\cdot e^{-gx}\cos(nt - gx) = 2Ag^2 e^{-gx}\cos(nt - gx)$$

$$\therefore n = 2g^2$$

Example 8

If $u = e^{x^y}$*, find* $\dfrac{\partial^2 u}{\partial y\,\partial x}$.

Solution

$$u = e^{x^y}$$

Differentiating u partially w.r.t. x,

$$\begin{aligned}\frac{\partial u}{\partial x} &= e^{x^y}\,\frac{\partial}{\partial x}(x^y)\\ &= e^{x^y}\cdot yx^{y-1}\end{aligned}$$

Differentiating $\dfrac{\partial u}{\partial x}$ partially w.r.t. y,

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) &= e^{x^y}\,\frac{\partial}{\partial y}(x^y)\cdot yx^{y-1} + e^{x^y}\,x^{y-1} + e^{x^y}\,y\frac{\partial}{\partial y}(x^{y-1})\\ \frac{\partial^2 u}{\partial y\,\partial x} &= e^{x^y}\,x^y\log x\cdot yx^{y-1} + e^{x^y}\,x^{y-1} + e^{x^y}\,yx^{y-1}\log x\\ &= e^{x^y}\,x^{y-1}(yx^y\log x + 1 + y\log x)\end{aligned}$$

Example 9

If $z^3 - zx - y = 0$*, find* $\dfrac{\partial^2 z}{\partial x\partial y}$.

Solution

$$z^3 - zx - y = 0 \qquad \text{...(1)}$$

Differentiating Eq. (1) partially w.r.t. y,

$$\begin{aligned}3z^2\frac{\partial z}{\partial y} - x\frac{\partial z}{\partial y} - 1 &= 0\\ \frac{\partial z}{\partial y} &= \frac{1}{3z^2 - x}\end{aligned}$$

Differentiating Eq. (1) partially w.r.t. x,

$$\begin{aligned}3z^2\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial x} - z &= 0\\ \frac{\partial z}{\partial x} &= \frac{z}{3z^2 - x}\end{aligned}$$

Differentiating $\frac{\partial z}{\partial y}$ partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{(3z^2 - x)^2}\left(6z\frac{\partial z}{\partial x} - 1\right)$$

$$= -\frac{1}{(3z^2 - x)^2}\left(\frac{6z^2}{3z^2 - x} - 1\right)$$

$$= -\frac{3z^2 + x}{(3z^2 - x)^3}$$

Example 10

If $u = \tan^{-1}\left(\frac{xy}{\sqrt{1 + x^2 + y^2}}\right)$, *show that* $\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{(1 + x^2 + y^2)^{\frac{3}{2}}}$.

Solution

$$u = \tan^{-1}\left(\frac{xy}{\sqrt{1 + x^2 + y^2}}\right)$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial x} = \frac{1}{1 + \frac{x^2 y^2}{1 + x^2 + y^2}} \frac{\partial}{\partial x}\left(\frac{xy}{\sqrt{1 + x^2 + y^2}}\right)$$

$$= \frac{1 + x^2 + y^2}{1 + x^2 + y^2 + x^2 y^2}\left(\frac{\sqrt{1 + x^2 + y^2}\, y - xy \cdot \frac{1}{2\sqrt{1 + x^2 + y^2}} 2x}{1 + x^2 + y^2}\right)$$

$$= \frac{1 + x^2 + y^2}{1 + x^2 + y^2(1 + x^2)}\left[\frac{(1 + x^2 + y^2 - x^2)y}{(1 + x^2 + y^2)\sqrt{1 + x^2 + y^2}}\right]$$

$$= \frac{(1 + y^2)y}{(1 + x^2)(1 + y^2)\sqrt{1 + x^2 + y^2}}$$

$$= \frac{y}{(1 + x^2)\sqrt{1 + x^2 + y^2}}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left[\frac{y}{(1+x^2)\sqrt{1+x^2+y^2}}\right]$$

$$= \frac{1}{1+x^2}\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{1+x^2+y^2}}\right)$$

$$= \frac{1}{1+x^2}\left[\frac{\sqrt{1+x^2+y^2}(1) - y\frac{2y}{2\sqrt{1+x^2+y^2}}}{1+x^2+y^2}\right]$$

$$= \frac{1}{1+x^2}\left[\frac{1+x^2+y^2-y^2}{(1+x^2+y^2)\sqrt{1+x^2+y^2}}\right]$$

$$= \frac{1}{(1+x^2+y^2)^{\frac{3}{2}}}$$

Example 11

If $u = e^{xyz}$, *show that* $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$.

Solution

$$u = e^{xyz}$$

Differentiating u partially w.r.t. z,

$$\frac{\partial u}{\partial z} = e^{xyz} \cdot xy$$

Differentiating $\frac{\partial u}{\partial z}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial^2 u}{\partial y \, \partial z} = xe^{xyz} + x^2yze^{xyz}$$

Differentiating $\frac{\partial^2 u}{\partial y \, \partial z}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \, \partial z}\right) = e^{xyz} + xyze^{xyz} + 2xyze^{xyz} + x^2y^2z^2e^{xyz}$$

$$\frac{\partial^3 u}{\partial x \, \partial y \, \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$$

Example 12

If $u = e^{3xyz}$*, show that* $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = (3 + 27xyz + 27x^2y^2z^2)e^{3xyz}$.

[Summer 2014]

Solution

$$u = e^{3xyz}$$

Differentiating u partially w.r.t. z,

$$\frac{\partial u}{\partial z} = 3xy\, e^{3xyz}$$

Differentiating $\dfrac{\partial u}{\partial z}$ partially w.r.t. y,

$$\begin{aligned}\frac{\partial^2 u}{\partial y \partial z} &= 3x\left(e^{3xyz} + ye^{3xy} \cdot 3xz\right)\\ &= e^{3xyz}(3x + 9x^2yz)\end{aligned}$$

Differentiating $\dfrac{\partial^2 u}{\partial y \partial z}$ partially w.r.t. x,

$$\begin{aligned}\frac{\partial^3 u}{\partial x \partial y \partial z} &= e^{3xyz}(3 + 18xyz) + (3x + 9x^2yz)e^{3xyz} \cdot 3yz\\ &= e^{3xyz}(3 + 18xyz + 9xyz + 27x^2y^2z^2)\\ &= e^{3xyz}(3 + 27xyz + 27x^2y^2z^2)\end{aligned}$$

Example 13

If $u = \log(x^2 + y^2)$*, verify that* $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$.

Solution

$$u = \log(x^2 + y^2)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2}(2x) = \frac{2x}{x^2 + y^2}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2}(2y) = \frac{2y}{x^2 + y^2}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y\,\partial x} = 2x\left[-\frac{1}{(x^2+y^2)^2}\right]2y = -\frac{4xy}{(x^2+y^2)^2} \qquad \ldots(1)$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x\,\partial y} = 2y\left[-\frac{1}{(x^2+y^2)^2}\right]2x = -\frac{4xy}{(x^2+y^2)^2} \qquad \ldots(2)$$

From Eqs (1) and (2),

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$$

Example 14

If $u = x^3 y + e^{xy^2}$, *prove that* $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.

Solution

$$u = x^3 y + e^{xy^2}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = x^3 + e^{xy^2}\cdot 2xy$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 3x^2 + 2ye^{xy^2} + 2xye^{xy^2}\cdot y^2$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = 3x^2 + 2ye^{xy^2}(1+xy^2) \qquad \ldots(1)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = 3x^2 y + e^{xy^2}\cdot y^2$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 3x^2 + 2ye^{xy^2} + y^2 e^{xy^2}\cdot 2xy$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = 3x^2 + 2ye^{xy^2}(1 + xy^2) \qquad \text{...(2)}$$

From Eqs (1) and (2),

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}$$

Example 15

If $z = x + y^x$, prove that $\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x}$. **[Winter 2013]**

Solution

$$z = x + y^x$$

Differentiating z partially w.r.t. x,

$$\frac{\partial z}{\partial x} = 1 + y^x \log y$$

Differentiating z partially w.r.t. y,

$$\frac{\partial z}{\partial y} = xy^{x-1}$$

Differentiating $\dfrac{\partial z}{\partial x}$ partially w.r.t. y,

$$\begin{aligned}\frac{\partial^2 z}{\partial y\,\partial x} &= y^x \cdot \frac{1}{y} + \log y \cdot xy^{x-1}\\ &= y^{x-1} + x\log y(y^{x-1})\\ &= y^{x-1}(1 + x\log y)\end{aligned} \qquad \text{...(1)}$$

Differentiating $\dfrac{\partial z}{\partial y}$ partially w.r.t. x,

$$\begin{aligned}\frac{\partial^2 z}{\partial x\,\partial y} &= y^{x-1} \cdot 1 + xy^{x-1}\log y\\ &= y^{x-1}(1 + x\log y)\end{aligned} \qquad \text{...(2)}$$

From Eqs (1) and (2),

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial x\,\partial y}$$

Example 16

If $z = x^y + y^x$, *prove that* $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$.

Solution

$$z = x^y + y^x$$

$$z = e^{\log x^y} + e^{\log y^x}$$

$$= e^{y\log x} + e^{x\log y}$$

Differentiating z partially w.r.t. y,

$$\frac{\partial z}{\partial y} = e^{y\log x}\cdot \log x + e^{x\log y}\cdot \frac{x}{y}$$

Differentiating $\frac{\partial z}{\partial y}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = e^{y\log x}\cdot\frac{y}{x}\log x + e^{y\log x}\cdot\frac{1}{x} + e^{x\log y}\cdot\frac{1}{y} + e^{x\log y}\log y\cdot\frac{x}{y}$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{e^{y\log x}}{x}(y\log x + 1) + \frac{e^{x\log y}}{y}(1 + x\log y) \qquad \ldots(1)$$

Differentiating z partially w.r.t. x,

$$\frac{\partial z}{\partial x} = e^{y\log x}\cdot\frac{y}{x} + e^{x\log y}\cdot\log y$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{1}{x}(e^{y\log x} + e^{y\log x}\,y\log x) + e^{x\log y}\cdot\frac{x}{y}\log y + e^{x\log y}\cdot\frac{1}{y}$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{e^{y\log x}}{x}(1 + y\log x) + \frac{e^{x\log y}}{y}(x\log y + 1) \qquad \ldots(2)$$

From Eqs (1) and (2),

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial x\,\partial y}$$

Example 17

If $u = (3xy - y^3) - (y^2 - 2x)^{\frac{3}{2}}$, *show that* $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.

Solution

$$u = (3xy - y^3) - (y^2 - 2x)^{\frac{3}{2}}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = 3x - 3y^2 - \frac{3}{2}(y^2 - 2x)^{\frac{1}{2}}(2y)$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 3 - \frac{3y}{2}(y^2 - 2x)^{-\frac{1}{2}}(-2)$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = 3 + \frac{3y}{\sqrt{y^2 - 2x}} \qquad \ldots(1)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = 3y - \frac{3}{2}(y^2 - 2x)^{\frac{1}{2}}(-2)$$

$$= 3y + 3(y^2 - 2x)^{\frac{1}{2}}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 3 + \frac{3}{2}(y^2 - 2x)^{-\frac{1}{2}}(2y)$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = 3 + \frac{3y}{\sqrt{y^2 - 2x}} \qquad \ldots(2)$$

From Eqs (1) and (2),

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$$

Example 18

If $z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, *prove that* $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x} = \frac{x^2 - y^2}{x^2 + y^2}$.

Solution

$$z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$$

Differentiating z partially w.r.t. y,

$$\frac{\partial z}{\partial y} = x^2 \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) - y^2 \frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right) - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{y^2+x^2} - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$= x - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

Differentiating $\frac{\partial z}{\partial y}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 1 - 2y\frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = 1 - \frac{2y^2}{y^2+x^2} = \frac{y^2+x^2-2y^2}{y^2+x^2}$$

$$= \frac{x^2-y^2}{x^2+y^2} \qquad \ldots(1)$$

Differentiating z partially w.r.t. x,

$$\frac{\partial z}{\partial y} = 2x\tan^{-1}\left(\frac{y}{x}\right) + x^2 \cdot \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) - \frac{y^2}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2 y}{x^2+y^2} - \frac{y^3}{x^2+y^2}$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - y$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 2x \cdot \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) - 1$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{2x^2}{x^2+y^2} - 1$$

$$= \frac{2x^2-x^2-y^2}{x^2+y^2}$$

$$= \frac{x^2-y^2}{x^2+y^2} \qquad \ldots(2)$$

From Eqs (1) and (2),

$$\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial^2 z}{\partial y\,\partial x}=\frac{x^2-y^2}{x^2+y^2}$$

Example 19

If $u=\log(x^2+y^2+z^2)$, *prove that* $x\dfrac{\partial^2 u}{\partial y\,\partial z}=y\dfrac{\partial^2 u}{\partial z\,\partial x}=z\dfrac{\partial^2 u}{\partial x\,\partial y}$.

Solution

$$u=\log(x^2+y^2+z^2)$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y}=\frac{1}{x^2+y^2+z^2}\cdot 2y$$

Differentiating $\dfrac{\partial u}{\partial y}$ partially w.r.t. z,

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right)=-\frac{2y}{(x^2+y^2+z^2)^2}\cdot 2z$$

$$x\frac{\partial^2 u}{\partial z\,\partial y}=-\frac{4xyz}{(x^2+y^2+z^2)^2}$$

or

$$x\frac{\partial^2 u}{\partial y\,\partial z}=-\frac{4xyz}{(x^2+y^2+z^2)^2} \qquad \ldots(1)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x}=\frac{1}{x^2+y^2+z^2}\cdot 2x$$

Differentiating $\dfrac{\partial u}{\partial x}$ partially w.r.t. z,

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right)=-\frac{2x}{(x^2+y^2+z^2)^2}\cdot 2z$$

$$y\frac{\partial^2 u}{\partial z\,\partial x}=-\frac{4xyz}{(x^2+y^2+z^2)^2} \qquad \ldots(2)$$

Differentiating $\dfrac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=-\frac{2x}{(x^2+y^2+z^2)^2}\cdot 2y$$

$$z\frac{\partial^2 u}{\partial x\,\partial y}=-\frac{4xyz}{(x^2+y^2+z^2)^2} \qquad \ldots(3)$$

From Eqs (1), (2) and (3),

$$x\frac{\partial^2 u}{\partial y\,\partial z}=y\frac{\partial^2 u}{\partial z\,\partial x}=z\frac{\partial^2 u}{\partial x\,\partial y}$$

Example 20

If $z = \tan(y+ax)+(y-ax)^{\frac{3}{2}}$, *find the value of* $\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2}$.

Solution

$$z = \tan(y+ax)+(y-ax)^{\frac{3}{2}}$$

Differentiating z partially w.r.t. x,

$$\frac{\partial z}{\partial x} = a\sec^2(y+ax) - \frac{3}{2}a(y-ax)^{\frac{1}{2}}$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x^2} = 2a^2\sec^2(y+ax)\tan(y+ax) + \frac{3}{4}a^2(y-ax)^{-\frac{1}{2}} \qquad \ldots(1)$$

Differentiating z partially w.r.t. y,

$$\frac{\partial z}{\partial y} = \sec^2(y+ax) + \frac{3}{2}(y-ax)^{\frac{1}{2}}$$

Differentiating $\frac{\partial z}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 z}{\partial y^2} = 2\sec^2(y+ax)\tan(y+ax) + \frac{3}{4}(y-ax)^{-\frac{1}{2}} \qquad \ldots(2)$$

From Eqs (1) and (2),

$$\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = 0$$

Example 21

If $a^2x^2 + b^2y^2 = c^2z^2$, *evaluate* $\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2}$.

Solution

$$a^2x^2 + b^2y^2 = c^2z^2$$

Differentiating partially w.r.t. x,

$$2a^2x = 2c^2z\cdot\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{a^2x}{c^2z}$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x^2} = \frac{a^2}{c^2}\left(\frac{1}{z} - \frac{x}{z^2}\cdot\frac{\partial z}{\partial x}\right)$$

$$= \frac{a^2}{c^2 z}\left(1 - \frac{x}{z}\cdot\frac{a^2 x}{c^2 z}\right)$$

$$\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2 z}\left(1 - \frac{a^2 x^2}{c^2 z^2}\right)$$

Similarly,

$$\frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2 z}\left(1 - \frac{b^2 y^2}{c^2 z^2}\right)$$

Hence,

$$\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2 z}\left(2 - \frac{a^2 x^2 + b^2 y^2}{c^2 z^2}\right)$$

$$= \frac{1}{c^2 z}\left(2 - \frac{c^2 z^2}{c^2 z^2}\right)$$

$$= \frac{1}{c^2 z}(2-1)$$

$$= \frac{1}{c^2 z}$$

Example 22

If $u = \log(x^3 + y^3 - x^2 y - xy^2)$, *prove that* $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}$.

Solution

$$u = \log(x^3 + y^3 - x^2 y - xy^2)$$

$$= \log[(x+y)(x^2 - xy + y^2) - xy(x+y)]$$

$$= \log[(x+y)(x^2 - xy + y^2 - xy)]$$

$$= \log[(x+y)(x-y)^2]$$

$$= \log(x+y) + 2\log(x-y)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{1}{x+y} + \frac{2}{x-y}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{(x+y)^2} - \frac{2}{(x-y)^2}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = \frac{1}{x+y} - \frac{2}{x-y}$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = -\frac{1}{(x+y)^2} - \frac{2}{(x-y)^2}$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x\,\partial y} = -\frac{1}{(x+y)^2} + \frac{2}{(x-y)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}$$

Example 23

If $u = \log(x^3 + y^3 + z^3 - 3xyz)$*, prove that* $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

Solution

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) v$$

where

$$v = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

Differentiating u partially w.r.t. x, y, and z,

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$v = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= \frac{3(x^2 + y^2 + z^2) - 3(xy + yz + zx)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz} \cdot \frac{(x + y + z)}{(x + y + z)}$$

$$= \frac{3(x^3 + y^3 + z^3 - 3xyz)}{(x^3 + y^3 + z^3 - 3xyz)(x + y + z)}$$

$$= \frac{3}{x + y + z}$$

Hence, $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x + y + z}\right)$$

$$= -\frac{3}{(x + y + z)^2} - \frac{3}{(x + y + z)^2} - \frac{3}{(x + y + z)^2}$$

$$= -\frac{9}{(x + y + z)^2}$$

Example 24

If $u = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$ *and* $a^2 + b^2 + c^2 = 1$, *show that* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$

Solution

$$u = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = 6(ax + by + cz)a - 2x$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = 6a \cdot a - 2$$

$$= 6a^2 - 2$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = 6(ax + by + cz)b - 2y$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= 6b \cdot b - 2 \\ &= 6b^2 - 2\end{aligned}$$

Differentiating u partially w.r.t. z,

$$\frac{\partial u}{\partial z} = 6(ax + by + cz)c - 2z$$

Differentiating $\frac{\partial u}{\partial z}$ partially w.r.t. z,

$$\begin{aligned}\frac{\partial^2 u}{\partial z^2} &= 6c \cdot c - 2 \\ &= 6c^2 - 2\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} &= 6(a^2 + b^2 + c^2) - 6 \\ &= 6(1) - 6 \qquad [\because a^2 + b^2 + c^2 = 1] \\ &= 0\end{aligned}$$

Example 25

If $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$, *find the value of* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Solution

$$u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

Differentiating u partially w.r.t. x,

$$\begin{aligned}\frac{\partial u}{\partial x} &= -\frac{1}{2(x^2 + y^2 + z^2)^{\frac{3}{2}}} \cdot 2x \\ &= -\frac{x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}\end{aligned}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = -\left[\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} - \frac{3x \cdot 2x}{2(x^2+y^2+z^2)^{\frac{5}{2}}}\right]$$

$$= -\frac{1}{(x^2+y^2+z^2)^{\frac{5}{2}}}(x^2+y^2+z^2-3x^2)$$

$$= -\frac{(-2x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = -\frac{(x^2-2y^2+z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

and

$$\frac{\partial^2 u}{\partial z^2} = -\frac{(x^2+y^2-2z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

Hence,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{(-2x^2+2y^2+2z^2+2x^2-2y^2-2z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$= 0$$

Example 26

If $u = z\tan^{-1}\left(\frac{x}{y}\right)$, *find the value of* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Solution

$$u = z\tan^{-1}\left(\frac{x}{y}\right)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = z\frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y}$$

$$= \frac{zy}{y^2+x^2}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{yz \cdot 2x}{(x^2+y^2)^2}$$

$$= -\frac{2xyz}{(x^2+y^2)^2}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y}=\frac{z}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right)$$

$$=-\frac{xz}{y^2+x^2}$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2}=\frac{xz\cdot 2y}{(x^2+y^2)^2}$$

$$=\frac{2xyz}{(x^2+y^2)^2}$$

Differentiating u partially w.r.t. z,

$$\frac{\partial u}{\partial z}=\tan^{-1}\left(\frac{x}{y}\right)$$

Differentiating $\frac{\partial u}{\partial z}$ partially w.r.t. z,

$$\frac{\partial^2 u}{\partial z^2}=0$$

Hence, $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-\frac{2xyz}{(x^2+y^2)^2}+\frac{2xyz}{(x^2+y^2)^2}$$

$$=0$$

Example 27

If $v=(1-2xy+y^2)^{-\frac{1}{2}}$, *find the value of* $\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial v}{\partial x}\right]+\frac{\partial}{\partial y}\left(y^2\frac{\partial v}{\partial y}\right)$.

Solution

$$v=(1-2xy+y^2)^{-\frac{1}{2}}$$

Differentiating v partially w.r.t. x,

$$\frac{\partial v}{\partial x}=-\frac{1}{2}(1-2xy+y^2)^{-\frac{3}{2}}(-2y)$$

$$(1-x^2)\frac{\partial v}{\partial x}=y(1-x^2)(1-2xy+y^2)^{-\frac{3}{2}}$$

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial v}{\partial x}\right]=y\frac{\partial}{\partial x}\left[(1-x^2)(1-2xy+y^2)^{-\frac{3}{2}}\right]$$

$$=y\left[(-2x)(1-2xy+y^2)^{-\frac{3}{2}}-\frac{3}{2}(1-x^2)(1-2xy+y^2)^{-\frac{5}{2}}(-2y)\right]$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}[-2x(1-2xy+y^2)+3y(1-x^2)]$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}(-2x+4x^2y-2xy^2+3y-3x^2y)$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}(-2x+x^2y-2xy^2+3y) \quad \ldots(1)$$

Differentiating v partially w.r.t. y,

$$\frac{\partial v}{\partial y} = -\frac{1}{2}(1-2xy+y^2)^{-\frac{3}{2}}(-2x+2y)$$

$$y^2\frac{\partial v}{\partial y} = -y^2(-x+y)(1-2xy+y^2)^{-\frac{3}{2}}$$

$$\frac{\partial}{\partial y}\left(y^2\frac{\partial v}{\partial y}\right) = -2y(-x+y)(1-2xy+y^2)^{-\frac{3}{2}} - y^2(1-2xy+y^2)^{-\frac{3}{2}} + \frac{3y^2}{2}(-x+y)(1-2xy+y^2)^{-\frac{5}{2}}(-2x+2y)$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}[2(x-y)(1-2xy+y^2) - y(1-2xy+y^2) + 3y(-x+y)^2]$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}(2x-4x^2y+2xy^2-2y+4xy^2-2y^3-y+2xy^2-y^3+3yx^2+3y^3-6xy^2)$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}(2x-x^2y+2xy^2-3y) \quad \ldots(2)$$

Adding Eqs (1) and (2),

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial y}\left(y^2\frac{\partial v}{\partial y}\right) = 0.$$

Example 28

If $u = (ar^n + br^{-n})(\cos n\theta + \sin n\theta)$, *show that* $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0.$

Solution

$$u = (ar^n + br^{-n})(\cos n\theta + \sin n\theta)$$

Differentiating u partially w.r.t. r,

$$\frac{\partial u}{\partial r} = (nar^{n-1} - bnr^{-n-1})(\cos n\theta + \sin n\theta)$$

Differentiating $\dfrac{\partial u}{\partial r}$ partially w.r.t. r,

$$\frac{\partial^2 u}{\partial r^2} = n[a(n-1)r^{n-2} + b(n+1)r^{-n-2}](\cos n\theta + \sin n\theta)$$

Differentiating u partially w.r.t. θ,

$$\frac{\partial u}{\partial \theta} = (ar^n + br^{-n})(-n\sin n\theta + n\cos n\theta)$$

Differentiating $\frac{\partial u}{\partial \theta}$ partially w.r.t. θ,

$$\begin{aligned}\frac{\partial^2 u}{\partial \theta^2} &= (ar^n + br^{-n})(-n^2\cos n\theta - n^2\sin n\theta)\\ &= -n^2(ar^n + br^{-n})(\cos n\theta + \sin n\theta)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= n[a(n-1)r^{n-2} + b(n+1)r^{-n-2}](\cos n\theta + \sin n\theta)\\ &\quad + n(ar^{n-2} - br^{-n-2})(\cos n\theta + \sin n\theta) - \frac{n^2}{r^2}(ar^n + br^{-n})(\cos n\theta + \sin n\theta)\\ &= (\cos n\theta + \sin n\theta)r^{n-2}(an^2 - an + bn^2 + bn + an - bn - an^2 - bn^2)\\ &= 0\end{aligned}$$

Example 29

If $x = r\cos\theta$, $y = r\sin\theta$, *show that* $\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1.$

Solution

$$x = r\cos\theta,\ y = r\sin\theta$$

$$x^2 + y^2 = r^2 \qquad \ldots(1)$$

Differentiating Eq. (1) partially w.r.t. x,

$$2x = 2r\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Differentiating Eq. (1) partially w.r.t. y,

$$2y = 2r\frac{\partial r}{\partial y}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

Hence, $$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

Example 30

If $x = \cos\theta - r\sin\theta$, $y = \sin\theta + r\cos\theta$, *show that*

(i) $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$ *(ii)* $\dfrac{\partial \theta}{\partial x} = -\dfrac{\cos\theta}{r}$.

Solution

(i)
$$x = \cos\theta - r\sin\theta \qquad \text{...(1)}$$
$$y = \sin\theta + r\cos\theta \qquad \text{...(2)}$$
$$x^2 = \cos^2\theta - 2r\cos\theta\sin\theta + r^2\sin^2\theta$$
$$y^2 = \sin^2\theta + 2r\sin\theta\cos\theta + r^2\cos^2\theta$$
$$x^2 + y^2 = 1 + r^2 \qquad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t. x,
$$2x = 2r\frac{\partial r}{\partial x}$$
$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

(ii) Multiplying Eq. (1) by $\cos\theta$ and Eq. (2) by $\sin\theta$,
$$x\cos\theta = \cos^2\theta - r\sin\theta\cos\theta \qquad \text{...(4)}$$
$$y\sin\theta = \sin^2\theta + r\cos\theta\sin\theta \qquad \text{...(5)}$$

Adding Eqs (4) and (5),
$$x\cos\theta + y\sin\theta = 1$$
$$x\cot\theta + y = \operatorname{cosec}\theta \qquad \text{...(6)}$$

Differentiating Eq. (6) partially w.r.t. x,
$$\cot\theta - x\operatorname{cosec}^2\theta\frac{\partial\theta}{\partial x} = -\cot\theta\operatorname{cosec}\theta\frac{\partial\theta}{\partial x}$$
$$\cot\theta + \cot\theta\operatorname{cosec}\theta\frac{\partial\theta}{\partial x} = x\operatorname{cosec}^2\theta\frac{\partial\theta}{\partial x}$$
$$= (\cos\theta - r\sin\theta)\operatorname{cosec}^2\theta\frac{\partial\theta}{\partial x}$$
$$= \cos\theta\operatorname{cosec}^2\theta\frac{\partial\theta}{\partial x} - r\sin\theta\operatorname{cosec}^2\theta\frac{\partial\theta}{\partial x}$$
$$= \cot\theta\operatorname{cosec}\theta\frac{\partial\theta}{\partial x} - r\operatorname{cosec}\theta\frac{\partial\theta}{\partial x}$$
$$\cot\theta = -r\operatorname{cosec}\theta\frac{\partial\theta}{\partial x}$$

$$\frac{\partial\theta}{\partial x}=-\frac{\cot\theta}{r\operatorname{cosec}\theta}=-\frac{\cos\theta}{r}$$

Example 31

Show that $\frac{\partial x}{\partial r}=\frac{1}{r}\frac{\partial y}{\partial\theta}$ *and* $\frac{\partial y}{\partial r}=-\frac{1}{r}\frac{\partial x}{\partial\theta}$ *and hence, show that* $\frac{\partial^2 x}{\partial r^2}+\frac{1}{r}\frac{\partial x}{\partial r}+\frac{1}{r^2}\frac{\partial^2 x}{\partial\theta^2}=0$ *if* $x=e^{r\cos\theta}\cos(r\sin\theta)$ *and* $y=e^{r\cos\theta}\sin(r\sin\theta)$.

Solution

$$x=e^{r\cos\theta}\cos(r\sin\theta)$$

Differentiating x partially w.r.t. r,

$$\frac{\partial x}{\partial r}=e^{r\cos\theta}\cdot\cos\theta\cos(r\sin\theta)+e^{r\cos\theta}[-\sin(r\sin\theta)]\sin\theta$$
$$=e^{r\cos\theta}[\cos\theta\cos(r\sin\theta)-\sin\theta\sin(r\sin\theta)]$$
$$=e^{r\cos\theta}\cos(\theta+r\sin\theta) \qquad \ldots(1)$$

Now, $y=e^{r\cos\theta}\sin(r\sin\theta)$

Differentiating y partially w.r.t. r,

$$\frac{\partial y}{\partial r}=e^{r\cos\theta}\cos\theta\sin(r\sin\theta)+e^{r\cos\theta}\cos(r\sin\theta)\sin\theta$$
$$=e^{r\cos\theta}\sin(r\sin\theta+\theta) \qquad \ldots(2)$$

Differentiating x partially w.r.t. θ,

$$\frac{\partial x}{\partial\theta}=e^{r\cos\theta}(-r\sin\theta)\cos(r\sin\theta)+e^{r\cos\theta}[-\sin(r\sin\theta)]\cdot(r\cos\theta)$$
$$=-re^{r\cos\theta}\sin(\theta+r\sin\theta) \qquad \ldots(3)$$

Differentiating y partially w.r.t. θ,

$$\frac{\partial y}{\partial\theta}=e^{r\cos\theta}(-r\sin\theta)\sin(r\sin\theta)+e^{r\cos\theta}\cos(r\sin\theta)\cdot r\cos\theta$$
$$=re^{r\cos\theta}\cos(\theta+r\sin\theta) \qquad \ldots(4)$$

From Eqs (1) and (4),

$$\frac{\partial x}{\partial r}=\frac{1}{r}\frac{\partial y}{\partial\theta} \qquad \ldots(5)$$

From Eqs (2) and (3),

$$\frac{\partial y}{\partial r}=-\frac{1}{r}\frac{\partial x}{\partial\theta}$$
$$\frac{\partial x}{\partial\theta}=-r\frac{\partial y}{\partial r} \qquad \ldots(6)$$

Differentiating Eq. (5) partially w.r.t. r,

$$\frac{\partial^2 x}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial y}{r\partial \theta}\right)$$

$$= \frac{-1}{r^2}\frac{\partial y}{\partial \theta} + \frac{1}{r}\frac{\partial^2 y}{\partial r \partial \theta}$$

Differentiating Eq. (6) partially w.r.t. θ,

$$\frac{\partial^2 x}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(-r\frac{\partial y}{\partial r}\right)$$

$$= -r\frac{\partial^2 y}{\partial \theta \partial r}$$

$$= -r\frac{\partial^2 y}{\partial r \partial \theta}$$

Hence, $\dfrac{\partial^2 x}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial x}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 x}{\partial \theta^2} = \dfrac{-1}{r^2}\dfrac{\partial y}{\partial \theta} + \dfrac{1}{r}\dfrac{\partial^2 y}{\partial r \partial \theta} + \dfrac{1}{r^2}\dfrac{\partial y}{\partial \theta} - \dfrac{1}{r}\dfrac{\partial^2 y}{\partial r \partial \theta}$

$$= 0$$

Example 32

If $\theta = t^n e^{\frac{-r^2}{4t}}$ *then find n so that* $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial \theta}{\partial r}\right) = \dfrac{\partial \theta}{\partial t}$.

[Winter 2016; Summer 2016]

Solution

$$\theta = t^n e^{\frac{-r^2}{4t}}$$

Differentiating θ partially w.r.t. t,

$$\frac{\partial \theta}{\partial t} = nt^{n-1}e^{\frac{-r^2}{4t}} + t^n e^{\frac{-r^2}{4t}}\left(\frac{r^2}{4t^2}\right)$$

$$= e^{\frac{-r^2}{4t}}\left(nt^{n-1} + \frac{1}{4}r^2 t^{n-2}\right)$$

Differentiating θ partially w.r.t. r,

$$\frac{\partial \theta}{\partial r} = t^n e^{\frac{-r^2}{4t}}\left(\frac{-2r}{4t}\right)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial}{\partial r}\left(-\frac{t^{n-1}}{2}r^3 e^{\frac{-r^2}{4t}}\right)$$

$$= -\frac{t^{n-1}}{2}\left[3r^2 e^{\frac{-r^2}{4t}} + r^3 e^{\frac{-r^2}{4t}}\left(\frac{-2r}{4t}\right)\right]$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=e^{\frac{-r^2}{4t}}\left(-\frac{3}{2}t^{n-1}+\frac{r^2}{4}t^{n-2}\right)$$

Given
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t},$$

$$e^{\frac{-r^2}{4t}}\left(-\frac{3}{2}t^{n-1}+\frac{r^2}{4}t^{n-2}\right)=e^{\frac{-r^2}{4t}}\left(nt^{n-1}+\frac{1}{4}r^2t^{n-2}\right)$$

$$-\frac{3}{2}t^{n-1}=nt^{n-1}$$

$$n=-\frac{3}{2}$$

Example 33

Find the value of n so that $v = r^n (3\cos^2\theta - 1)$ *satisfies the equation* $\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)=0.$

Solution

$$v = r^n (3\cos^2\theta - 1)$$

Differentiating v partially w.r.t. r,

$$\frac{\partial v}{\partial r}=nr^{n-1}(3\cos^2\theta-1)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)=\frac{\partial}{\partial r}[nr^{n+1}(3\cos^2\theta-1)]$$

$$=n(n+1)r^n(3\cos^2\theta-1) \qquad \ldots(1)$$

Differentiating v partially w.r.t. θ,

$$\frac{\partial v}{\partial\theta}=r^n\cdot 6\cos\theta(-\sin\theta)$$

$$=-3r^n\sin 2\theta$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)=\frac{\partial}{\partial\theta}(-3r^n\sin\theta\cdot\sin 2\theta)$$

$$=-3r^n(\cos\theta\sin 2\theta+2\sin\theta\cos 2\theta)$$

$$=-3r^n[\cos\theta\cdot 2\sin\theta\cos\theta+2\sin\theta(2\cos^2\theta-1)]$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)=-3r^n(2\cos^2\theta+4\cos^2\theta-2)$$

$$=-6r^n(3\cos^2\theta-1)$$

Given $$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)=0,$$

$$n(n+1)\,r^n(3\cos^2\theta-1)-6r^n(3\cos^2\theta-1)=0$$
$$n(n+1)-6=0$$
$$n^2+n-6=0$$
$$(n+3)(n-2)=0$$
$$n=-3,\ 2$$

Example 34

If $x^x y^y z^z = c$, show that at $x = y = z$,

(*i*) $\dfrac{\partial^2 z}{\partial x\,\partial y}=-(x\log ex)^{-1}$ (*ii*) $\dfrac{\partial^2 z}{\partial x^2}-2xy\dfrac{\partial^2 z}{\partial x\,\partial y}+\dfrac{\partial^2 z}{\partial y^2}=\dfrac{2(x^2-2)}{x(1+\log x)}$.

Solution

(i) $$x^x y^y z^z = c$$

Taking logarithm on both the sides,

$$\log x^x+\log y^y+\log z^z=\log c$$
$$x\log x+y\log y+z\log z=\log c \qquad \ldots(1)$$

Differentiating Eq. (1) partially w.r.t. x,

$$x\cdot\frac{1}{x}+\log x+\frac{\partial z}{\partial x}\cdot\log z+z\cdot\frac{1}{z}\frac{\partial z}{\partial x}=0 \qquad [\because z=f(x,y)]$$
$$\frac{\partial z}{\partial x}=-\frac{1+\log x}{1+\log z}$$

Differentiating $\dfrac{\partial z}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=-(1+\log x)\left[-\frac{1}{(1+\log z)^2}\cdot\frac{1}{z}\frac{\partial z}{\partial x}\right]$$
$$\frac{\partial^2 z}{\partial y\,\partial x}=\frac{(1+\log x)}{z(1+\log z)^2}\left(-\frac{1+\log x}{1+\log z}\right)$$
$$\frac{\partial^2 z}{\partial x\,\partial y}=-\frac{(1+\log x)^2}{z(1+\log z)^3}$$

At $x=y=z$,

$$\frac{\partial^2 z}{\partial x\,\partial y}=-\frac{(1+\log x)^2}{x(1+\log x)^3}$$

$$= -\frac{1}{x(1+\log x)}$$
$$= -[x(\log e + \log x)]^{-1} \qquad [\because \log e = 1]$$
$$= -(x \log ex)^{-1}.$$

(ii) Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{1+\log x}{1+\log z}\right)$$
$$= \frac{(1+\log x)}{(1+\log z)^2}\cdot\frac{1}{z}\frac{\partial z}{\partial x} - \frac{1}{x(1+\log z)}$$
$$= -\frac{(1+\log x)}{z(1+\log z)^2}\cdot\frac{(1+\log x)}{(1+\log z)} - \frac{1}{x(1+\log z)}$$

At $x = y = z$,

$$\frac{\partial^2 z}{\partial x^2} = -\frac{2}{x(1+\log x)}$$

Similarly,
$$\frac{\partial^2 z}{\partial y^2} = \frac{-(1+\log y)^2}{z(1+\log z)^3} - \frac{1}{y(1+\log z)}$$

At $x = y = z$,

$$\frac{\partial^2 z}{\partial y^2} = -\frac{2}{x(1+\log x)}$$

Hence, $$\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 z}{\partial y^2} = -\frac{2}{x(1+\log x)} - 2xy\left[-\frac{1}{x(1+\log x)}\right] + \left[-\frac{2}{x(1+\log x)}\right]$$
$$= \frac{2(xy-2)}{x(1+\log x)}$$
$$= \frac{2(x^2-2)}{x(1+\log x)} \qquad [\because x = y = z]$$

Example 35

If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, *prove that*

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right).$$

Solution

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1 \qquad \text{...(1)}$$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{2x}{a^2+u}-\frac{x^2}{(a^2+u)^2}\frac{\partial u}{\partial x}-\frac{y^2}{(b^2+u)^2}\frac{\partial u}{\partial x}-\frac{z^2}{(c^2+u)^2}\frac{\partial u}{\partial x}=0$$

$$\frac{\partial u}{\partial x}\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]=\frac{2x}{a^2+u}$$

$$\frac{\partial u}{\partial x}\cdot p=\frac{2x}{(a^2+u)}$$

where
$$p=\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}$$

$$\frac{\partial u}{\partial x}=\frac{2x}{(a^2+u)p}$$

Similarly,
$$\frac{\partial u}{\partial y}=\frac{2y}{(b^2+u)p}$$

$$\frac{\partial u}{\partial z}=\frac{2z}{(c^2+u)p}$$

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\frac{4}{p^2}\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]$$

$$=\frac{4}{p^2}(p)$$

$$=\frac{4}{p} \qquad \ldots(2)$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{2}{p}\left(\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right)$$

$$=\frac{2}{p}(1)$$

$$=\frac{2}{p} \qquad \ldots(3)$$

From Eqs (2) and (3),

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)$$

Example 36

If $z=e^{ax+by}f(ax-by)$, *prove that* $b\dfrac{\partial z}{\partial x}+a\dfrac{\partial z}{\partial y}=2abz$.

Solution

$$z = e^{ax+by} f(ax - by)$$

Differentiating z partially w.r.t. x,

$$\frac{\partial z}{\partial x} = ae^{ax+by} f(ax - by) + ae^{ax+by} f'(ax - by)$$
$$= az + ae^{ax+by} f'(ax - by)$$

Differentiating z partially w.r.t. y,

$$\frac{\partial z}{\partial y} = be^{ax+by} f(ax - by) - be^{ax+by} f'(ax - by)$$
$$= bz - be^{ax+by} f'(ax - by)$$

Hence, $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y} = abz + abe^{ax+by} f'(ax - by) + abz - abe^{ax+by} f'(ax - by)$

$$= 2abz$$

Example 37

If $u = \phi(x + ky) + \psi(x - ky)$, *show that* $\dfrac{\partial^2 u}{\partial y^2} = k^2 \dfrac{\partial^2 u}{\partial x^2}$.

Solution

$$u = \phi(x + ky) + \psi(x - ky)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \phi'(x + ky) \cdot 1 + \psi'(x - ky) \cdot 1$$

Differentiating $\dfrac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = \phi''(x + ky) + \psi''(x - ky) \quad \ldots(1)$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = \phi'(x + ky) \cdot k + \psi'(x - ky) \cdot (-k)$$

Differentiating $\dfrac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = \phi''(x + ky) \cdot k^2 + \psi''(x - ky)(-k)^2$$
$$= k^2[\phi''(x + ky) + \psi''(x - ky)] \quad \ldots(2)$$

From Eqs (1) and (2),

$$\frac{\partial^2 u}{\partial y^2} = k^2 \frac{\partial^2 u}{\partial x^2}$$

Example 38

If $u = xf(x + y) + y\phi(x + y)$, *show that* $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$

Solution

$$u = xf(x + y) + y\,\phi(x + y),$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = f(x+y) + xf'(x+y) + y\phi'(x+y)$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= f'(x+y) + f'(x+y) + xf''(x+y) + y\phi''(x+y)\\ &= 2f'(x+y) + xf''(x+y) + y\phi''(x+y)\end{aligned}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial x\,\partial y} = f'(x+y) + xf''(x+y) + y\phi''(x+y) + \phi'(x+y)$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = xf'(x+y) + \phi(x+y) + y\phi'(x+y)$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= xf''(x+y) + \phi'(x+y) + \phi'(x+y) + y\phi''(x+y)\\ &= xf''(x+y) + 2\phi'(x+y) + y\phi''(x+y)\end{aligned}$$

Hence, $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2}$

$$\begin{aligned}&= 2f'(x+y) + xf''(x+y) + y\phi''(x+y) - 2f'(x+y) - 2xf''(x+y)\\ &\quad - 2y\phi''(x+y) - 2\phi'(x+y) + xf''(x+y) + 2\phi'(x+y) + y\phi''(x+y)\\ &= 0\end{aligned}$$

Example 39

If $u = f\left(\sqrt{x^2+y^2}\right)$, *prove that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{\sqrt{x^2+y^2}} f'\left(\sqrt{x^2+y^2}\right) + f''\left(\sqrt{x^2+y^2}\right).$$

Solution

Let $\sqrt{x^2+y^2} = r$

$$u = f(r)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} f(r)$$

$$= \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$= f'(r) \cdot \frac{\partial}{\partial x} \sqrt{x^2+y^2}$$

$$= f'(r) \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = f''(r) \frac{\partial r}{\partial x} \cdot \frac{x}{\sqrt{x^2+y^2}} + \frac{f'(r)}{\sqrt{x^2+y^2}} - xf'(r) \frac{1}{2(x^2+y^2)^{\frac{3}{2}}} \cdot 2x$$

$$= f''(r) \frac{x}{\sqrt{x^2+y^2}} \cdot \frac{x}{\sqrt{x^2+y^2}} + \frac{f'(r)}{\sqrt{x^2+y^2}} - \frac{x^2 f'(r)}{(x^2+y^2)^{\frac{3}{2}}}$$

$$= f''(r) \frac{x^2}{x^2+y^2} + \frac{f'(r)}{\sqrt{x^2+y^2}} - \frac{x^2 f'(r)}{(x^2+y^2)^{\frac{3}{2}}}$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = f''(r) \frac{y^2}{x^2+y^2} + \frac{f'(r)}{\sqrt{x^2+y^2}} - \frac{y^2 f'(r)}{(x^2+y^2)^{\frac{3}{2}}}$$

Hence, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) \frac{(x^2+y^2)}{x^2+y^2} + \frac{2f'(r)}{\sqrt{x^2+y^2}} - \frac{(x^2+y^2) f'(r)}{(x^2+y^2)^{\frac{3}{2}}}$$

$$= f''(r) + \frac{2f'(r)}{\sqrt{x^2+y^2}} - \frac{f'(r)}{\sqrt{x^2+y^2}}$$

$$= f''(r) + \frac{f'(r)}{\sqrt{x^2+y^2}}$$

$$= f''\left(\sqrt{x^2+y^2}\right) + \frac{f'\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}$$

Example 40

If $u = f\left(\frac{x^2}{y}\right)$, *show that* $x^2\frac{\partial^2 u}{\partial x^2} + 3xy\frac{\partial^2 u}{\partial x\,\partial y} + 2y^2\frac{\partial^2 u}{\partial y^2} = 0.$

Solution

$$u = f\left(\frac{x^2}{y}\right)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = f'\left(\frac{x^2}{y}\right)\frac{\partial}{\partial x}\left(\frac{x^2}{y}\right)$$

$$= f'\left(\frac{x^2}{y}\right)\left(\frac{2x}{y}\right)$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = \frac{2}{y}f'\left(\frac{x^2}{y}\right) + f''\left(\frac{x^2}{y}\right)\cdot\frac{\partial}{\partial x}\left(\frac{x^2}{y}\right)\cdot\left(\frac{2x}{y}\right)$$

$$= \frac{2}{y}f'\left(\frac{x^2}{y}\right) + f''\left(\frac{x^2}{y}\right)\cdot\left(\frac{2x}{y}\right)^2$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = f'\left(\frac{x^2}{y}\right)\cdot\frac{\partial}{\partial y}\left(\frac{x^2}{y}\right)$$

$$= f'\left(\frac{x^2}{y}\right)\cdot\left(-\frac{x^2}{y^2}\right)$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = \frac{2x^2}{y^3}f'\left(\frac{x^2}{y}\right) + \left(-\frac{x^2}{y^2}\right)f''\left(\frac{x^2}{y}\right)\cdot\frac{\partial}{\partial y}\left(\frac{x^2}{y}\right)$$

$$= \frac{2x^2}{y^3}f'\left(\frac{x^2}{y}\right) + \left(\frac{x^2}{y^2}\right)^2 f''\left(\frac{x^2}{y}\right)$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial x\,\partial y} = -\frac{2x}{y^2} f'\left(\frac{x^2}{y}\right) + \frac{2x}{y} f''\left(\frac{x^2}{y}\right)\cdot\left(-\frac{x^2}{y^2}\right)$$

$$\begin{aligned} x^2\frac{\partial^2 u}{\partial x^2} + 3xy\frac{\partial^2 u}{\partial x\,\partial y} + 2y^2\frac{\partial^2 u}{\partial y^2} &= \frac{2x^2}{y} f'\left(\frac{x^2}{y}\right) + \frac{4x^4}{y^2} f''\left(\frac{x^2}{y}\right) - \frac{6x^2}{y} f'\left(\frac{x^2}{y}\right) \\ &\quad - \frac{6x^4}{y^2} f''\left(\frac{x^2}{y}\right) + \frac{4x^2}{y} f'\left(\frac{x^2}{y}\right) + \frac{2x^4}{y^2} f''\left(\frac{x^2}{y}\right) \\ &= 0 \end{aligned}$$

Example 41

If $u = e^{xyz} f\left(\frac{xy}{z}\right)$, *prove that* $x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = 2xyzu$ *and* $y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2xyzu$ *and hence, show that* $x\frac{\partial^2 u}{\partial z\,\partial x} = y\frac{\partial^2 u}{\partial z\,\partial y}$.

Solution

$$u = e^{xyz} f\left(\frac{xy}{z}\right)$$

Differentiating u partially w.r.t. x, y and z,

$$\frac{\partial u}{\partial x} = e^{xyz} yz\cdot f\left(\frac{xy}{z}\right) + e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{y}{z}\right)$$

$$\frac{\partial u}{\partial y} = e^{xyz} xz\cdot f\left(\frac{xy}{z}\right) + e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{x}{z}\right)$$

$$\frac{\partial u}{\partial z} = e^{xyz} xy\cdot f\left(\frac{xy}{z}\right) + e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{-xy}{z^2}\right)$$

(i) $x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z}$

$$= e^{xyz} xyz\cdot f\left(\frac{xy}{z}\right) + \frac{xy}{z} e^{xyz} f'\left(\frac{xy}{z}\right) + e^{xyz} xyz\cdot f\left(\frac{xy}{z}\right) - \frac{xy}{z} e^{xyz} f'\left(\frac{xy}{z}\right)$$

$$= 2xyz e^{xyz}\cdot f\left(\frac{xy}{z}\right)$$

$$= 2xyzu.$$

(ii) $y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

$$= e^{xyz} xyz\cdot f\left(\frac{xy}{z}\right) + \frac{xy}{z} e^{xyz} f'\left(\frac{xy}{z}\right) + e^{xyz} xyz\cdot f\left(\frac{xy}{z}\right) - \frac{xy}{z} e^{xyz} f'\left(\frac{xy}{z}\right)$$

$$= 2xyze^{xyz} f\left(\frac{xy}{z}\right)$$

$$= 2xyzu$$

(iii) Differentiating $\frac{\partial u}{\partial z}$ w.r.t. x,

$$\frac{\partial^2 u}{\partial z\,\partial x} = e^{xyz} yz\cdot xy\cdot f\left(\frac{xy}{z}\right) + e^{xyz} y\cdot f\left(\frac{xy}{z}\right) + e^{xyz} xy\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{y}{z}\right)$$

$$+ e^{xyz} yz\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{-xy}{z^2}\right) + e^{xyz}\left[f''\left(\frac{xy}{z}\right)\right]\left(\frac{y}{z}\right)\left(\frac{-xy}{z^2}\right)$$

$$+ e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(-\frac{y}{z^2}\right)$$

$$x\frac{\partial^2 u}{\partial z\partial x} = e^{xyz}\left[x^2y^2z\cdot f\left(\frac{xy}{z}\right) + xy\cdot f\left(\frac{xy}{z}\right) - \frac{x^2y^2}{z^3} f''\left(\frac{xy}{z}\right) - \frac{xy}{z^2} f'\left(\frac{xy}{z}\right)\right] \qquad \ldots(1)$$

Differentiating $\frac{\partial u}{\partial z}$ w.r.t. y,

$$\frac{\partial^2 u}{\partial z\,\partial y} = e^{xyz} xz\cdot xy\cdot f\left(\frac{xy}{z}\right) + e^{xyz} x\cdot f\left(\frac{xy}{z}\right) + e^{xyz} xy\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{x}{z}\right)$$

$$+ e^{xyz} xz\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{-xy}{z^2}\right) + e^{xyz}\left[f''\left(\frac{xy}{z}\right)\right]\left(\frac{x}{z}\right)\left(\frac{-xy}{z^2}\right)$$

$$+ e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(-\frac{x}{z^2}\right)$$

$$y\frac{\partial^2 u}{\partial z\,\partial y} = e^{xyz}\left[x^2y^2z\cdot f\left(\frac{xy}{z}\right) + xy\cdot f\left(\frac{xy}{z}\right) - \frac{x^2y^2}{z^3} f''\left(\frac{xy}{z}\right) - \frac{xy}{z^2} f'\left(\frac{xy}{z}\right)\right] \qquad \ldots(2)$$

From Eqs (1) and (2),

$$x\frac{\partial^2 u}{\partial z\,\partial x} = y\frac{\partial^2 u}{\partial z\,\partial y}$$

Example 42

If $u = r^m, r = \sqrt{x^2 + y^2 + z^2}$, *show that* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = m(m+1)r^{m-2}$.

Solution

$$u = r^m$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = mr^{m-1}\frac{\partial r}{\partial x} \qquad \ldots(1)$$

But
$$r = \sqrt{x^2 + y^2 + z^2}$$
$$r^2 = x^2 + y^2 + z^2$$

Differentiating r^2 partially w.r.t. x,

$$2r\frac{\partial r}{\partial x} = 2x$$
$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Substituting in Eq. (1),
$$\frac{\partial u}{\partial x} = mr^{m-1}\frac{x}{r}$$
$$= mr^{m-2}x$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = m\left[r^{m-2} + (m-2)r^{m-3}\frac{\partial r}{\partial x}x\right]$$
$$= m\left[r^{m-2} + (m-2)r^{m-3}\frac{x}{r}x\right]$$
$$= m[r^{m-2} + (m-2)r^{m-4}x^2] \qquad \text{...(2)}$$

Similarly,
$$\frac{\partial^2 u}{\partial y^2} = m[r^{m-2} + (m-2)r^{m-4}y^2] \qquad \text{...(3)}$$
$$\frac{\partial^2 u}{\partial z^2} = m[r^{m-2} + (m-2)r^{m-4}z^2] \qquad \text{...(4)}$$

Adding Eqs (2), (3) and (4),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 3mr^{m-2} + m(m-2)r^{m-4}(x^2 + y^2 + z^2)$$
$$= 3mr^{m-2} + m(m-2)r^{m-4} \cdot r^2$$
$$= r^{m-2}(3m + m^2 - 2m)$$
$$= r^{m-2}(m + m^2)$$
$$= m(m+1)r^{m-2}$$

Example 43

If $u = f(r)$ and $r^2 = x^2 + y^2 + z^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r}f'(r).$$ **[Summer 2015]**

Solution

$$u = f(r)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} f(r)$$
$$= \frac{\partial}{\partial r} f(r) \cdot \frac{\partial r}{\partial x}$$
$$= f'(r) \cdot \frac{\partial r}{\partial x} \quad \text{...(1)}$$

But $\quad r^2 = x^2 + y^2 + z^2$

Differentiating r^2 partially w.r.t. x,

$$2r\frac{\partial r}{\partial x} = 2x$$
$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Subsituting in Eq. (1),

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{x}{r}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right]$$
$$= f''(r)\frac{\partial r}{\partial x} \cdot \frac{x}{r} + \frac{f'(r)}{r} + xf'(r)\left(-\frac{1}{r^2}\right) \cdot \frac{\partial r}{\partial x}$$
$$= f''(r)\frac{x}{r}\frac{x}{r} + \frac{f'(r)}{r} - \frac{x}{r^2} f'(r) \cdot \frac{x}{r}$$
$$= f''(r)\frac{x^2}{r^2} + \frac{f'(r)}{r} - \frac{x^2}{r^3} f'(r) \quad \text{...(2)}$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = f''(r)\frac{y^2}{r^2} + \frac{f'(r)}{r} - \frac{y^2}{r^3} f'(r) \quad \text{...(3)}$$

and $$\frac{\partial^2 u}{\partial z^2} = f''(r)\frac{z^2}{r^2} + \frac{f'(r)}{r} - \frac{z^2}{r^3} f'(r) \quad \text{...(4)}$$

Adding Eqs (2), (3) and (4),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{f''(r)}{r^2}(x^2 + y^2 + z^2) + \frac{3f'(r)}{r} - \frac{(x^2 + y^2 + z^2)}{r^3} f'(r)$$
$$= \frac{f''(r)}{r^2} \cdot r^2 + \frac{3f'(r)}{r} - \frac{r^2}{r^3} f'(r)$$
$$= f''(r) + \frac{2f'(r)}{r}$$

Example 44

If $u = f(r^2)$ *where* $r^2 = x^2 + y^2 + z^2$, *prove that*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=4r^2 f''(r^2)+6f'(r^2).$$

Solution

$$u = f(r^2)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}f(r^2)$$

$$=\frac{\partial}{\partial x}f(l), \quad \text{where } r^2 = l$$

$$=\frac{\partial}{\partial l}f(l)\cdot\frac{\partial l}{\partial x}$$

$$=f'(l)\frac{\partial l}{\partial x}$$

$$=f'(r^2)\frac{\partial r^2}{\partial x}$$

$$\therefore \quad \frac{\partial u}{\partial x}=f'(r^2)\cdot 2r\frac{\partial r}{\partial x} \qquad \text{...(1)}$$

But $\qquad r^2 = x^2 + y^2 + z^2$

Differentiating r^2 partially w.r.t. x,

$$2r\frac{\partial r}{\partial x}=2x$$

$$\frac{\partial r}{\partial x}=\frac{x}{r}$$

Substituting in Eq. (1),

$$\frac{\partial u}{\partial x}=f'(r^2)\cdot 2r\frac{x}{r}$$

$$=2x\,f'(r^2)$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2}=2f'(r^2)+2x\frac{\partial f'(r^2)}{\partial x}$$

$$=2f'(r^2)+2x\,f''(r^2)\cdot 2r\frac{\partial r}{\partial x}$$

$$=2f'(r^2)+2x\,f''(r^2)\cdot 2r\frac{x}{r} \qquad \text{...(2)}$$

$$=2f'(r^2)+4x^2 f''(r^2)$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = 2f'(r^2) + 4y^2 f''(r^2) \qquad \ldots(3)$$

and $$\frac{\partial^2 u}{\partial z^2} = 2f'(r^2) + 4z^2 f''(r^2) \qquad \ldots(4)$$

Adding Eqs (2), (3) and (4),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 6f'(r^2) + 4(x^2 + y^2 + z^2) f''(r^2)$$

$$= 6f'(r^2) + 4r^2 f''(r^2)$$

Example 45

If $v = x \log(x + r) - r$ *where* $r^2 = x^2 + y^2$, *prove that* $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = \dfrac{1}{x + r}$.

Solution

$$v = x \log(x + r) - r$$

Differentiating v partially w.r.t. x,

$$\frac{\partial v}{\partial x} = \log(x + r) + \frac{x}{x + r}\left(1 + \frac{\partial r}{\partial x}\right) - \frac{\partial r}{\partial x}$$

But $$r^2 = x^2 + y^2$$

Differentiating r^2 partially w.r.t. x,

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\therefore \quad \frac{\partial v}{\partial x} = \log(x + r) + \frac{x}{x + r}\left(1 + \frac{x}{r}\right) - \frac{x}{r}$$

$$= \log(x + r) + \frac{x}{(x + r)} \cdot \frac{(r + x)}{r} - \frac{x}{r}$$

$$= \log(x + r) + \frac{x}{r} - \frac{x}{r}$$

$$= \log(x + r)$$

Differentiating $\frac{\partial v}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{x+r}\left(1+\frac{\partial r}{\partial x}\right)$$

$$= \frac{1}{x+r}\left(1+\frac{x}{r}\right)$$

$$= \frac{1}{r}$$

Differentiating v partially w.r.t. y,

$$\frac{\partial v}{\partial y} = \frac{x}{x+r}\cdot\frac{\partial r}{\partial y} - \frac{\partial r}{\partial y}$$

$$= \frac{x}{x+r}\cdot\frac{y}{r} - \frac{y}{r}$$

$$= \frac{y}{r}\left(\frac{x-x-r}{x+r}\right)$$

$$= -\frac{y}{x+r}$$

Differentiating $\frac{\partial v}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 v}{\partial y^2} = -\frac{1}{x+r} + \frac{y}{(x+r)^2}\cdot\frac{\partial r}{\partial y}$$

$$= -\frac{1}{x+r}\left(1-\frac{y}{x+r}\cdot\frac{y}{r}\right)$$

$$= -\frac{1}{x+r}\left[\frac{rx+r^2-y^2}{r(x+r)}\right]$$

$$= -\frac{1}{x+r}\left[\frac{rx+x^2}{r(x+r)}\right]$$

$$= -\frac{x}{r(x+r)}$$

Hence,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{r}\left(1-\frac{x}{x+r}\right)$$

$$= \frac{1}{r}\left(\frac{x+r-x}{x+r}\right)$$

$$= \frac{1}{x+r}$$

EXERCISE 8.2

1. If $u = \cos\left(\sqrt{x} + \sqrt{y}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \dfrac{1}{2}\left(\sqrt{x} + \sqrt{y}\right)\sin\left(\sqrt{x} + \sqrt{y}\right) = 0.$

2. If $u = 2(ax + by)^2 - k(x^2 + y^2)$ and $a^2 + b^2 = k$, find the value of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$.

[Ans. : 0]

3. If $e^u = \tan x + \tan y$, show that $\sin 2x \dfrac{\partial u}{\partial x} + \sin 2y \dfrac{\partial u}{\partial y} = 2.$

4. If $z^3 - 3yz - 3x = 0$, show that

(i) $z\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y}$ (ii) $z\left[\dfrac{\partial^2 z}{\partial x \partial y} + \left(\dfrac{\partial z}{\partial x}\right)^2\right] = \dfrac{\partial^2 z}{\partial y^2}$

5. If $z(z^2 + 3x) + 3y = 0$, prove that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{2z(x-1)}{(z^2 + x)^3}$.

6. If $u = \log(x^2 + y^2) + \tan^{-1}\left(\dfrac{y}{x}\right)$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

7. If $u(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$, find the value of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$.

$$\left[\textbf{Ans.: } \frac{2}{(x^2 + y^2 + z^2)^2}\right]$$

8. If $x = e^{r\cos\theta}\cos(r\sin\theta)$ and $y = e^{r\cos\theta}\sin(r\sin\theta)$, prove that

$$\frac{\partial x}{\partial r} = \frac{1}{r}\frac{\partial y}{\partial \theta}, \quad \frac{\partial y}{\partial r} = -\frac{1}{r}\frac{\partial x}{\partial \theta}$$

Hence, deduce that $\dfrac{\partial^2 x}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial x}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 x}{\partial \theta^2} = 0.$

9. If $v = (x^2 - y^2)f(x, y)$, prove that $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = (x^4 - y^4)f''(x, y).$

10. If $u = f(ax^2 + 2hxy + by^2)$ and $v = \phi(ax^2 + 2hxy + by^2)$, show that

$$\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right).$$

11. If $x = \frac{r}{2}(e^{\theta} + e^{-\theta}), y = \frac{r}{2}(e^{\theta} - e^{-\theta})$, prove that $\left(\frac{\partial x}{\partial r}\right)_{\theta} = \left(\frac{\partial r}{\partial x}\right)_{y}$.

$$\left[\textbf{Hint: } x = r\cosh\theta, y = r\sinh\theta,\ x^2 - y^2 = r^2\right]$$

12. If $\log_e \theta = r - x, r^2 = x^2 + y^2$, show that $\frac{\partial^2 \theta}{\partial y^2} = \frac{\theta(x^2 + ry^2)}{r^3}$.

$$\left[\textbf{Hint: } \theta = e^{r-x}, \frac{\partial r}{\partial y} = \frac{y}{r}\right]$$

13. If $u = e^{ax} \sin by$, prove that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.

14. If $u = \tan^{-1}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)$, prove that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{1}{(1+x^2+y^2)^{\frac{3}{2}}}$.

15. If $u = \frac{1}{\sqrt{y}} e^{-\frac{(x-a)^2}{4y}}$, prove that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.

16. If $u = \tan(y + ax) - (y - ax)^{\frac{3}{2}}$, prove that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.

17. If $u = \frac{xy}{2x + y}$, prove that $\frac{\partial^3 u}{\partial y\,\partial z^2} = \frac{\partial^3 u}{\partial z^2\,\partial y}$.

18. If $u = x^m y^n$, prove that $\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = \frac{\partial^3 u}{\partial y\,\partial x^2}$.

19. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for the following functions:

(i) $\sqrt{x+y-1}$ (ii) $\sqrt{1-x^2-y^2}$ (iii) y^x (iv) $\log_{10}(ax+by)$ (v) $(y-ax)^{\frac{3}{2}}$

$$\left[\begin{array}{ll} \textbf{Ans.: } \text{(i) } \frac{1}{\sqrt{x+y-1}}, \frac{1}{\sqrt{x+y-1}} & \text{(ii) } \frac{-x}{\sqrt{1-x^2-y^2}}, \frac{-y}{\sqrt{1-x^2-y^2}} \\ \text{(iii) } y^x \log y,\ xy^{x-1} & \text{(iv) } \frac{a}{(\log_e 10)(ax+by)}, \frac{b}{(\log_e 10)(ax+by)} \\ \text{(v) } -\frac{3a}{2}(y-ax)^{\frac{1}{2}}, \frac{3}{2}(y-ax)^{\frac{1}{2}} & \end{array}\right]$$

20. If $x^4 - xy^2 + yz^2 - z^4 = 6$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\left[\textbf{Ans.}: \frac{y^2 - 4x^3}{2yz - 4z^3}, \frac{2xy - z^2}{2yz - 4z^3}\right]$$

21. If $z^3 + xy - y^2 z = 6$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at (0, 1, 2).

$$\left[\textbf{Ans.}: -\frac{1}{11}, \frac{4}{11}\right]$$

22. Find the value of for which $u = t^n e^{-\frac{r^2}{4kt}}$ satisfies the partial differential equation $\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right)$.

$$\left[\textbf{Ans.}: n = -\frac{3}{2}\right]$$

23. If $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$, find $\frac{\partial r}{\partial x}, \frac{\partial \theta}{\partial x}$ in terms of r, θ, ϕ.

$$\left[\textbf{Hint}: r^2 = x^2 + y^2 + z^2, \phi = \tan^{-1}\frac{y}{x}, \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}\right]$$

$$\left[\textbf{Ans.}: \sin\theta\cos\phi, \frac{\cos\theta\cos\phi}{r}, \frac{-\sin\phi}{r\sin\theta}\right]$$

24. If $u = x^2 (y - z) + y^2 (z - x) + z^2 (x - y)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

25. If $u = e^x(x\cos y - y\sin y)$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

26. For the function $f(x, y, z) = z\tan^{-1}\frac{y}{x}$, prove that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$.

27. If $z(x + y) = x^2 + y^2$, prove that $x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x\,\partial y} = 2\frac{\partial z}{\partial x}$.

28. If $\frac{x^2}{a + u} + \frac{y^2}{b + u} = 1$, prove that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$

29. If $u = x^y$, prove that $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x\,\partial y\,\partial x}$

30. If $\frac{x^2}{2+u}+\frac{y^2}{4+u}+\frac{z^2}{6+u}=1$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$

31. If $u=(x^2-y^2)\,f(r)$, where $r=xy$, show that

$$\frac{\partial^2 u}{\partial x\,\partial y}=(x^2-y^2)[3f'(r)+rf''(r)].$$

32. If $z=f(x^2, y)$, prove that $x\frac{\partial z}{\partial x}=2y\frac{\partial z}{\partial y}$.

33. Prove that $z=\frac{1}{r}[f(ct+r)+\phi(ct-r)]$ satisfies the partial differential equation $\frac{\partial^2 u}{\partial t^2}=\frac{c^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)$ where is constant.

34. If $u+iv=f(x+iy)$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0.$

$$\left[\begin{array}{l}\textbf{Hint: } u+iv=f(x+iy),\ u-iv=f(x-iy)\\ \qquad u=\frac{1}{2}[f(x+iy)+f(x-iy)],\ v=\frac{1}{2i}[f(x+iy)-f(x-iy)]\end{array}\right]$$

35. If u, v, w are function of x, y, z given as $x=u+v+w$, $y=u^2+v^2+w^2$, $z=u^3+v^3+w^3$, prove that $\frac{\partial u}{\partial x}=\frac{vw(w-v)}{(u-v)(v-w)(w-u)}$.

[**Hint**: Differentiate x, y, z w.r.t. x and solve the equations using Cramer's rule.]

36. If $u=(x^2+y^2+z^2)^{\frac{n}{2}}$, find the value of which satisfies the equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0.$

[**Ans.**: 0, –1]

37. If $u=\log(e^x+e^y)$, show that $\left(\frac{\partial^2 u}{\partial x^2}\right)\left(\frac{\partial^2 u}{\partial y^2}\right)-\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2=0.$

38. If $z=yf(x^2-y^2)$, show that $y\left(\frac{\partial z}{\partial x}\right)+x\left(\frac{\partial z}{\partial y}\right)=\frac{xy}{y}.$

8.6 TOTAL DERIVATIVES

8.6.1 Chain Rule

If $z = f(u)$, where u is again a function of two variables x and y, i.e., $u = \phi(x, y)$ then

$$\frac{\partial z}{\partial x} = \frac{dz}{du}\cdot\frac{\partial u}{\partial x} \quad \text{or} \quad \frac{df}{du}\cdot\frac{\partial u}{\partial x} \quad \text{or} \quad f'(u)\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du}\cdot\frac{\partial u}{\partial y} \quad \text{or} \quad \frac{df}{du}\cdot\frac{\partial u}{\partial y} \quad \text{or} \quad f'(u)\frac{\partial u}{\partial y}$$

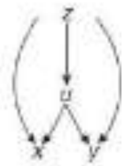


Fig. 8.1

8.6.2 Composite Function of One Variable

If $u = f(x, y)$, where $x = \phi(t)$, $y = \psi(t)$ then u is a function of t and is called the *composite function of a single variable t* and

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

is called the *total derivative* of *u*.

Fig. 8.2

If $u = f(x, y, z)$ and $x = \phi(t)$, $y = \psi(t)$, $z = \xi(t)$ then total derivative of u is given as

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}.$$

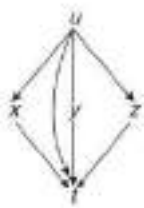


Fig. 8.3

Example 1

If $u = y^2 - 4ax$, $x = at^2$, $y = 2at$, *find* $\frac{du}{dt}$.

Solution

$$u = y^2 - 4ax,\ x = at^2,\ y = 2at$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= (-4a)2at + (2y)2a$$

Substituting y,

$$\frac{du}{dt} = -8a^2t + 2(2at)(2a)$$

$$= -8a^2t + 8a^2t$$

$$= 0$$

Fig. 8.4

Example 2

If $u = \sin\left(\frac{x}{y}\right)$ *where* $x = e^t$, $y = t^2$, *find* $\frac{du}{dt}$.

Solution

$$u = \sin\left(\frac{x}{y}\right), x = e^t, y = t^2$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{1}{y}\cos\left(\frac{x}{y}\right)\cdot e^t + \left(-\frac{x}{y^2}\right)\cos\left(\frac{x}{y}\right)\cdot 2t$$

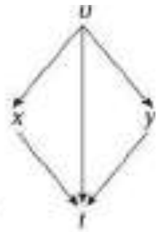


Fig. 8.5

Substituting x and y,

$$\frac{du}{dt} = \frac{1}{t^2}\cos\left(\frac{e^t}{t^2}\right)\cdot e^t - \frac{e^t}{(t^2)^2}\cos\left(\frac{e^t}{t^2}\right)\cdot 2t$$

$$= \frac{1}{t^2}e^t\cos\left(\frac{e^t}{t^2}\right)\left(1 - \frac{2}{t}\right)$$

Example 3

If $u = x^2y^3$, $x = \log t$, $y = e^t$, *find* $\frac{du}{dt}$.

Solution

$$u = x^2y^3, x = \log t, y = e^t$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= (2xy^3)\frac{1}{t} + (3x^2y^2)e^t$$

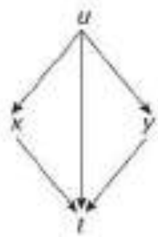


Fig. 8.6

Substituting x and y,

$$\frac{du}{dt} = 2(\log t)e^{3t}\cdot\frac{1}{t} + 3(\log t)^2e^{2t}\cdot e^t$$

$$= \frac{2}{t}\log t\, e^{3t} + 3(\log t)^2e^{3t}$$

Example 4

If $u = xy + yz + zx$ *where* $x = \frac{1}{t}, y = e^t, z = e^{-t}$, *find* $\frac{du}{dt}$.

Solution

$$u = xy + yz + zx, \quad x = \frac{1}{t}, y = e^t, z = e^{-t}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$= (y+z)\left(-\frac{1}{t^2}\right) + (x+z)e^t + (y+x)(-e^{-t})$$

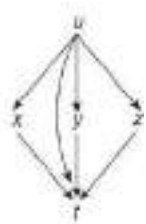


Fig. 8.7

Substituting x, y and z,

$$\frac{du}{dt} = -\frac{1}{t^2}(e^t + e^{-t}) + \left(\frac{1}{t} + e^{-t}\right)e^t - \left(e^t + \frac{1}{t}\right)e^{-t}$$
$$= -\frac{1}{t^2}(e^t + e^{-t}) + \frac{1}{t}(e^t - e^{-t})$$

Example 5

If $z = xy^2 + x^2y, x = at^2, y = 2at,$ *find* $\frac{dz}{dt}$.

Solution

$$z = xy^2 + x^2y, x = at^2, y = 2at$$
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$
$$= (y^2 + 2xy)2at + (2xy + x^2)2a$$

Fig. 8.8

Substituting x, y and z,

$$\frac{dz}{dt} = (4a^2t^2 + 2at^2\cdot 2at)2at + (2at^2\cdot 2at + a^2t^4)2a$$
$$= 4a^2t^2(1+t)2at + a^2t^3(4+t)2a$$
$$= 8a^3t^3(1+t) + 2a^3t^3(4+t)$$
$$= 2a^3t^3(8+5t)$$

Example 6

If $z = \sin^{-1}(x - y), x = 3t, y = 4t^3,$ *prove that* $\frac{dz}{dt} = \frac{3}{\sqrt{1-t^2}}$.

[Summer 2015]

Solution

$$z = \sin^{-1}(x-y), x = 3t, y = 4t^3$$
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$
$$= \frac{1}{\sqrt{1-(x-y)^2}}\cdot 3 + \frac{1(-1)}{\sqrt{1-(x-y)^2}}\cdot 12t^2$$
$$= \frac{3-12t^2}{\sqrt{1-x^2-y^2+2xy}}$$

Fig. 8.9

Substituting x and y,

$$\begin{aligned}
\frac{dz}{dt} &= \frac{3(1-4t^2)}{\sqrt{1-9t^2-16t^6+24t^4}} \\
&= \frac{3(1-4t^2)}{\sqrt{1-8t^2-t^2-16t^6+16t^4+8t^4}} \\
&= \frac{3(1-4t^2)}{\sqrt{1-8t^2+16t^4-t^2-16t^6+8t^4}} \\
&= \frac{3(1-4t^2)}{\sqrt{(1-4t^2)^2-t^2(1+16t^4-8t^2)}} \\
&= \frac{3(1-4t^2)}{\sqrt{(1-4t^2)^2-t^2(1-4t^2)^2}} \\
&= \frac{3(1-4t^2)}{(1-4t^2)\sqrt{1-t^2}} \\
&= \frac{3}{\sqrt{1-t^2}}
\end{aligned}$$

Example 7

If $u = \tan^{-1}\left(\frac{y}{x}\right), x = e^t - e^{-t}, y = e^t + e^{-t}$, *find* $\frac{du}{dt}$.

Solution

$$u = \tan^{-1}\left(\frac{y}{x}\right), x = e^t - e^{-t}, y = e^t + e^{-t}$$

$$\begin{aligned}
\frac{du}{dt} &= \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \\
&= \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)(e^t + e^{-t}) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)(e^t - e^{-t}) \\
&= -\frac{y}{x^2+y^2}\cdot y + \frac{x}{x^2+y^2}\cdot x \\
&= \frac{x^2-y^2}{x^2+y^2} \\
&= \frac{(e^t-e^{-t})^2-(e^t+e^{-t})^2}{(e^t-e^{-t})^2+(e^t+e^{-t})^2}
\end{aligned}$$

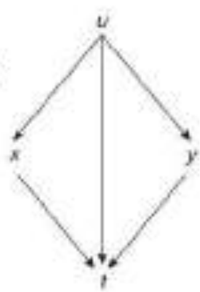


Fig. 8.10

$$= -\frac{4}{2(e^{2t} + e^{-2t})}$$

$$= -\frac{2}{e^{2t} + e^{-2t}}$$

Example 8

For $z = \tan^{-1}\left(\frac{x}{y}\right)$, $x = u \cos v$, $y = u \sin v$, *evaluate* $\frac{\partial z}{\partial u}$ *and* $\frac{\partial z}{\partial v}$ *at the point* $\left(1.3, \frac{\pi}{6}\right)$. **[Winter 2015]**

Solution

$$z = \tan^{-1}\left(\frac{x}{y}\right), \quad x = u \cos v, \; y = u \sin v$$

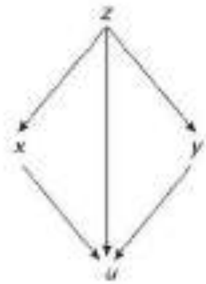


Fig. 8.11

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= \frac{1}{1 + \frac{x^2}{y^2}}\left(\frac{1}{y}\right) \cdot \cos v + \frac{1}{1 + \frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right) \cdot \sin v$$

$$= \frac{y \cos v}{y^2 + x^2} - \frac{x \sin v}{y^2 + x^2}$$

$$= \frac{u \sin v \cos v - u \cos v \sin v}{x^2 + y^2}$$

$$= 0$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

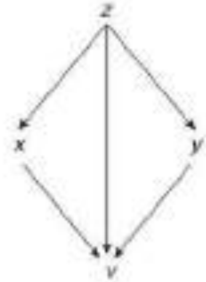


Fig. 8.12

$$= \frac{1}{1 + \frac{x^2}{y^2}}\left(\frac{1}{y}\right)(-u \sin v) + \frac{1}{1 + \frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right)(u \cos v)$$

$$= -\frac{yu \sin v}{y^2 + x^2} - \frac{xu \cos v}{y^2 + x^2}$$

$$= -\frac{(y^2 + x^2)}{y^2 + x^2}$$

$$= -1$$

Example 9

If $u = x^2 + y^2 + z^2$, *where* $x = e^t, y = e^t \sin t, z = e^t \cos t$, *find* $\frac{du}{dt}$.

Solution

$$u = x^2 + y^2 + z^2,\ x = e^t,\ y = e^t \sin t,\ z = e^t \cos t$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$= 2xe^t + 2y(e^t \sin t + e^t \cos t) + 2z(e^t \cos t - e^t \sin t)$$

$$= 2e^t \cdot e^t + 2e^t \sin t \cdot e^t(\sin t + \cos t) + 2e^t \cos t \cdot e^t(\cos t - \sin t)$$

$$= 2e^{2t}(1 + \sin^2 t + \sin t \cos t + \cos^2 t - \cos t \sin t)$$

$$= 4e^{2t}$$

Fig. 8.13

Example 10

If $z = e^{xy}, x = t\cos t, y = t \sin t$, *find* $\frac{dz}{dt}$ *at* $t = \frac{\pi}{2}$.

Solution

$$z = e^{xy},\ x = t\cos t,\ y = t\sin t$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= e^{xy}\, y(\cos t - t\sin t) + e^{xy}\, x(\sin t + t\cos t)$$

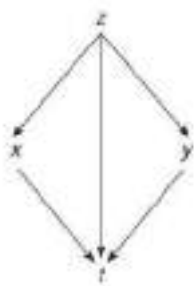


Fig. 8.14

At $t = \frac{\pi}{2}, x = 0, y = \frac{\pi}{2}$

Hence,
$$\left.\frac{dz}{dt}\right|_{t=\frac{\pi}{2}} = e^0\left[\frac{\pi}{2}\left(0 - \frac{\pi}{2}\right) + 0\right]$$

$$= -\frac{\pi^2}{4}$$

Example 11

If $z = x^2y + 3xy^4$ *where* $x = \sin 2t$ *and* $y = \cos t$, *find* $\frac{dz}{dt}$ *when* $t = 0$.

[Winter 2013]

Solution

$$z = x^2y + 3xy^4,\ x = \sin 2t,\ y = \cos t$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

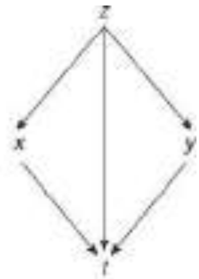


Fig. 8.15

At $t = 0,\ x = 0,\ y = 1$

Hence, $\left.\frac{dz}{dt}\right|_{t=0} = (0+3)[2(1)] + 0 = 6$

Example 12

If $u = x^2 + y^2 + z^2 - 2xyz = 1$, *show that* $\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0.$

Solution

$$u = x^2 + y^2 + z^2 - 2xyz = 1$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0$$

$$(2x - 2yz)dx + (2y - 2xz)dy + (2z - 2xy)dz = 0$$

$$(x - yz)dx + (y - xz)dy + (z - xy)dz = 0 \quad \ldots(1)$$

We have,

$$x^2 + y^2 + z^2 - 2xyz = 1$$

$$x^2 - 2xyz = 1 - y^2 - z^2$$

$$x^2 - 2xyz + y^2z^2 = 1 - y^2 - z^2 + y^2z^2$$

$$(x - yz)^2 = (1 - y^2)(1 - z^2)$$

$$x - yz = \sqrt{1-y^2}\cdot\sqrt{1-z^2}$$

Similarly,

$$y - xz = \sqrt{1-x^2}\cdot\sqrt{1-z^2}$$

and

$$z - xy = \sqrt{1-x^2}\cdot\sqrt{1-y^2}$$

Substituting in Eq. (1),

$$\sqrt{1-y^2}\cdot\sqrt{1-z^2}\,dx + \sqrt{1-x^2}\cdot\sqrt{1-z^2}\,dy + \sqrt{1-x^2}\cdot\sqrt{1-y^2}\,dz = 0$$

$$\sqrt{1-x^2}\sqrt{1-y^2}\sqrt{1-z^2}\left(\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}}\right) = 0$$

Hence, $$\frac{dx}{\sqrt{1-x^2}}+\frac{dy}{\sqrt{1-y^2}}+\frac{dz}{\sqrt{1-z^2}}=0$$

8.6.3 Composite Function of Two Variables

If $z = f(x, y)$, where $x = \phi(u, v)$, $y = \psi(u, v)$ then z is a function of u, v and is called the composite function of two variables u and v.

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

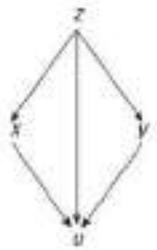


Fig. 8.16

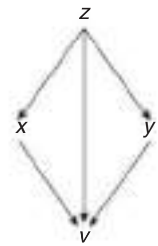


Fig. 8.17

Example 1

If $z = f(u, v), u=\log(x^2+y^2), v=\frac{y}{x}$, *show that*

$$x\frac{\partial z}{\partial y}-y\frac{\partial z}{\partial x}=(1+v^2)\frac{\partial z}{\partial v}.$$

Solution

$$z=f(u,v), u=\log(x^2+y^2), v=\frac{y}{x},$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=\frac{\partial z}{\partial u}\cdot\frac{1}{x^2+y^2}\cdot 2x+\frac{\partial z}{\partial v}\left(-\frac{y}{x^2}\right)$$

$$y\frac{\partial z}{\partial x}=\frac{2xy}{x^2+y^2}\cdot\frac{\partial z}{\partial u}-\frac{y^2}{x^2}\cdot\frac{\partial z}{\partial v} \quad ...(1)$$

Fig. 8.18

and $$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$=\frac{\partial z}{\partial u}\cdot\frac{2y}{x^2+y^2}+\frac{\partial z}{\partial v}\cdot\frac{1}{x}$$

$$x\frac{\partial z}{\partial y}=\frac{2xy}{x^2+y^2}\cdot\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v} \quad ...(2)$$

Fig. 8.19

Subtracting Eq. (1) from Eq. (2),

Hence, $x\dfrac{\partial z}{\partial y} - y\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial v} + \dfrac{y^2}{x^2}\dfrac{\partial z}{\partial v}$

$$= (1+v^2)\frac{\partial z}{\partial v}$$

Example 2

If $w = \phi(u, v), u = x^2 - y^2 - 2xy, v = y$, *prove that* $\dfrac{\partial w}{\partial v} = 0$ *is equivalent to* $(x+y)\dfrac{\partial w}{\partial x} + (x-y)\dfrac{\partial w}{\partial y} = 0.$

Solution

$$w = \phi(u, v), u = x^2 - y^2 - 2xy, v = y$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial w}{\partial u}(2x-2y) + \frac{\partial w}{\partial v}\cdot 0$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}(2x-2y)$$

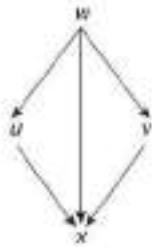


Fig. 8.20

and

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{\partial w}{\partial u}(-2y-2x) + \frac{\partial w}{\partial v}\cdot 1$$

$$\frac{\partial w}{\partial y} = -2(x+y)\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}$$

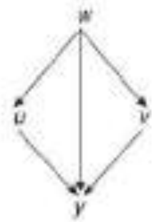


Fig. 8.21

$$(x+y)\frac{\partial w}{\partial x} + (x-y)\frac{\partial w}{\partial y} = (x+y)2(x-y)\frac{\partial w}{\partial u} - (x-y)2(x+y)\frac{\partial w}{\partial u} + (x-y)\frac{\partial w}{\partial v}$$

$$= (x-y)\frac{\partial w}{\partial v}$$

If $\dfrac{\partial w}{\partial v} = 0$ then $(x+y)\dfrac{\partial w}{\partial x} + (x-y)\dfrac{\partial w}{\partial y} = 0.$

Hence, $\dfrac{\partial w}{\partial v} = 0$ is equivalent to $(x+y)\dfrac{\partial w}{\partial x} + (x-y)\dfrac{\partial w}{\partial y} = 0.$

Example 3

If $z = f(x, y)$ *and* $x = e^u + e^{-v}$ *and* $y = e^{-u} - e^v$, *show that*

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

[Summer 2014]

Solution

$$z = f(x, y),\ x = e^u + e^{-v},\ y = e^{-u} - e^v$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}e^u + \frac{\partial z}{\partial y}(-e^{-u}) \quad ...(1)$$

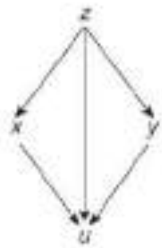


Fig. 8.22

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^v) \quad ...(2)$$

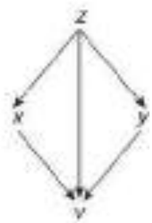


Fig. 8.23

Subtracting Eq. (2) from Eq. (1),

Hence,

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^u + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} - e^v)$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Example 4

If $z = f(u, v)$ *and* $u = x\cos\theta - y\sin\theta,\ v = x\sin\theta + y\cos\theta$, *show that*

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}.$$

Solution

$$z = f(u, v),\ u = x\cos\theta - y\sin\theta,\ v = x\sin\theta + y\cos\theta$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}\cos\theta + \frac{\partial z}{\partial v}\sin\theta$$

$$x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}x\cos\theta + \frac{\partial z}{\partial v}x\sin\theta \quad ...(1)$$

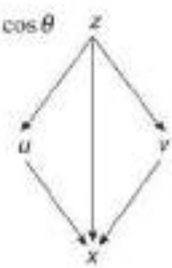
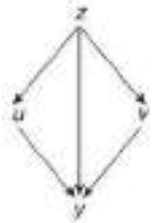


Fig. 8.24

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}(-\sin\theta) + \frac{\partial z}{\partial v}\cos\theta$$

$$y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}(-y\sin\theta) + \frac{\partial z}{\partial v}(y\cos\theta) \qquad \text{...(2)}$$



Adding Eqs (1) and (2),

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}(x\cos\theta - y\sin\theta) + \frac{\partial z}{\partial v}(x\sin\theta + y\cos\theta)$$

Fig. 8.25

$$= u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$$

Example 5

If $z = f(x, y)$, $x = uv$, $y = \dfrac{u+v}{u-v}$, *prove that* $u\dfrac{\partial z}{\partial u} + v\dfrac{\partial z}{\partial v} = 2x\dfrac{\partial z}{\partial x}$.

Solution

$$z = f(x, y),\ x = uv,\ y = \frac{u+v}{u-v}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}v + \frac{\partial z}{\partial y}\left\{\frac{(u-v)-(u+v)}{(u-v)^2}\right\}$$

$$= \frac{\partial z}{\partial x}v - \frac{2v}{(u-v)^2}\frac{\partial z}{\partial y}$$

$$u\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}uv - \frac{2uv}{(u-v)^2}\frac{\partial z}{\partial y} \qquad \text{...(1)}$$

Fig. 8.26

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}u + \frac{\partial z}{\partial y}\left\{\frac{(u-v)-(u+v)(-1)}{(u-v)^2}\right\}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}u + \frac{2u}{(u-v)^2}\cdot\frac{\partial z}{\partial y}$$

$$v\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}uv + \frac{2uv}{(u-v)^2}\cdot\frac{\partial z}{\partial y} \qquad \text{...(2)}$$

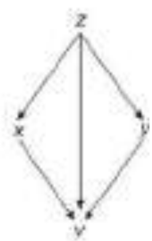


Fig. 8.27

Adding Eqs (1) and (2),

$$u\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}=2uv\frac{\partial z}{\partial x}$$
$$=2x\frac{\partial z}{\partial x}$$

Example 6

If $z=f(x,y)$, $x=u\cosh v$, $y=u\sinh v$, *prove that*

$$\left(\frac{\partial z}{\partial u}\right)^2-\frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2.$$

Solution

$$z=f(x,y),\quad x=u\cosh v,\quad y=u\sinh v$$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$
$$=\frac{\partial z}{\partial x}\cosh v+\frac{\partial z}{\partial y}\sinh v \qquad \ldots(1)$$

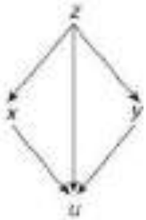


Fig. 8.28

and

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$
$$=\frac{\partial z}{\partial x}u\sinh v+\frac{\partial z}{\partial y}u\cosh v$$
$$\frac{1}{u}\cdot\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\sinh v+\frac{\partial z}{\partial y}\cosh v \qquad \ldots(2)$$

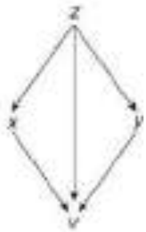


Fig. 8.29

Squaring and subtracting Eq. (2) from Eq. (1),

$$\left(\frac{\partial z}{\partial u}\right)^2-\frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2\cosh^2 v+\left(\frac{\partial z}{\partial y}\right)^2\sinh^2 v+2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cosh v\sinh v$$
$$-\left(\frac{\partial z}{\partial x}\right)^2\sinh^2 v-\left(\frac{\partial z}{\partial y}\right)^2\cosh^2 v-2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cosh v\sinh v$$
$$=\left(\frac{\partial z}{\partial x}\right)^2(\cosh^2 v-\sinh^2 v)-\left(\frac{\partial z}{\partial y}\right)^2(\cosh^2 v-\sinh^2 v)$$
$$=\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2$$

Example 7

If $x = r\cosh\theta,\ y = r\sinh\theta,$ *show that* $(x-y)(z_x - z_y) = r\,z_r - z_\theta.$

Solution

$$z = f(x, y),\quad x = r\cosh\theta,\quad y = r\sinh\theta$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial z}{\partial x}\cdot\cosh\theta + \frac{\partial z}{\partial y}\cdot\sinh\theta$$

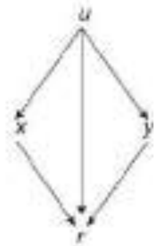


Fig. 8.30

and $$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial z}{\partial x}\cdot r\sinh\theta + \frac{\partial z}{\partial y}\cdot r\cosh\theta$$

$$r\frac{\partial z}{\partial r} - \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot r\cosh\theta + \frac{\partial z}{\partial y}\cdot r\sinh\theta - \frac{\partial z}{\partial x}\cdot r\sinh\theta - \frac{\partial z}{\partial y}\cdot r\cosh\theta$$

$$= \frac{\partial z}{\partial x}(r\cosh\theta - r\sinh\theta) + \frac{\partial z}{\partial y}(r\sinh\theta - r\cosh\theta)$$

$$= \frac{\partial z}{\partial x}(x-y) - \frac{\partial z}{\partial y}(x-y)$$

$$= (x-y)\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

Fig. 8.31

Hence, $(x-y)(z_x - z_y) = r\,z_r - z_\theta$

Example 8

If $z = f(x, y)$ *where* $x^2 = au + bv,\ y^2 = au - bv$ *then prove that*

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = \frac{1}{2}\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

Solution

$$z = f(x, y),\quad x^2 = au + bv,\quad y^2 = au - bv$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}\cdot\frac{a}{2x} + \frac{\partial z}{\partial y}\cdot\frac{a}{2y}$$

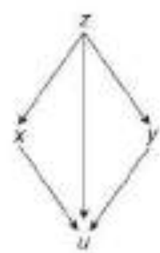


Fig. 8.32

$$u\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{au}{2x} + \frac{\partial z}{\partial y}\cdot\frac{au}{2y} \qquad \text{...(1)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}\cdot\frac{b}{2x} + \frac{\partial z}{\partial y}\left(-\frac{b}{2y}\right)$$

$$v\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{bv}{2x} - \frac{\partial z}{\partial y}\cdot\frac{bv}{2y} \quad \text{...(2)}$$

Fig. 8.33

Adding Eqs (1) and (2),

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{au}{2x} + \frac{\partial z}{\partial y}\frac{au}{2y} + \frac{\partial z}{\partial x}\frac{bv}{2x} - \frac{\partial z}{\partial y}\frac{bv}{2y}$$

$$= \frac{\partial z}{\partial x}\left(\frac{au+bv}{2x}\right) + \frac{\partial z}{\partial y}\left(\frac{au-bv}{2y}\right)$$

$$= \frac{\partial z}{\partial x}\left(\frac{x^2}{2x}\right) + \frac{\partial z}{\partial y}\left(\frac{y^2}{2y}\right)$$

$$= \frac{1}{2}\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

Example 9

If $u = f(x, y)$ where $x = r\cos\theta$ and $y = r\sin\theta$, prove that

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

Solution

$$u = f(x, y),\ x = r\cos\theta,\ y = r\sin\theta$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \quad \text{...(1)}$$

Fig. 8.34

and

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}r\cos\theta$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial u}{\partial y} \quad \text{...(2)}$$

Squaring and adding Eqs (1) and (2),

$$\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2=\cos^2\theta\left(\frac{\partial u}{\partial x}\right)^2+\sin^2\theta\left(\frac{\partial u}{\partial y}\right)^2+2\sin\theta\cos\theta\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$$

$$+\sin^2\theta\left(\frac{\partial u}{\partial x}\right)^2+\cos^2\theta\left(\frac{\partial u}{\partial y}\right)^2-2\sin\theta\cos\theta\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$$

$$=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$$

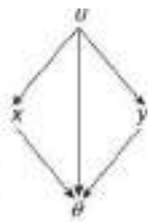


Fig. 8.35

Example 10

If $z=f(u,v)$ where $u=x^2+y^2$, $v=2xy$, *show that*

$$x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=2\sqrt{u^2-v^2}\left(\frac{\partial z}{\partial u}\right).$$

Solution

$$z=f(u,v),\quad u=x^2+y^2,\quad v=2xy$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=\frac{\partial z}{\partial u}\cdot 2x+\frac{\partial z}{\partial v}\cdot 2y$$

$$x\frac{\partial z}{\partial x}=2x^2\frac{\partial z}{\partial u}+2xy\frac{\partial z}{\partial v} \quad ...(1)$$

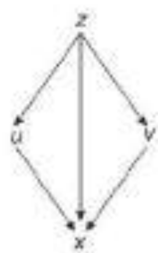


Fig. 8.36

and

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$=\frac{\partial z}{\partial u}\cdot 2y+\frac{\partial z}{\partial v}\cdot 2x$$

$$y\frac{\partial z}{\partial y}=2y^2\frac{\partial z}{\partial u}+2xy\frac{\partial z}{\partial v} \quad ...(2)$$

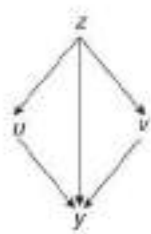


Fig. 8.37

Substracting Eq. (2) from Eq. (1),

$$x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=2(x^2-y^2)\frac{\partial z}{\partial u}$$

$$=2\sqrt{(x^2+y^2)^2-4x^2y^2}\left(\frac{\partial z}{\partial u}\right)$$

$$=2\sqrt{(u^2-v^2)}\left(\frac{\partial z}{\partial u}\right)$$

Example 11

If $z = f(u, v)$, *and* $u = x^2 - y^2, v = 2xy$, *show that*

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4(u^2 + v^2)^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right].$$

Solution

$z = f(u, v)$, and $u = x^2 - y^2, v = 2xy$

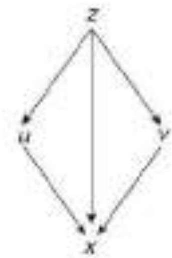


Fig. 8.38

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}\cdot 2x + \frac{\partial z}{\partial v}\cdot 2y$$

$$= 2\left(x\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}\right) \qquad \text{...(1)}$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}(-2y) + \frac{\partial z}{\partial v}(2x)$$

$$= 2\left(-y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}\right) \qquad \text{...(2)}$$

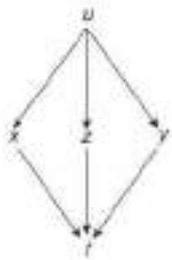


Fig. 8.39

Squaring and adding Eq. (1) and Eq. (2),

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4\left(x\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}\right)^2 + 4\left(-y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}\right)^2$$

$$= 4\left[x^2\left(\frac{\partial z}{\partial u}\right)^2 + y^2\left(\frac{\partial z}{\partial v}\right)^2 + 2xy\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v}\right.$$

$$\left. + y^2\left(\frac{\partial z}{\partial u}\right)^2 + x^2\left(\frac{\partial z}{\partial v}\right)^2 - 2xy\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v}\right]$$

$$= 4(x^2 + y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4\left[(x^2 + y^2)^2\right]^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4\left[(x^2 - y^2)^2 + 4x^2y^2\right]^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4(u^2 + v^2)^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

Example 12

If $x = e^u \operatorname{cosec} v$, $y = e^u \cot v$ *then show that*

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - \sin^2 v\left(\frac{\partial z}{\partial v}\right)^2\right].$$

Solution

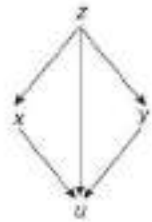


Fig. 8.40

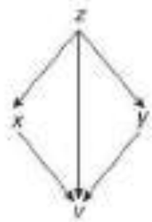


Fig. 8.41

$$z = f(x, y), \quad x = e^u \operatorname{cosec} v, \quad y = e^u \cot v$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x} e^u \operatorname{cosec} v + \frac{\partial z}{\partial y} e^u \cot v$$

and
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(-e^u \operatorname{cosec} v \cot v) + \frac{\partial z}{\partial y}(-e^u \operatorname{cosec}^2 v)$$

$$e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - \sin^2 v\left(\frac{\partial z}{\partial v}\right)^2\right] = e^{-2u}\left[\left(\frac{\partial z}{\partial x}\right)^2 e^{2u}\operatorname{cosec}^2 v + \left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\cot^2 v\right.$$

$$+ 2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} e^{2u}\operatorname{cosec} v \cot v$$

$$+ (-\sin^2 v)\left(\frac{\partial z}{\partial x}\right)^2 (e^{2u}\operatorname{cosec}^2 v \cot^2 v)$$

$$+ (-\sin^2 v)\left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\operatorname{cosec}^4 v$$

$$\left. + (-\sin^2 v) 2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} e^{2u}\operatorname{cosec}^3 v \cot v\right]$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 (\operatorname{cosec}^2 v - \cot^2 v) + \left(\frac{\partial z}{\partial y}\right)^2 (\cot^2 v - \operatorname{cosec}^2 v)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$$

Example 13

If $z = f(x, y)$, $x = e^u \cos v$, $y = e^u \sin v$, *prove that*

(i) $x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^{2u}\frac{\partial z}{\partial y}$ (ii) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$

Solution

$$z = f(x, y),\ x = e^u \cos v,\ y = e^u \sin v$$

(i)
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$
$$= \frac{\partial z}{\partial x}\cdot e^u \cos v + \frac{\partial z}{\partial y} e^u \sin v$$
$$= x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$
$$y\frac{\partial z}{\partial u} = xy\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} \qquad \text{...(1)}$$

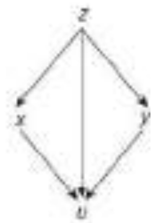


Fig. 8.42

and
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$
$$= \frac{\partial z}{\partial x}(-e^u \sin v) + \frac{\partial z}{\partial y} e^u \cos v$$
$$= -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$$
$$x\frac{\partial z}{\partial v} = -xy\frac{\partial z}{\partial x} + x^2\frac{\partial z}{\partial y} \qquad \text{...(2)}$$

Fig. 8.43

Adding Eqs (1) and (2),
$$x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = (y^2 + x^2)\frac{\partial z}{\partial y}$$
$$= e^{2u}\frac{\partial z}{\partial y}$$

(ii)
$$e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] = e^{-2u}\left[x^2\left(\frac{\partial z}{\partial x}\right)^2 + y^2\left(\frac{\partial z}{\partial y}\right)^2 + 2xy\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\right]$$
$$+\left[y^2\left(\frac{\partial z}{\partial x}\right)^2 + x^2\left(\frac{\partial z}{\partial y}\right)^2 - 2xy\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\right]$$
$$= e^{-2u}\left[(x^2 + y^2)\left(\frac{\partial z}{\partial x}\right)^2 + (x^2 + y^2)\left(\frac{\partial z}{\partial y}\right)^2\right]$$
$$= e^{-2u}e^{2u}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

Example 14

If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, *show that* $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$

[Summer 2016]

Solution

Let
$$l = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}, m = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$$

$$\frac{\partial l}{\partial x} = -\frac{1}{x^2}, \qquad \frac{\partial m}{\partial x} = -\frac{1}{x^2},$$

$$\frac{\partial l}{\partial y} = \frac{1}{y^2}, \qquad \frac{\partial m}{\partial y} = 0,$$

$$\frac{\partial l}{\partial z} = 0, \qquad \frac{\partial m}{\partial z} = \frac{1}{z^2}$$

$$u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right) = f(l, m)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x}$$

$$= \frac{\partial u}{\partial l}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial m}\left(-\frac{1}{x^2}\right)$$

$$x^2\frac{\partial u}{\partial x} = -\left(\frac{\partial u}{\partial l} + \frac{\partial u}{\partial m}\right) \qquad \ldots(1)$$

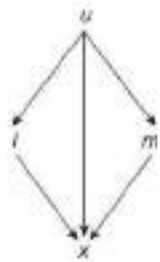


Fig. 8.44

Also,
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y}$$

$$= \frac{\partial u}{\partial l}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial m}\cdot 0$$

$$y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l} \qquad \ldots(2)$$

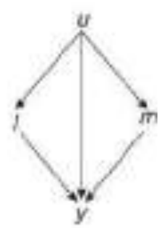


Fig. 8.45

and
$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z}$$

$$= \frac{\partial u}{\partial l}\cdot 0 + \frac{\partial u}{\partial m}\left(\frac{1}{z^2}\right)$$

$$z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial m} \qquad \ldots(3)$$

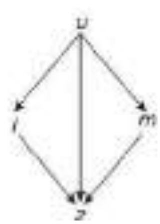


Fig. 8.46

Adding Eqs (1), (2) and (3),

$$x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=-\left(\frac{\partial u}{\partial l}+\frac{\partial u}{\partial m}\right)+\frac{\partial u}{\partial l}+\frac{\partial u}{\partial m}=0$$

Example 15

If $u=f(x-y,\,y-z,\,z-x)$ *then show that* $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=0.$

Solution

Let $x-y=l, \qquad y-z=m, \qquad z-x=n$

$$\frac{\partial l}{\partial x}=1,\qquad \frac{\partial m}{\partial x}=0,\qquad \frac{\partial n}{\partial x}=-1,$$
$$\frac{\partial l}{\partial y}=-1,\qquad \frac{\partial m}{\partial y}=1,\qquad \frac{\partial n}{\partial y}=0,$$
$$\frac{\partial l}{\partial z}=0\qquad \frac{\partial m}{\partial z}=-1\qquad \frac{\partial n}{\partial z}=1$$

$$u=f(x-y,\,y-z,\,z-x)=f(l,m,n)$$

$$\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial u}{\partial l}\frac{\partial l}{\partial x}+\frac{\partial u}{\partial m}\frac{\partial m}{\partial x}+\frac{\partial u}{\partial n}\frac{\partial n}{\partial x}\\&=\frac{\partial u}{\partial l}\cdot 1+\frac{\partial u}{\partial m}\cdot 0+\frac{\partial u}{\partial n}(-1)\\&=\frac{\partial u}{\partial l}-\frac{\partial u}{\partial n}\end{aligned} \qquad \text{... (1)}$$

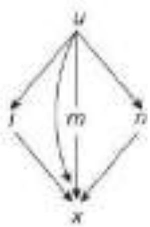


Fig. 8.47

also,

$$\begin{aligned}\frac{\partial u}{\partial y}&=\frac{\partial u}{\partial l}\frac{\partial l}{\partial y}+\frac{\partial u}{\partial m}\frac{\partial m}{\partial y}+\frac{\partial u}{\partial n}\frac{\partial n}{\partial y}\\&=\frac{\partial u}{\partial l}(-1)+\frac{\partial u}{\partial m}(1)+\frac{\partial u}{\partial n}\cdot 0\\&=-\frac{\partial u}{\partial l}+\frac{\partial u}{\partial m}\end{aligned} \qquad \text{... (2)}$$

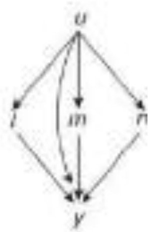


Fig. 8.48

and

$$\begin{aligned}\frac{\partial u}{\partial z}&=\frac{\partial u}{\partial l}\frac{\partial l}{\partial z}+\frac{\partial u}{\partial m}\frac{\partial m}{\partial z}+\frac{\partial u}{\partial n}\frac{\partial n}{\partial z}\\&=\frac{\partial u}{\partial l}\cdot 0+\frac{\partial u}{\partial m}(-1)+\frac{\partial u}{\partial n}(1)\\&=-\frac{\partial u}{\partial m}+\frac{\partial u}{\partial n}\end{aligned} \qquad \text{... (3)}$$

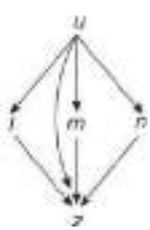


Fig. 8.49

Adding Eqs (1), (2) and (3),

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

Example 16

Find $\frac{\partial w}{\partial r}, \frac{\partial w}{\partial s}$ *in terms of r and s if* $w = x + 2y + z^2$, *where* $x=\frac{r}{s}$, $y = r^2 + \log s$, $z = 2r$. **[Winter 2014]**

Solution

$$x=\frac{r}{s}, \qquad y=r^2+\log s, \qquad z=2r$$

$$\frac{\partial x}{\partial r}=\frac{1}{s}, \qquad \frac{\partial y}{\partial r}=2r, \qquad \frac{\partial z}{\partial r}=2$$

$$\frac{\partial x}{\partial s}=-\frac{r}{s^2}, \qquad \frac{\partial y}{\partial s}=\frac{1}{s}, \qquad \frac{\partial z}{\partial s}=0$$

$$w = x + 2y + z^2$$

Fig. 8.50

$$\begin{aligned}\frac{\partial w}{\partial r}&=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}\\&=(1)\left(\frac{1}{s}\right)+(2)(2r)+(2z)(2)\\&=\frac{1}{s}+4r+4z\\&=\frac{1}{s}+4r+4(2r)\\&=\frac{1}{s}+12r\end{aligned}$$

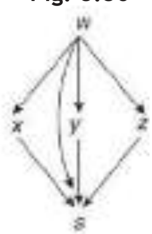


Fig. 8.51

$$\begin{aligned}\frac{\partial w}{\partial s}&=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}\\&=(1)\left(-\frac{r}{s^2}\right)+(2)\left(\frac{1}{s}\right)+(2z)(0)\\&=-\frac{r}{s^2}+\frac{2}{s}\end{aligned}$$

Example 17

If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, *prove that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$ **[Winter 2013]**

Solution

Let $\frac{x}{y} = l, \quad \frac{y}{z} = m, \quad \frac{z}{x} = n$

$$\frac{\partial l}{\partial x} = \frac{1}{y}, \quad \frac{\partial m}{\partial x} = 0, \quad \frac{\partial n}{\partial x} = -\frac{z}{x^2},$$

$$\frac{\partial l}{\partial y} = \frac{-x}{y^2}, \quad \frac{\partial m}{\partial y} = \frac{1}{z} \quad \frac{\partial n}{\partial y} = 0,$$

$$\frac{\partial l}{\partial z} = 0 \quad \frac{\partial m}{\partial z} = \frac{-y}{z^2}, \quad \frac{\partial n}{\partial z} = \frac{1}{x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x}$$

$$= \frac{\partial u}{\partial l}\cdot\frac{1}{y} + \frac{\partial u}{\partial m}\cdot 0 + \frac{\partial u}{\partial n}\left(\frac{-z}{x^2}\right)$$

$$x\frac{\partial u}{\partial x} = \frac{x}{y}\cdot\frac{\partial u}{\partial l} - \frac{z}{x}\cdot\frac{\partial u}{\partial n} \quad \text{...(1)}$$

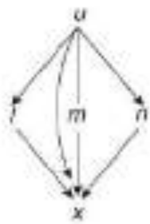


Fig. 8.52

Also,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}$$

$$= \frac{\partial u}{\partial l}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial m}\cdot\frac{1}{z} + \frac{\partial u}{\partial n}\cdot 0$$

$$y\frac{\partial u}{\partial y} = -\frac{x}{y}\cdot\frac{\partial u}{\partial l} + \frac{y}{z}\cdot\frac{\partial u}{\partial m} \quad \text{...(2)}$$

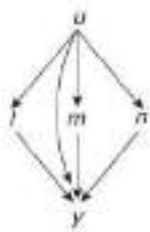


Fig. 8.53

and

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$

$$= \frac{\partial u}{\partial l}\cdot 0 + \frac{\partial u}{\partial m}\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial n}\cdot\frac{1}{x}$$

$$z\frac{\partial u}{\partial z} = \frac{-y}{z}\frac{\partial u}{\partial m} + \frac{z}{x}\cdot\frac{\partial u}{\partial n} \quad \text{...(3)}$$

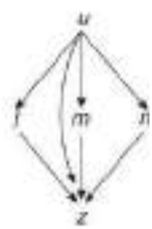


Fig. 8.54

Adding Eqs (1), (2) and (3),

Hence, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

Example 18

If $u = f\left(x^2 - y^2, y^2 - z^2, z^2 - x^2\right)$, *prove that* $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0.$

Solution

Let

$$l = x^2 - y^2,\ m = y^2 - z^2,\ n = z^2 - x^2$$

$$\frac{\partial l}{\partial x} = 2x, \quad \frac{\partial m}{\partial x} = 0, \quad \frac{\partial n}{\partial x} = -2x$$

$$\frac{\partial l}{\partial y} = -2y, \quad \frac{\partial m}{\partial y} = 2y, \quad \frac{\partial n}{\partial y} = 0$$

$$\frac{\partial l}{\partial z} = 0, \quad \frac{\partial m}{\partial z} = -2z, \quad \frac{\partial n}{\partial z} = 2z$$

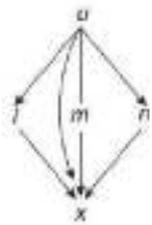


Fig. 8.55

$$u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2) = f(l, m, n)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x}$$

$$= \frac{\partial u}{\partial l}\cdot 2x + \frac{\partial u}{\partial m}\cdot 0 + \frac{\partial u}{\partial n}\cdot(-2x)$$

$$\frac{1}{x}\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial l} - 2\frac{\partial u}{\partial n} \qquad \text{...(1)}$$

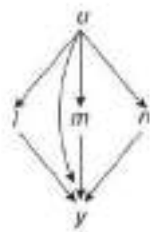


Fig. 8.56

Also,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}$$

$$= \frac{\partial u}{\partial l}(-2y) + \frac{\partial u}{\partial m}(2y) + \frac{\partial u}{\partial n}(0)$$

$$\frac{1}{y}\frac{\partial u}{\partial y} = -2\frac{\partial u}{\partial l} + 2\frac{\partial u}{\partial m} \qquad \text{...(2)}$$

and

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$

$$= \frac{\partial u}{\partial l}\cdot 0 + \frac{\partial u}{\partial m}(-2z) + \frac{\partial u}{\partial n}(2z)$$

$$\frac{1}{z}\frac{\partial u}{\partial z} = -2\frac{\partial u}{\partial m} + 2\frac{\partial u}{\partial n} \qquad \text{...(3)}$$

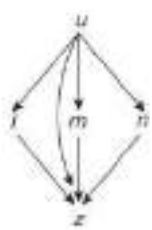


Fig. 8.57

Adding Eqs (1), (2) and (3),

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$$

Example 19

If $u = f(x^2 + 2yz,\ y^2 + 2zx)$, *prove that*

$$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0.$$

Solution

Let $\quad x^2 + 2yz = l, \quad y^2 + 2zx = m$

$$\frac{\partial l}{\partial x} = 2x, \quad \frac{\partial l}{\partial y} = 2z, \quad \frac{\partial l}{\partial z} = 2y$$

$$\frac{\partial m}{\partial x} = 2z, \quad \frac{\partial m}{\partial y} = 2y, \quad \frac{\partial m}{\partial z} = 2x$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x}$$

$$= \frac{\partial u}{\partial l}\cdot 2x + \frac{\partial u}{\partial m}\cdot 2z$$

$$(y^2 - zx)\frac{\partial u}{\partial x} = (2xy^2 - 2x^2 z)\frac{\partial u}{\partial l} + (2y^2 z - 2z^2 x)\frac{\partial u}{\partial m} \quad \text{... (1)}$$

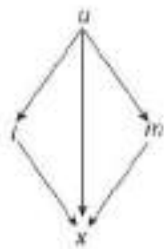


Fig. 8.58

Also,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y}$$

$$= \frac{\partial u}{\partial l}\cdot 2z + \frac{\partial u}{\partial m}\cdot 2y$$

$$(x^2 - yz)\frac{\partial u}{\partial y} = (2x^2 z - 2yz^2)\frac{\partial u}{\partial l} + (2x^2 y - 2y^2 z)\frac{\partial u}{\partial m} \quad \text{... (2)}$$

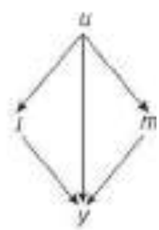


Fig. 8.59

and

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z}$$

$$= \frac{\partial u}{\partial l}\cdot 2y + \frac{\partial u}{\partial m}\cdot 2x$$

$$(z^2 - xy)\frac{\partial u}{\partial z} = (2yz^2 - 2xy^2)\frac{\partial u}{\partial l} + (2z^2 x - 2x^2 y)\frac{\partial u}{\partial m} \quad \text{... (3)}$$

Adding Eqs (1), (2) and (3),

Hence, $\quad (y^2 - zx)\dfrac{\partial u}{\partial x} + (x^2 - yz)\dfrac{\partial u}{\partial y} + (z^2 - xy)\dfrac{\partial u}{\partial z} = 0$

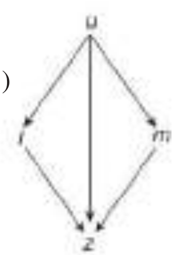


Fig. 8.60

Example 20

If $u = f(e^{y-z}, e^{z-x}, e^{x-y})$, *show that* $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0.$

Solution

Let $\qquad l = e^{y-z}, \qquad m = e^{z-x}, \qquad n = e^{x-y}$

$$\frac{\partial l}{\partial x} = 0, \qquad \frac{\partial m}{\partial x} = -e^{z-x} = -m, \qquad \frac{\partial n}{\partial x} = e^{x-y} = n$$

$$\frac{\partial l}{\partial y} = e^{y-z} = l, \qquad \frac{\partial m}{\partial y} = 0, \qquad \frac{\partial n}{\partial y} = -e^{x-y} = -n$$

$$\frac{\partial l}{\partial z} = -e^{y-z} = -l, \qquad \frac{\partial m}{\partial z} = e^{z-x} = m, \qquad \frac{\partial n}{\partial z} = 0$$

$$u = f(e^{y-z}, e^{z-x}, e^{x-y}) = f(l, m, n).$$

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x}\\ &= \frac{\partial u}{\partial l}\cdot 0 + \frac{\partial u}{\partial m}\cdot(-m) + \frac{\partial u}{\partial n}\cdot n\\ &= -m\frac{\partial u}{\partial m} + n\frac{\partial u}{\partial n} \qquad \text{...(1)}\end{aligned}$$

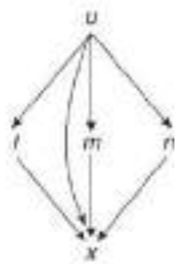


Fig. 8.61

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}\\ &= \frac{\partial u}{\partial l}\cdot l + \frac{\partial u}{\partial m}\cdot 0 + \frac{\partial u}{\partial n}\cdot(-n)\\ &= l\frac{\partial u}{\partial l} - n\frac{\partial u}{\partial n} \qquad \text{...(2)}\end{aligned}$$

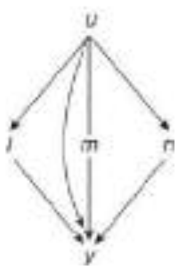


Fig. 8.62

and
$$\begin{aligned}\frac{\partial u}{\partial z} &= \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z}+\frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z}+\frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}\\ &= \frac{\partial u}{\partial l}(-l)+\frac{\partial u}{\partial m}\cdot m+\frac{\partial u}{\partial n}\cdot 0\\ &= -l\frac{\partial u}{\partial l}+m\frac{\partial u}{\partial m} \qquad \text{...(3)}\end{aligned}$$

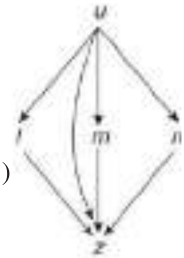


Fig. 8.63

Adding Eqs (1), (2) and (3),

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

Example 21

If $x=\sqrt{vw},\ y=\sqrt{wu},\ z=\sqrt{uv}$ *and* ϕ *is a function of x, y and z, prove that* $x\frac{\partial \phi}{\partial x}+y\frac{\partial \phi}{\partial y}+z\frac{\partial \phi}{\partial z}=u\frac{\partial \phi}{\partial u}+v\frac{\partial \phi}{\partial v}+w\frac{\partial \phi}{\partial w}$.

Solution

$$x=\sqrt{vw}, \qquad y=\sqrt{wu}, \qquad z=\sqrt{uv}$$

$$\frac{\partial x}{\partial u}=0, \qquad \frac{\partial y}{\partial u}=\frac{1}{2}\sqrt{\frac{w}{u}}, \qquad \frac{\partial z}{\partial u}=\frac{1}{2}\sqrt{\frac{v}{u}}$$

$$\frac{\partial x}{\partial v}=\frac{1}{2}\sqrt{\frac{w}{v}}, \qquad \frac{\partial y}{\partial v}=0, \qquad \frac{\partial z}{\partial v}=\frac{1}{2}\sqrt{\frac{u}{v}}$$

$$\frac{\partial x}{\partial w}=\frac{1}{2}\sqrt{\frac{v}{w}}, \qquad \frac{\partial y}{\partial w}=\frac{1}{2}\sqrt{\frac{u}{w}}, \qquad \frac{\partial z}{\partial w}=0$$

$$\begin{aligned}\frac{\partial \phi}{\partial u} &= \frac{\partial \phi}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial \phi}{\partial y}\cdot\frac{\partial y}{\partial u}+\frac{\partial \phi}{\partial z}\cdot\frac{\partial z}{\partial u}\\ &= \frac{\partial \phi}{\partial x}\cdot 0+\frac{\partial \phi}{\partial y}\cdot\frac{1}{2}\sqrt{\frac{w}{u}}+\frac{\partial \phi}{\partial z}\cdot\frac{1}{2}\sqrt{\frac{v}{u}}\\ u\frac{\partial \phi}{\partial u} &= \frac{1}{2}\left[\frac{\partial \phi}{\partial y}\sqrt{uw}+\frac{\partial \phi}{\partial z}\sqrt{uv}\right]\\ &= \frac{1}{2}\left(y\frac{\partial \phi}{\partial y}+z\frac{\partial \phi}{\partial z}\right) \qquad \text{...(1)}\end{aligned}$$

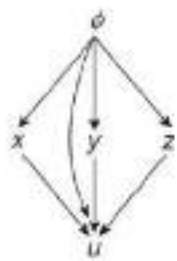


Fig. 8.64

Also,
$$\frac{\partial \phi}{\partial v}=\frac{\partial \phi}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial \phi}{\partial y}\cdot\frac{\partial y}{\partial v}+\frac{\partial \phi}{\partial z}\cdot\frac{\partial z}{\partial v}$$

$$= \frac{\partial \phi}{\partial x} \cdot \frac{1}{2}\sqrt{\frac{w}{v}} + \frac{\partial \phi}{\partial y} \cdot 0 + \frac{\partial \phi}{\partial z} \frac{1}{2}\sqrt{\frac{u}{v}}$$

$$v\frac{\partial \phi}{\partial v} = \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\sqrt{vw} + \frac{\partial \phi}{\partial z}\sqrt{uv}\right)$$

$$= \frac{1}{2}\left(x\frac{\partial \phi}{\partial x} + z\frac{\partial \phi}{\partial z}\right) \qquad \text{...(2)}$$

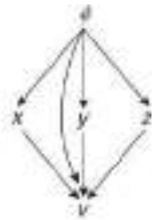


Fig. 8.65

and

$$\frac{\partial \phi}{\partial w} = \frac{\partial \phi}{\partial x} \cdot \frac{\partial x}{\partial w} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial y}{\partial w} + \frac{\partial \phi}{\partial z} \cdot \frac{\partial z}{\partial w}$$

$$= \frac{\partial \phi}{\partial x} \cdot \frac{1}{2}\sqrt{\frac{v}{w}} + \frac{\partial \phi}{\partial y} \frac{1}{2}\sqrt{\frac{u}{w}} + \frac{\partial \phi}{\partial z} \cdot 0$$

$$w\frac{\partial \phi}{\partial w} = \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\sqrt{vw} + \frac{\partial \phi}{\partial y}\sqrt{uw}\right)$$

$$= \frac{1}{2}\left(x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y}\right) \qquad \text{...(3)}$$

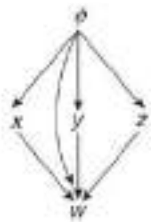


Fig. 8.66

Adding Eqs (1), (2) and (3),

$$u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w} = x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z}$$

Example 22

If $f(xy^2, z-2x) = 0$, *show that* $2x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 4x$.

Solution

Let

$$l = xy^2, \; m = z - 2x,$$

$$f(l, m) = 0$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial l} \cdot \frac{\partial l}{\partial x} + \frac{\partial f}{\partial m} \cdot \frac{\partial m}{\partial x}$$

$$0 = \frac{\partial f}{\partial l}(y^2) + \frac{\partial f}{\partial m}\left(\frac{\partial z}{\partial x} - 2\right) \qquad \left[\because f(xy^2, z-2x) = 0\right]$$

$$\frac{\frac{\partial f}{\partial l}}{\frac{\partial f}{\partial m}} = \frac{2 - \frac{\partial z}{\partial x}}{y^2} \qquad \text{...(1)}$$

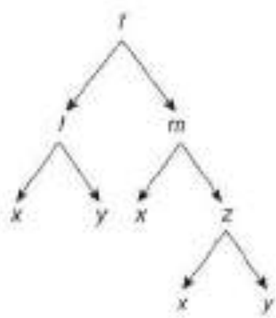


Fig. 8.67

and $$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial l}\cdot\frac{\partial l}{\partial y}+\frac{\partial f}{\partial m}\cdot\frac{\partial m}{\partial y}=0$$

$$\frac{\partial f}{\partial l}(2xy)+\frac{\partial f}{\partial m}\left(\frac{\partial z}{\partial y}\right)=0$$

$$\frac{\frac{\partial f}{\partial l}}{\frac{\partial f}{\partial m}}=-\frac{\frac{\partial z}{\partial y}}{2xy} \qquad \text{...(2)}$$

From Eqs (1) and (2),

$$\frac{2-\frac{\partial z}{\partial x}}{y^2}=-\frac{\frac{\partial z}{\partial y}}{2xy}$$

$$4x-2x\frac{\partial z}{\partial x}=-y\frac{\partial z}{\partial y}$$

Hence, $$2x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=4x$$

Example 23

If $f\left(\frac{z}{x^3},\frac{y}{x}\right)=0$, *prove that* $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=3z.$

Solution

Let $$l=\left(\frac{z}{x^3}\right),\ m=\frac{y}{x}$$

$$f(l,m)=0.$$

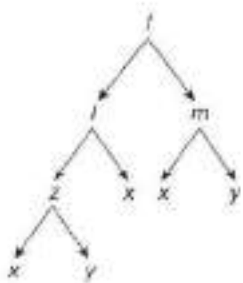


Fig. 8.68

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial l}\cdot\frac{\partial l}{\partial x}+\frac{\partial f}{\partial m}\cdot\frac{\partial m}{\partial x}$$

$$0=\frac{\partial f}{\partial l}\left(\frac{-3z}{x^4}+\frac{1}{x^3}\frac{\partial z}{\partial x}\right)+\frac{\partial f}{\partial m}\left(-\frac{y}{x^2}\right) \qquad \left[\because\ f\left(\frac{z}{x^3},\frac{y}{x}\right)=0\right]$$

$$\frac{\partial f}{\partial l}\left(-\frac{3z}{x^4}+\frac{1}{x^3}\frac{\partial z}{\partial x}\right)=\frac{\partial f}{\partial m}\left(\frac{y}{x^2}\right)$$

$$\frac{\partial f}{\partial l}\left(-3z+x\frac{\partial z}{\partial x}\right)=\frac{\partial f}{\partial m}(x^2y) \qquad \text{...(1)}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial l} \cdot \frac{\partial l}{\partial y} + \frac{\partial f}{\partial m} \cdot \frac{\partial m}{\partial y}$$

and
$$0 = \frac{\partial f}{\partial l}\left(\frac{1}{x^3}\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial m}\left(\frac{1}{x}\right)$$

$$\frac{\partial f}{\partial l}\left(\frac{1}{x^3}\frac{\partial z}{\partial y}\right) = -\frac{\partial f}{\partial m}\left(\frac{1}{x}\right)$$

$$\frac{\partial f}{\partial l}\left(\frac{\partial z}{\partial y}\right) = -\frac{\partial f}{\partial m}(x^2) \qquad \text{...(2)}$$

Dividing Eq. (1) by Eq. (2),

$$\frac{-3z + x\dfrac{\partial z}{\partial x}}{\dfrac{\partial z}{\partial y}} = \frac{x^2 y}{-x^2}$$

$$-3z + x\frac{\partial z}{\partial x} = -y\frac{\partial z}{\partial y}$$

Hence,
$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$$

Example 24

If $f(lx + my + nz,\ x^2 + y^2 + z^2) = 0$, *prove that*

$$(ly - mx) + (ny - mz)\frac{\partial z}{\partial x} + (lz - nx)\frac{\partial z}{\partial y} = 0.$$

Solution

Let
$$u = lx + my + nz, \quad v = x^2 + y^2 + z^2$$

$$f(u, v) = 0$$

Fig. 8.69

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$0 = \frac{\partial f}{\partial u}\left(l + n\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(2x + 2z\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial f}{\partial u}\left(l + n\frac{\partial z}{\partial x}\right) = -\frac{\partial f}{\partial v}\left(2x + 2z\frac{\partial z}{\partial x}\right) \qquad \text{...(1)}$$

and
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$0 = \frac{\partial f}{\partial u}\left(m + n\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(2y + 2z\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial f}{\partial u}\left(m+n\frac{\partial z}{\partial y}\right)=-\frac{\partial f}{\partial v}\left(2y+2z\frac{\partial z}{\partial y}\right) \qquad \ldots(2)$$

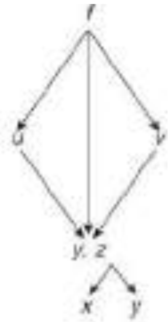


Fig. 8.70

Dividing Eq. (1) by Eq. (2),

$$\frac{l+n\frac{\partial z}{\partial x}}{m+n\frac{\partial z}{\partial y}}=\frac{x+z\frac{\partial z}{\partial x}}{y+z\frac{\partial z}{\partial y}}$$

$$ly+lz\frac{\partial z}{\partial y}+ny\frac{\partial z}{\partial x}+nz\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}=mx+nx\frac{\partial z}{\partial y}+mz\frac{\partial z}{\partial x}+nz\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}$$

Hence, $(ly-mx)+(ny-mz)\frac{\partial z}{\partial x}+(lz-nx)\frac{\partial z}{\partial y}=0$

Example 25

If u is a function of x and y and x and y are functions of r and θ given by $x = e^r\cos\theta$, $y = e^r\sin\theta$, *show that*

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=e^{-2r}\left[\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial u}{\partial \theta}\right)^2\right]$$

Solution

$$u=f(x,y),\ x=e^r\cos\theta,\ y=e^r\sin\theta$$

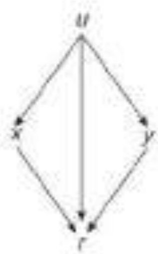


Fig. 8.71

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$=\frac{\partial u}{\partial x}e^r\cos\theta+\frac{\partial u}{\partial y}e^r\sin\theta$$

$$=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} \qquad \ldots(1)$$

Fig. 8.72

Again,

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$=\frac{\partial u}{\partial x}(-e^r\sin\theta)+\frac{\partial u}{\partial y}e^r\cos\theta$$

$$=-y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y} \qquad \ldots(2)$$

Squaring and adding Eqs (1) and (2),

$$\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial u}{\partial \theta}\right)^2 = x^2\left(\frac{\partial u}{\partial x}\right)^2+2xy\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+y^2\left(\frac{\partial u}{\partial y}\right)^2+y^2\left(\frac{\partial u}{\partial x}\right)^2$$

$$-2xy\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+x^2\left(\frac{\partial u}{\partial y}\right)^2$$

$$=(x^2+y^2)\left[\left(\frac{\partial u}{\partial x}\right)^2\right]+(x^2+y^2)\left[\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$=(x^2+y^2)\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$=e^{2r}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]=e^{-2r}\left[\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial u}{\partial \theta}\right)^2\right]$$

Example 26

If $x+y=2e^{\theta}\cos\phi$ *and* $x-y=2ie^{\theta}\sin\phi$, *prove that*

$$\frac{\partial v}{\partial \theta}+i\frac{\partial v}{\partial \phi}=2y\frac{\partial v}{\partial y}$$

Solution

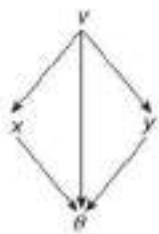


Fig. 8.73

$$x+y=2e^{\theta}\cos\phi,\quad x-y=2ie^{\theta}\sin\phi$$

$$2x=2e^{\theta}(\cos\phi+i\sin\phi)$$

$$x=e^{\theta}e^{i\phi}=e^{\theta+i\phi}$$

$$\frac{\partial x}{\partial \theta}=e^{\theta+i\phi}=x$$

$$\frac{\partial x}{\partial \phi}=ie^{\theta+i\phi}=ix$$

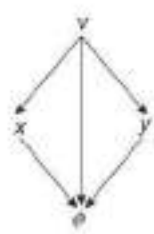


Fig. 8.74

And
$$2y=2e^{\theta}\left(\cos\theta-i\sin\phi\right)$$

$$y=e^{\theta}e^{-i\phi}=e^{\theta-i\phi}$$

$$\frac{\partial y}{\partial \theta}=e^{\theta-i\phi}=y$$

$$\frac{\partial y}{\partial \phi}=-ie^{\theta-i\phi}=-iy$$

Let $v = f(x, y)$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial v}{\partial x}x + \frac{\partial v}{\partial y}y \quad \ldots(1)$$

$$\frac{\partial v}{\partial \phi} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \phi} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \phi}$$

$$= \frac{\partial v}{\partial x}(ix) + \frac{\partial v}{\partial y}(-iy)$$

$$= i\left(x\frac{\partial v}{\partial x} - y\frac{\partial v}{\partial y}\right) \quad \ldots(2)$$

$$\frac{\partial v}{\partial \theta} + i\frac{\partial v}{\partial \phi} = x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + i^2\left(x\frac{\partial v}{\partial x} - y\frac{\partial v}{\partial y}\right)$$

$$= x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} - x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}$$

$$= 2y\frac{\partial v}{\partial y}$$

Example 27

If $z = f(x, y)$ *where* $x = \log u$, $y = \log v$, *show that* $\frac{\partial^2 z}{\partial x\,\partial y} = uv\frac{\partial^2 z}{\partial u\,\partial v}$.

Solution

$$z = f(x, y), \quad x = \log u, \quad y = \log v$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}\cdot\frac{1}{u} + \frac{\partial z}{\partial y}\cdot 0$$

$$= \frac{1}{u}\frac{\partial z}{\partial x}$$

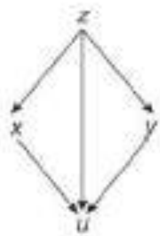


Fig. 8.75

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}\cdot 0 + \frac{\partial z}{\partial y}\frac{1}{v}$$

$$= \frac{1}{v}\frac{\partial z}{\partial y}$$

$$\frac{\partial}{\partial v} = \frac{1}{v}\frac{\partial}{\partial y}$$

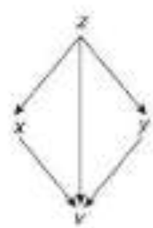


Fig. 8.76

$$\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)=\left(\frac{1}{v}\frac{\partial}{\partial y}\right)\left(\frac{1}{u}\frac{\partial z}{\partial x}\right)$$

Now,

$$\frac{\partial^2 z}{\partial v \partial u}=\frac{1}{uv}\frac{\partial^2 z}{\partial y \partial x}$$

$$\frac{\partial^2 z}{\partial x \partial y}=uv\frac{\partial^2 z}{\partial u \partial v}$$

EXERCISE 8.3

1. If $z=\tan^{-1}\left(\frac{x}{y}\right)$, where $x = 2t$, $y = 1 - t^2$, prove that $\frac{dz}{dt}=\frac{2}{1+t^2}$.

2. If $u = x^3 + y^3$, where $x = a\cos t$, $y = b\sin t$, find $\frac{du}{dt}$.

[**Ans.:** $-3a^3\cos^2 t\sin t+3b^3\sin^2 t\cos t$]

3. If $u = xe^y z$, where $y=\sqrt{a^2-x^2}$, $z = \sin^3 x$, find $\frac{du}{dx}$.

$$\left[\textbf{Ans.:}\ e^y z\left(1-\frac{x^2}{y}+3x\cot x\right)\right]$$

4. If $u=e^{\frac{t-x}{l}}$, where $r^2 = x^2 + y^2$ and l is a constant, show that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{2}{l}\cdot\frac{\partial u}{\partial x}=\frac{u}{lr}.$$

5. If $u = \log r$ and $r=\sqrt{(x-a)^2+(y-b)^2}$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ if a, b are constants.

6. If $u^2(x^2 + y^2 + z^2) = 1$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0.$

$$\left[\textbf{Hint:} \text{Let } x^2+y^2+z^2=r^2, u=\frac{1}{r}\right]$$

7. If $u = r^m$, where $r=\sqrt{x^2+y^2+z^2}$, find the value of $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$.

[**Ans.:** $m(m+1)r^{m-2}$]

8. If $u = f(r)$, where is given by the relation $x = r\cos\theta$, $y = r\sin\theta$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{d^2 f}{dr^2}+\frac{1}{r}\frac{df}{dr}.$

9. If $z = f(u,v)$, where $u = x^2 + y^2$, $y = 2$ then show that

$$x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=2\sqrt{u^2+v^2}\left(\frac{\partial z}{\partial u}\right).$$

10. If $z = f(u,v)$, where $u = x^2 + y^2$, $v = x^2 - y^2$ then show that

(i) $y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = 4xy\dfrac{\partial z}{\partial u}$.

(ii) $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 = 4u\left[\left(\dfrac{\partial z}{\partial u}\right)^2 + \left(\dfrac{\partial z}{\partial v}\right)^2\right] + 8v\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial z}{\partial v}$

11. If $w = z\sin^{-1}\left(\dfrac{y}{x}\right)$, where $x = 3u^2 + 2v$, $y = 4u - 2v^3$, $z = 2u^2 - 3v^2$, find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$.

12. If $w = (x^2 + y - 2)^4 + (x - y + 2)^3$, where $x = u - 2v + 1$ and $y = 2u + v - 2$, find $\dfrac{\partial w}{\partial v}$ at $u = 0$, $v = 0$.

[Ans.: -882**]**

13. If $w = x + 2y + z^2$, $x = \dfrac{u}{v}$, $y = u^2 + e^v$, $z = 2u$, show that

$u\dfrac{\partial w}{\partial u} + v\dfrac{\partial w}{\partial v} = 12u^2 + 2ve^v$.

14. If F is a function of x, y, z then show that

$u\dfrac{\partial F}{\partial u} + v\dfrac{\partial F}{\partial v} + w\dfrac{\partial F}{dw} = x\dfrac{\partial F}{\partial x} + 2y\dfrac{\partial F}{\partial y} + 3z\dfrac{\partial F}{dz}$, where $x = u + v + w$, $y = uv + vw + wu$, $z = uvw$.

15. If $z = f(x,y)$, $x = uv$, $y = \dfrac{u}{v}$, prove that

$\dfrac{\partial z}{\partial x} = \dfrac{1}{2v}\dfrac{\partial z}{\partial u} + \dfrac{1}{2u}\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial y} = \dfrac{v}{2}\dfrac{\partial z}{\partial u} - \dfrac{v^2}{2u}\dfrac{\partial z}{\partial v}$.

16. If $x = u + v$, $y = uv$ and F is a function of x, y, prove that

$\dfrac{\partial^2 F}{\partial u^2} - 2\dfrac{\partial^2 F}{\partial u\,\partial v} + \dfrac{\partial^2 F}{\partial v^2} = (x^2 - 4y)\dfrac{\partial^2 y}{\partial y^2} - 2\dfrac{\partial v}{\partial y}$.

$\left[\textbf{Hint: LHS} = \left(\dfrac{\partial}{\partial u} - \dfrac{\partial}{\partial v}\right)\left(\dfrac{\partial F}{\partial u} - \dfrac{\partial F}{\partial v}\right)\right]$

17. If $u = f(x^n - y^n, y^n - z^n, z^n - x^n)$, prove that

$\dfrac{1}{x^{n-1}}\dfrac{\partial u}{\partial x} + \dfrac{1}{y^{n-1}}\dfrac{\partial u}{\partial y} + \dfrac{1}{z^{n-1}}\dfrac{\partial u}{\partial z} = 0$.

18. If $z = f(x,y)$, where $x = u - av$, $y = u + av$, prove that

$a^2\dfrac{\partial^2 z}{\partial u^2} - \dfrac{\partial^2 z}{\partial v^2} = 4a^2\dfrac{\partial^2 z}{\partial x\,\partial y}$.

19. If $z = f(u, v)$, where $u = lx + my, v = ly - mx$, prove that
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right).$$

20. If $x = u + av$ and $y = u + bv$, transform the equation $2\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x\,\partial y} + 3\frac{\partial^2 z}{\partial y^2} = 0$ into the equation $\frac{\partial^2 z}{\partial u\,\partial v} = 0$, find the values of *a* and *b*.

$$\left[\text{Ans.}: a = 1, b = \frac{2}{3}\right]$$

21. If $z = f(x, y), y = e^x, v = e^y$, prove that $\frac{\partial^2 z}{\partial x\,\partial y} = uv\frac{\partial^2 z}{\partial u\,\partial v}$.

22. If $z = f(x, y), x = \frac{\cos u}{v}, y = \frac{\sin u}{v}$, prove that
$$v\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} = (y - x)\frac{\partial z}{\partial x} - (y + x)\frac{\partial z}{\partial y}.$$

23. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = 0.$

24. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that
$$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0.$$

25. If $u = f(ax^2 + 2hxy + by^2)$ $v = \phi(ax^2 + 2hxy + by^2)$ show that
$$\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right).$$

26. If $x + y = 2e^{\theta}\cos\phi$ and $x - y = 2ie^{\theta}\sin\phi$, prove that $\frac{\partial v}{\partial \theta} + i\frac{\partial v}{\partial \phi} = 2y\frac{\partial v}{\partial y}$.

27. Find the values of the constants and such that $u = x + y$ and $v = x + by$ transform the equation $9\frac{\partial^2 f}{\partial x^2} - 9\frac{\partial^2 f}{\partial x\,\partial y} + 2\frac{\partial^2 f}{\partial y^2} = 0$ into $\frac{\partial^2 f}{\partial u\,\partial v} = 0$, where *f* is a function of *x* and *y*.

$$\left[\text{Ans.}: a = \frac{3}{2}, b = 3\right]$$

28. If $x = y\cosh\theta$, $y = r\sinh\theta$ and $z = f(x, y)$, prove that

(i) $(x - y)\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = r\frac{\partial z}{\partial r} - \frac{\partial z}{\partial \theta}$

(ii) $(x^2 - y^2)\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) = r^2\frac{\partial^2 z}{\partial r^2} + r\frac{\partial z}{\partial r} - \frac{\partial^2 z}{\partial \theta^2}$

29. If $x = e^v \sec u, y = e^v \tan u$ and $z = f(x, y)$, prove that

$$\cos u\left(\frac{\partial^2 z}{\partial u\, \partial v} - \frac{\partial z}{\partial u}\right) = xy\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + (x^2 + y^2)\frac{\partial^2 z}{\partial x\, \partial y}.$$

30. If $f(x^2y^3, z - 3x) = 0$, prove that $3x\dfrac{\partial z}{\partial x} - 2y\dfrac{\partial z}{\partial y} = 9x.$

31. If $f(y + z, x^2 + y^2 + z^2) = 0$, prove that $(y - z)\dfrac{\partial z}{\partial x} - x\dfrac{\partial z}{\partial y} = x.$

32. If $f(cx - az, cy - bz) = 0$, prove that $a\dfrac{\partial z}{\partial x} - b\dfrac{\partial z}{\partial y} = c.$

33. If $x = \dfrac{\cos\theta}{u}, y = \dfrac{\sin\theta}{u}$, prove that $\left(\dfrac{\partial x}{\partial u}\right)_\theta \left(\dfrac{\partial u}{\partial x}\right)_y = \cos^2\theta.$

34. If $x^2 = au + bv, y^2 = au - bv$, prove that

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u.$$

35. If $u = ax + by, v = bx - ay$, prove that

(i) $\left(\dfrac{\partial y}{\partial v}\right)_x \left(\dfrac{\partial v}{\partial y}\right)_u = \dfrac{a^2 + b^2}{a^2}$ (ii) $\left(\dfrac{\partial u}{\partial x}\right)_y \left(\dfrac{\partial x}{\partial u}\right)_v = \dfrac{a^2}{a^2 + b^2}$

8.7 IMPLICIT DIFFERENTIATION

Any function of the type $f(x, y) = c$ is called an implicit function, where y is a function of x and c is a constant.

If $\quad f(x, y) = c \quad$ then $\quad \dfrac{dy}{dx} = -\dfrac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

Proof If $f(x, y)$ is a function of x and y, where y is a function of x then total differential coefficient of f w.r.t. x is given by

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

But $\quad f(x, y) = c$

$$\frac{df}{dx} = 0$$

$$\frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0$$

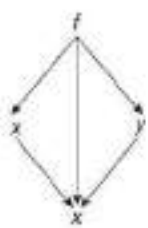


Fig. 8.77

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

Example 1

If $y\log(\cos x) = x\log(\sin y)$, find $\frac{dy}{dx}$.

Solution

Let $$f(x, y) = y\log(\cos x) - x\log(\sin y)$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{y\frac{1}{\cos x}(-\sin x) - \log(\sin y)}{\log\cos x - \frac{x}{\sin y}\cos y}$$

$$= \frac{y\tan x + \log\sin y}{\log\cos x - x\cot y}$$

Example 2

If $x^3 + y^3 = 3axy$, *find* $\frac{dy}{dx}$.

Solution

Let $$f(x, y) = x^3 + y^3 - 3axy$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{3x^2 - 3ay}{3y^2 - 3ax}$$

$$= -\frac{x^2 - ay}{y^2 - ax}$$

$$= \frac{ay - x^2}{y^2 - ax}$$

Example 3

If $y\sin x = x\cos y$, *find* $\frac{dy}{dx}$.

Solution

Let $$f(x, y) = y \sin x - x \cos y$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{y\cos x - \cos y}{\sin x + x\sin y}$$

$$= \frac{\cos y - y\cos x}{\sin x + x\sin y}$$

Example 4

If $x^y + y^x = c$, *find* $\frac{dy}{dx}$.

Solution

Let $$f(x, y) = x^y + y^x - c$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$$

Example 5

If $(\cos x)^y = (\sin y)^x$, *find* $\frac{dy}{dx}$. **[Winter 2014]**

Solution

$$(\cos x)^y = (\sin y)^x$$

Taking logarithm of both the sides,

$$y\log\cos x = x\log\sin y$$

Let $$f(x, y) = y\log\cos x - x\log\sin y$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{\frac{y}{\cos x}(-\sin x) - \log \sin y}{\log \cos x - \frac{x}{\sin y}(\cos y)}$$

$$= \frac{y \tan x + \log \sin y}{\log \cos x - x \cot y}$$

Example 6

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, *find* $\frac{dy}{dx}$.

Solution

Let $$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{2(ax + hy + g)}{2(hx + by + f)}$$

$$= -\frac{ax + hy + g}{hx + by + f}$$

Example 7

If $u = \sin(x^2 + y^2)$ *and* $a^2x^2 + b^2y^2 = c^2$, *find* $\frac{du}{dx}$.

Solution

$$u = \sin(x^2 + y^2) \text{ and } a^2x^2 + b^2y^2 = c^2$$

$$\frac{\partial u}{\partial x} = \cos(x^2 + y^2) \cdot 2x$$

$$\frac{\partial u}{\partial y} = \cos(x^2 + y^2) \cdot 2y$$

Let $$f(x, y) = a^2x^2 + b^2y^2 - c^2$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{2a^2x}{2b^2y}$$

$$= -\frac{a^2x}{b^2y}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= 2x\cos(x^2+y^2) + 2y\cos(x^2+y^2)\left(-\frac{a^2x}{b^2y}\right)$$

$$= 2x\cos(x^2+y^2)\cdot\left(1-\frac{a^2}{b^2}\right)$$

Example 8

If $u = x\log(xy)$ *where* $x^3+y^3+3xy=1$, *find* $\frac{du}{dx}$.

Solution

$$u = x\log(xy)$$

$$= x(\log x + \log y)$$

$$\frac{\partial u}{\partial x} = x\cdot\frac{1}{x} + (\log x + \log y)$$

$$= 1 + \log x + \log y$$

$$\frac{\partial u}{\partial y} = x\frac{1}{y} = \frac{x}{y}$$

Let $$f(x, y) = x^3 + y^3 + 3xy - 1$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{3x^2+3y}{3y^2+3x}$$

$$= -\frac{x^2+y}{y^2+x}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= 1 + \log x + \log y + \frac{x}{y}\left(-\frac{x^2+y}{y^2+x}\right)$$

$$= 1 + \log(xy) - \frac{x}{y}\left(\frac{x^2+y}{y^2+x}\right)$$

Example 9

If $u = \tan^{-1}\left(\frac{x}{y}\right)$ *where* $x^2 + y^2 = a^2$, *find* $\frac{du}{dx}$.

Solution

$$u = \tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y} = \frac{y}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right) = -\frac{x}{x^2 + y^2}$$

Let $f(x, y) = x^2 + y^2 - a^2$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{2x}{2y} = -\frac{x}{y}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$= \frac{y}{x^2 + y^2} - \frac{x}{x^2 + y^2}\left(-\frac{x}{y}\right)$$

$$= \frac{y}{x^2 + y^2} + \frac{x^2}{y(x^2 + y^2)}$$

$$= \frac{y^2 + x^2}{y(x^2 + y^2)}$$

$$= \frac{1}{y}$$

Example 10

If $u = \phi(x, y)$ *and* $f(x, y) = 0$, *prove that* $\frac{du}{dx} = \frac{\phi_x f_y - \phi_y f_x}{f_y}$.

Solution

$$f(x, y) = 0$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}$$

$$u = \phi(x, y)$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= \phi_x + \phi_y \frac{dy}{dx}$$

$$= \phi_x + \phi_y\left(-\frac{f_x}{f_y}\right)$$

$$= \frac{\phi_x f_y - \phi_y f_x}{f_y}$$

Example 11

If $f(x, y) = 0$, $\phi(x, z) = 0$, *show that* $\frac{\partial \phi}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot\frac{dy}{dz} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial z}$.

Solution

$$f(x, y) = 0 \text{ and } \phi(x, z) = 0$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \text{ and } \frac{dz}{dx} = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial z}}$$

$$\frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{\left(-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right)}{\left(-\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial z}}\right)}$$

$$\frac{dy}{dz} = \frac{\left(\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right)}{\left(\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial z}}\right)}$$

Hence, $$\frac{\partial\phi}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot\frac{dy}{dz}=\frac{\partial f}{\partial x}\cdot\frac{\partial\phi}{\partial z}$$

Example 12

If $\phi(x, y, z)=0$, *prove that* $\left(\frac{\partial z}{\partial y}\right)_x\left(\frac{\partial x}{\partial z}\right)_y\left(\frac{\partial y}{\partial z}\right)_z=-1.$

Solution

$$\phi(x, y, z)=0$$

$$\left(\frac{dz}{dy}\right)_x=\left(\frac{\partial z}{\partial y}\right)_x=-\frac{\frac{\partial\phi}{\partial y}}{\frac{\partial\phi}{\partial z}} \quad ...(1)$$

$$\left(\frac{dx}{dz}\right)_y=\left(\frac{\partial x}{\partial z}\right)_y=-\frac{\frac{\partial\phi}{\partial z}}{\frac{\partial\phi}{\partial x}} \quad ...(2)$$

$$\left(\frac{dy}{dx}\right)_z=\left(\frac{\partial y}{\partial x}\right)_z=-\frac{\frac{\partial\phi}{\partial x}}{\frac{\partial\phi}{\partial y}} \quad ...(3)$$

From Eqs (1), (2) and (3),

$$\left(\frac{\partial z}{\partial y}\right)_x\left(\frac{\partial x}{\partial z}\right)_y\left(\frac{\partial y}{\partial x}\right)_z=\left(-\frac{\frac{\partial\phi}{\partial y}}{\frac{\partial\phi}{\partial z}}\right)\left(-\frac{\frac{\partial\phi}{\partial z}}{\frac{\partial\phi}{\partial x}}\right)\left(-\frac{\frac{\partial\phi}{\partial x}}{\frac{\partial\phi}{\partial y}}\right)$$
$$=-1$$

EXERCISE 8.4

1. If $x^3+y^3-3axy=0$, find $\frac{dy}{dx}$.

   $\left[\text{Ans.: } \frac{ay-x^2}{y^2-ax}\right]$

2. If $x^3+3x^2+6xy^2+y^3=1$, find $\frac{dy}{dx}$.

   $\left[\text{Ans.: } -\frac{(x^2+2x+2y^2)}{(4xy+y^2)}\right]$

3. If $x^y=y^x$, prove that $\frac{dy}{dx}=\frac{y(y-x\log y)}{x(x-y\log x)}$.

4. If $f(x, y) = x \sin(x - y) - (x + y) = 0$, find $\frac{dy}{dx}$.

$$\left[\text{Ans.: } \frac{[\sin(x-y)](1+x)-1}{x\cos(x-y)+1}\right]$$

5. If $y^{x^y} = \sin x$, find $\frac{dy}{dx}$.

$$\left[\text{Hint: } f = x^y \log y - \log \sin x, \text{ let } x^y = z, \log z = y \log x \text{ find } \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \text{ and then } \frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}\right]$$

$$\left[\text{Ans.: } \frac{-(yx^{y-1}\log y - \cot x)}{x^y \log x \log y + x^y y^{-1}}\right]$$

6. If $x^5 + y^5 = 5a^3x^2$, find $\frac{d^2y}{dx^2}$.

$$\left[\text{Ans.: } \frac{6a^3x^2(a^3+x^3)}{y^9}\right]$$

7. If $xy^3 - yx^3 = 6$ is the equation of curve, find the slope of the tangent at the point (1, 2).

$$\left[\text{Hint: Find } \frac{dy}{dx} \text{ at } (1, 2)\right]$$

$$\left[\text{Ans.: } -\frac{2}{11}\right]$$

8. Find $\frac{d^2y}{dx^2}$, if $x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$.

$$\left[\text{Ans.: } \frac{a}{(1-x^2)^{\frac{3}{2}}}\right]$$

9. If $u = x \log xy$ and $x^3 + y^3 + 3xy - 1 = 0$, find $\frac{du}{dx}$.

$$\left[\text{Hint: } \frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}, \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right]$$

$$\left[\text{Ans.: } 1 + \log xy - \frac{x}{y}\left(\frac{x^2+ay}{y^2+ax}\right)\right]$$

10. If $x^m + y^m = b^m$, show that $\frac{d^2y}{dx^2} = -(m-1)b^m \frac{x^{m-2}}{y^{2m-1}}$.

11. If $u = x^2y$ and $x^2 + xy + y^2 = 1$, find $\frac{du}{dx}$.

12. If $x^3 + y^3 = 3ax^2$, find $\frac{d^2y}{dx^2}$.

$$\left[\text{Ans.: } -\frac{2a^2x^2}{y^5}\right]$$

8.8 GRADIENT AND DIRECTIONAL DERIVATIVE

8.8.1 Gradient

The gradient of a function $z = f(x, y)$ is written as grad f and is defined as

$$\text{grad } f = \nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

grad f is a vector function.

If $f(x, y, z)$ is a function of three independent variables, its total differential df is given as

$$\begin{aligned}
\mathrm{d}f &= \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z \\
&= \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)\cdot(\hat{i}\mathrm{d}x + \hat{j}\mathrm{d}y + \hat{k}\mathrm{d}z) \\
&= \nabla f \cdot \mathrm{d}\overline{r} \qquad \left[\begin{array}{l} \because \; \overline{r} = x\hat{i} + y\hat{j} + z\hat{k} \\ \quad \mathrm{d}\overline{r} = \hat{i}\mathrm{d}x + \hat{j}\mathrm{d}y + \hat{k}\mathrm{d}z \end{array}\right] \\
&= |\nabla f||\mathrm{d}\overline{r}|\cos\theta
\end{aligned}$$

where θ is the angle between the vectors ∇f and $\mathrm{d}\overline{r}$. If $\mathrm{d}\overline{r}$ and $\nabla\phi$ are in the same direction, then $\theta = 0$.

$$\mathrm{d}f = |\nabla f||\mathrm{d}\overline{r}|$$

$\cos\theta = 1$ is the maximum value of $\cos\theta$. Hence, df is maximum at $\theta = 0$.

8.8.2 Directional Derivative

The rate of change of a function of several variables in the direction of the coordinate axes is called as directional derivative.

For a function $z = f(x, y, z)$, $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are the directional derivative of f in the direction of coordinate axes.

The directional derivative of a function $f(x, y, z)$ in the direction of vector $\overline{a}$ is the component of ∇f in the direction of $\overline{a}$. If $\hat{a}$ is the unit vector in the direction of $\overline{a}$, the directional derivative of f in the direction of $\overline{a}$ is given by

$$D_a f = \nabla f \cdot \hat{a} = \nabla f \cdot \frac{a}{|\overline{a}|}$$

Example 1

Find ∇f *at* $(1, -2, 1)$ *if* $f = 3x^2y - y^3z^2$.

Solution

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}(6xy - 0) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(0 - 2y^3z)$$

At $x = 1$, $y = -2$, $z = 1$,

$$\nabla f = \hat{i}(-12) + \hat{j}(3 - 12) + \hat{k}(16)$$

$$\nabla f \text{ at } (1, -2, 1) = -12\hat{i} - 9\hat{j} + 16\hat{k}$$

Example 2

Find the directional derivatives of $f = xy^2 + yz^2$ *at the point* (2, –1, 1) *in the direction of the vector* $\hat{i} + 2\hat{j} + 2\hat{k}$.

Solution

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}y^2 + \hat{j}(2xy + z^2) + \hat{k}(2yz)$$

At the point (2, –1, 1),

$$\nabla f = \hat{i} + \hat{j}(-4 + 1) + \hat{k}(-2)$$

$$= \hat{i} - 3\hat{j} - 2\hat{k}$$

Directional derivative in the direction of the vector $\bar{a} = \hat{i} + 2\hat{j} + 2\hat{k}$ is

$$D_a f = \nabla f \cdot \frac{\bar{a}}{|\bar{a}|}$$

$$= (\hat{i} - 3\hat{j} - 2\hat{k}) \cdot \frac{(\hat{i} + 2\hat{j} + 2\hat{k})}{\sqrt{1 + 4 + 4}}$$

$$= \frac{(1 - 6 - 4)}{3}$$

$$= -3$$

Example 3

Find the directional derivatives of $f = \dfrac{1}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}$ *at the point* $P(1, -1, 1)$ *in the direction of* $\bar{a} = \hat{i} + \hat{j} + \hat{k}$.

Solution

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}\left[-\frac{2x}{2(x^2+y^2+z^2)^{\frac{3}{2}}}\right] + \hat{j}\left[-\frac{2y}{2(x^2+y^2+z^2)^{\frac{3}{2}}}\right] + \hat{k}\left[-\frac{2z}{2(x^2+y^2+z^2)^{\frac{3}{2}}}\right]$$

$$= -\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

At the point (1, –1, 1),

$$\Delta f = -\frac{(\hat{i}-\hat{j}+\hat{k})}{(3)^{\frac{3}{2}}}$$

Directional derivative in the direction of $\overline{a} = \hat{i}+\hat{j}+\hat{k}$ is

$$\begin{aligned} D_a f &= \nabla f \cdot \frac{\overline{a}}{|\overline{a}|} \\ &= -\frac{(\hat{i}-\hat{j}-\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})}{(3)^{\frac{3}{2}}\sqrt{1+1+1}} \\ &= \frac{-1+1+1}{3^2} \\ &= -\frac{1}{9} \end{aligned}$$

Example 4

Evaluate ∇e^{r^2}, where $r^2 = x^2 + y^2 + z^2$.

Solution

$$r^2 = x^2 + y^2 + z^2$$

Differentiating partially w.r.t. x, y, and z respectively,

$$2r\frac{\partial r}{\partial x} = 2x, \qquad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r\frac{\partial r}{\partial y} = 2y, \qquad \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r\frac{\partial r}{\partial z} = 2z, \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\begin{aligned} \nabla e^{r^2} &= \hat{i}\frac{\partial e^{r^2}}{\partial x} + \hat{j}\frac{\partial e^{r^2}}{\partial y} + \hat{k}\frac{\partial e^{r^2}}{\partial z} \\ &= \hat{i}\frac{\partial e^{r^2}}{\partial r}\cdot\frac{\partial r}{\partial x} + \hat{j}\frac{\partial e^{r^2}}{\partial r}\cdot\frac{\partial r}{\partial y} + \hat{k}\frac{\partial e^{r^2}}{\partial r}\cdot\frac{\partial r}{\partial z} \end{aligned}$$

$$= \hat{i}\left(e^{r^2}\cdot 2r\right)\frac{x}{r} + \hat{j}\left(e^{r^2}\cdot 2r\right)\frac{y}{r} + \hat{k}\left(e^{r^2}\cdot 2r\right)\frac{z}{r}$$

$$= 2e^{r^2}(x\hat{i} + y\hat{j} + z\hat{k})$$

Example 5

Prove that $\nabla r^n = nr^{n-2}\bar{r}, \bar{r} = x\hat{i} + y\hat{j} + z\hat{k}, r = |\bar{r}|$.

Solution

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}, \quad r^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla r^n = \hat{i}\frac{\partial r^n}{\partial x} + \hat{j}\frac{\partial r^n}{\partial y} + \hat{k}\frac{\partial r^n}{\partial z}$$

$$= \hat{i}\frac{\partial r^n}{\partial r}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r^n}{\partial r}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r^n}{\partial r}\frac{\partial r}{\partial z}$$

$$= \hat{i}nr^{n-1}\frac{x}{r} + \hat{j}nr^{n-1}\frac{y}{r} + \hat{k}nr^{n-1}\frac{z}{r}$$

$$= nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= nr^{n-2}\bar{r}$$

EXERCISE 8.5

1. Find ∇f if
   (i) $f = \log(x^2 + y^2 + z^2)$
   (ii) $f = (x^2 + y^2 + z^2)e^{-\sqrt{x^2+y^2+z^2}}$

   $\left[\text{Ans.: (i) } \frac{2\bar{r}}{r^2} \quad \text{(ii) } (2-r)e^{-r}\bar{r} \quad \text{where } \bar{r} = x\hat{i} + y\hat{j} + z\hat{k}, r = |\bar{r}|\right]$

2. Find ∇f and $|\nabla f|$ if
   (i) $f = 2xz^4 - x^2y$ at $(2, -2, -1)$
   (ii) $f = 2xz^2 - 3xy - 4x$ at $(1, -1, 2)$

   $\left[\text{Ans.: (i) } 10\hat{i} - 4\hat{j} - 16\hat{k}, \ 2\sqrt{93} \quad \text{(ii) } 7\hat{i} - 3\hat{j} + 8\hat{k}, \ 2\sqrt{29}\right]$

3. Find the directional derivative of $f = xy + yz + zx$ at (1, 2, 0) in the direction of vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

   $\left[\text{Ans.: } \frac{10}{3}\right]$

4. Find the directional derivative of $f = x^2yz + 4xz^2$ at (1, –2, –1) in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$.

$$\left[\textbf{Ans.}: \frac{37}{\sqrt{3}}\right]$$

5. Find the directional derivative of $f = x^2yz^2$ along the curve $x = e^{-t}$, $y = 2\sin t + 1$, $z = t - \cos t$ at $t = 0$.

$$\left[\textbf{Ans.}: -\frac{1}{\sqrt{6}}\right]$$

8.9 TANGENT PLANE AND NORMAL LINE

Let P be any point on the surface $f(x, y, z) = 0$ or $z = f(x, y)$ and let Q be any other point on it. Any curve is taken on the surface joining Q to P. As Q tends to P along this curve, the chord PQ tends, in general, to a definite straight line. This straight line is called a *tangent line* to the surface at P. Since different curves can be obtained by joining Q to P, different tangent lines are obtained at P. All these tangent lines lie in a plane called the *tangent plane* to the surface at P.

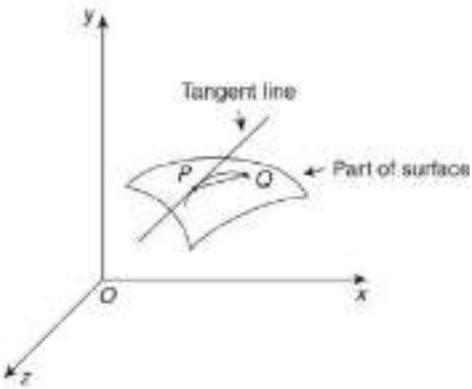


Fig. 8.78

The equation of the tangent plane at $P(x_0, y_0, z_0)$ to the surface $f(x, y, z) = 0$ is

$$(x - x_0)\, f_x(x_0, y_0, z_0) + (y - y_0)\, f_y(x_0, y_0, z_0) + (z - z_0)\, f_z(x_0, y_0, z_0) = 0$$

where

$$f_x(x_0, y_0, z_0) = \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0, z_0)}$$

$$f_y(x_0, y_0, z_0) = \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0, z_0)}$$

$$f_z(x_0, y_0, z_0) = \left.\frac{\partial f}{\partial z}\right|_{(x_0, y_0, z_0)}$$

The line through P, perpendicular to the tangent plane is called the *normal* to the surface at P. The equations of the normal line to the surface at $P(x_0, y_0, z_0)$ are

$$\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)} = \frac{z - z_0}{f_z(x_0, y_0, z_0)}$$

Example 1

Find the equations of the tangent plane and normal line to the surface $xyz = 6$ *at* (1, 2, 3). **[Winter 2014]**

Solution

Let $$f(xyz) = xyz - 6$$

$$f_x(x, y, z) = yz, \qquad f_x(1, 2, 3) = 6$$

$$f_y(x, y, z) = xz, \qquad f_y(1, 2, 3) = 3$$

$$f_z(x, y, z) = xy, \qquad f_z(1, 2, 3) = 2$$

Hence, the equation of the tangent plane at (1, 2, 3) is

$$6(x-1) + 3(y-2) + 2(z-3) = 0$$

$$6x - 6 + 3y - 6 + 2z - 6 = 0$$

$$6x + 3y + 2z = 18$$

The set of equations of the normal line is

$$\frac{x-1}{6} = \frac{y-2}{3} = \frac{z-3}{2}$$

Example 2

Find the equations of the tangent plane and normal line to the surface $x^2 + y^2 + z - 9 = 0$ *at the point* (1, 2, 4). **[Winter 2016]**

Solution

Let $$f(x, y, z) = x^2 + y^2 + z - 9$$

$$f_x(x, y, z) = 2x, \qquad f_x(1, 2, 4) = 2$$

$$f_y(x, y, z) = 2y, \qquad f_y(1, 2, 4) = 4$$

$$f_z(x, y, z) = 1, \qquad f_z(1, 2, 4) = 1$$

Hence, the equation of the tangent plane at (1, 2, 4) is

$$2(x-1) + 4(y-2) + 1(z-4) = 0$$

$$2x - 2 + 4y - 8 + z - 4 = 0$$

$$2x + 4y + z - 14 = 0$$

$$2x + 4y + z = 14$$

The set of equations of the normal line is

$$\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-4}{1}$$

Example 3

Find the equations of the tangent plane and normal line to the surface $x^2 + y^2 + z^2 = 3$ *at the point* (1, 1, 1).

Solution

Let $f(x, y, z) = x^2 + y^2 + z^2 - 3$

$$f_x(x, y, z) = 2x, \qquad f_x(1, 1, 1) = 2$$
$$f_y(x, y, z) = 2y, \qquad f_y(1, 1, 1) = 2$$
$$f_z(x, y, z) = 2z, \qquad f_z(1, 1, 1) = 2$$

Hence, the equation of the tangent plane at (1, 1, 1) is

$$2(x-1) + 2(y-1) + 2(z-1) = 0$$
$$2x - 2 + 2y - 2 + 2z - 2 = 0$$
$$2x + 2y + 2z = 6$$
$$x + y + z = 3$$

The set of equations of the normal line is

$$\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-1}{2}$$
$$x - 1 = y - 1 = z - 1$$

Example 4

Find the equations of the tangent plane and normal line to the surface $2x^2 + y + 2z = 3$ *at* (2, 1, – 3). **[Summer 2017]**

Solution

Let $f(x, y, z) = 2x^2 + y^2 + 2z - 3$

$$f_x(x, y, z) = 4x, \qquad f_x(2, 1, -3) = 8$$
$$f_y(x, y, z) = 2y, \qquad f_y(2, 1, -3) = 2$$
$$f_z(x, y, z) = 2, \qquad f_z(2, 1, -3) = 2$$

Hence, the equation of the tangent plane at (2, 1, –3) is

$$8(x-2) + 2(y-1) + 2(z+3) = 0$$
$$8x - 16 + 2y - 2 + 2z + 6 = 0$$
$$8x + 2y + 2z - 12 = 0$$
$$8x + 2y + 2z = 12$$
$$4x + y + z = 6$$

The set of equations of the normal line is

$$\frac{x-2}{8} = \frac{y-1}{2} = \frac{z+3}{2}$$
$$\frac{x-2}{4} = y - 1 = z + 3$$

Example 5

Find the equations of the tangent plane and normal line at the point (−2, 1, −3) *to the ellipsoid* $\frac{x^2}{4}+y^2+\frac{z^2}{9}=3$.

[Winter 2013; Summer 2016]

Solution

Let
$$f(x, y, z)=\frac{x^2}{4}+y^2+\frac{z^2}{9}-3$$

$$f_x(x, y, z)=\frac{x}{2}, \qquad f_x(-2, 1, -3)=-1$$
$$f_y(x, y, z)=2y, \qquad f_y(-2, 1, -3)=2$$
$$f_z(x, y, z)=\frac{2z}{9}, \qquad f_z(-2, 1, -3)=-\frac{2}{3}$$

Hence, the equation of the tangent plane at (−2, 1, −3) is

$$-1(x+2)+2(y-1)-\frac{2}{3}(z+3)=0$$
$$3(x+2)-6(y-1)+2(z+3)=0$$
$$3x+6-6y+6+2z+6=0$$
$$3x-6y+2z=-18$$

The set of equations of the normal line is

$$\frac{x+2}{-1}=\frac{y-1}{2}=\frac{z+3}{\frac{-2}{3}}$$

Example 6

Find the equations of the tangent plane and normal line to the surface $z=2x^2+y^2$ *at the point* (1, 1, 3).

Solution

Let
$$f(x, y, z)=2x^2+y^2-z$$

$$f_x(x, y, z)=4x, \qquad f_x(1, 1, 3)=4$$
$$f_y(x, y, z)=2y, \qquad f_y(1, 1, 3)=2$$
$$f_z(x, y, z)=-1, \qquad f_z(1, 1, 3)=-1$$

Hence, the equation of the tangent plane at (1, 1, 3) is

$$\begin{aligned} 4(x-1)+2(y-1)-1(z-3)&=0 \\ 4x-4+2y-2-z+3&=0 \\ 4x+2y-z&=3 \end{aligned}$$

The set of equations of the normal line is

$$\frac{x-1}{4}=\frac{y-1}{2}=\frac{z-3}{-1}$$

Example 7

Find the equations of the tangent plane and normal line to the surface $x^2yz+3y^2=2xz^2-8z$ *at the point* (1, 2, −1).

Solution

Let $$f(x,y,z)=x^2yz+3y^2-2xz^2+8z$$

$$\begin{aligned} f_x(x,y,z)&=2xyz-2z^2, & f_x(1,2,-1)&=2(1)(2)(-1)-2(-1)^2=-6 \\ f_y(x,y,z)&=x^2z+6y, & f_y(1,2,-1)&=(1)^2(-1)+6(2)=11 \\ f_z(x,y,z)&=x^2y-4xz+8, & f_z(1,2,-1)&=(1)^2(2)-4(1)(-1)+8=14 \end{aligned}$$

Hence, the equation of the tangent plane at (1, 2, −1) is

$$\begin{aligned} -6(x-1)+11(y-2)+14(z+1)&=0 \\ -6x+6+11y-22+14z+14&=0 \\ 6x-11y-14z+2&=0 \end{aligned}$$

The set of equations of the normal line is

$$\frac{x-1}{-6}=\frac{y-2}{11}=\frac{z+1}{14}$$

Example 8

Find the equations of the tangent plane and normal line to the surface $2xz^2-3xy-4x=7$ *at the point* (1, –1, 2). **[Summer 2015]**

Solution

Let $f(x, y, z) = 2xz^2 - 3xy - 4x = 7$

$$\begin{aligned} f_x(x,y,z)&=2z^2-3y-4, & f_x(1,-1,2)&=(2)^2-3(-1)-4=7 \\ f_y(x,y,z)&=-3x, & f_y(1,-1,2)&=-3(1)=-3 \\ f_z(x,y,z)&=4xz, & f_z(1,-1,2)&=4(1)(2)=8 \end{aligned}$$

Hence, the equation of the tangent plane at (1, –1, 2) is

$$7(x-1)+(-3)(y+1)+8(z-2)=0$$
$$7x-7-3y-3+8z-16=0$$
$$7x-3y+8z=26$$

The set of equations of the normal line is

$$\frac{x-1}{7}=\frac{y+1}{-3}=\frac{z-2}{8}$$

Example 9

Find the equation of the tangent plane and normal line to the surface $z+8=xe^y\cos z$ *at the point* (8, 0, 0).

Solution

Let $f(x,y,z)=xe^y\cos z-z-8$

$$f_x(x,y,z)=e^y\cos z \qquad f_x(8,0,0)=1$$
$$f_y(x,y,z)=xe^y\cos z \qquad f_y(8,0,0)=8$$
$$f_z(x,y,z)=\sin z-1 \qquad f_z(8,0,0)=-1$$

Hence, the equation of the tangent plane at (8,0,0) is

$$1(x-8)+8(y-0)-1(z-0)=0$$
$$x-8+8y-z=0$$
$$x+8y-z-8=0$$

The set of equations of the normal line is

$$\frac{x-8}{1}=\frac{y-0}{8}=\frac{z-0}{-1}$$
$$x-8=\frac{y}{8}=-z$$

Example 10

Find the equations of the tangent plane and normal line to the surface $\cos\pi x-x^2y+e^{xz}+yz=4$ *at the point* $P(0, 1, 2)$. **[Winter 2015]**

Solution

Let $f(x,y,z)=\cos\pi x-x^2y+e^{xz}+yz=4$

$$f_x(x,y,z)=-\pi\sin\pi x-2xy+ze^{xz}, \qquad f_x(0,1,2)=2$$
$$f_y(x,y,z)=-x^2+z, \qquad f_y(0,1,2)=2$$
$$f_z(x,y,z)=xe^{xz}+y, \qquad f_z(0,1,2)=1$$

Hence, the equation of the tangent plane at $P(0, 1, 2)$ is

$$\begin{aligned} 2(x-0)+2(y-1)+1(z-2)&=0 \\ 2x+2y-2+z-2&=0 \\ 2x+2y+z&=-4 \end{aligned}$$

The set of equations of the normal line is

$$\frac{x}{2}=\frac{y-1}{2}=\frac{z-2}{1}$$

Example 11

Show that the plane tangent to the surface $z=x^2+y^2$ at the point (a, b, c) intersects the z-axis at a point where z = −c.

Solution

Let $f(x, y, z)=x^2+y^2-z$

$$\begin{aligned} f_x(x, y, z)&=2x, & f_x(a, b, c)&=2a \\ f_y(x, y, z)&=2y, & f_y(a, b, c)&=2b \\ f_z(x, y, z)&=-1, & f_z(a, b, c)&=-1 \end{aligned}$$

Hence, the equation of the tangent plane at (a, b, c) is

$$\begin{aligned} 2a(x-a)+2b(y-b)-1(z-c)&=0 \\ 2ax-2a^2+2by-2b^2-z+c&=0 \\ 2ax+2by-z-2a^2-2b^2+c&=0 \qquad \ldots(1) \end{aligned}$$

The point of intersection of tangent plane with the z-axis is obtained by putting $x = 0$, $y = 0$ in Eq. (1),

$$\begin{aligned} -z-2a^2-2b^2+c&=0 \\ z&=c-2a^2-2b^2 \\ &=c-2(a^2+b^2) \qquad \ldots(2) \end{aligned}$$

Since the point (a, b, c) lies on the surface $x^2+y^2=z$, …(3)

$$a^2+b^2=c$$

Substituting Eq. (3) in Eq. (2),

$$z=c-2c=-c$$

Example 12

Show that the plane $4x-6y-z+14=0$ *touches the surface* $x^2+3y^2+2z=0$ *and find the point of contact.*

Solution

Let

$$f(x, y, z) = x^2 + 3y^2 + 2z$$
$$f_x(x, y, z) = 2x$$
$$f_y(x, y, z) = 6y$$
$$f_z(x, y, z) = 2$$

Let $P(x_0, y_0, z_0)$ be a point on the surface

$$f_x(x_0, y_0, z_0) = 2x_0$$
$$f_y(x_0, y_0, z_0) = 6y_0$$
$$f_z(x_0, y_0, z_0) = 2$$

Hence, the equation of the tangent plane at (x_0, y_0, z_0) is

$$2x_0(x-x_0)+6y_0(y-y_0)+2(z-z_0)=0$$
$$2xx_0-2x_0^2+6yy_0-6y_0^2+2z-2z_0=0$$
$$xx_0+3yy_0+z+z_0=0$$

Comparing with the plane equation $4x - 6y - z + 14 = 0$

$$\frac{x_0}{4}=\frac{3y_0}{-6}=\frac{1}{-1}=\frac{z_0}{14}$$
$$\therefore\ x_0 = -4,\ y_0 = 2,\ z_0 = -14$$

The point $(-4, 2, -14)$ satisfies the equation of the surface and the tangent plane. Hence, the point of contact is $(-4, 2, -14)$.

Example 13

Show that the surfaces $z = xy - 2$ *and* $x^2+y^2+z^2=3$ *have the same tangent plane at* $(1, 1, -1)$. **[Summer 2014]**

Solution

For the surface $f(x, y, z) = xy - z - 2$,

$$f_x(x, y, z) = y, \qquad f_x(1, 1, -1) = 1$$
$$f_y(x, y, z) = x, \qquad f_y(1, 1, -1) = 1$$
$$f_z(x, y, z) = -1, \qquad f_z(1, 1, -1) = -1$$

Hence, the equation of the tangent plane is

$$\begin{aligned} 1(x-1)+1(y-1)-1(z+1)&=0 \\ x-1+y-1-z-1&=0 \\ x+y-z-3&=0 \end{aligned}$$

For the surface $f(x, y, z)=x^2+y^2+z^2-3,$

$$\begin{aligned} f_x(x, y, z)&=2x & f_x(1, 1, -1)&=2 \\ f_y(x, y, z)&=2y & f_y(1, 1, -1)&=2 \\ f_z(x, y, z)&=2z & f_z(1, 1, -1)&=-2 \end{aligned}$$

Hence, the equation of the tangent plane is

$$\begin{aligned} 2(x-1)+2(y-1)-2(z+1)&=0 \\ 2x-2+2y-2-2z-2&=0 \\ x+y-z-3&=0 \end{aligned}$$

Hence, the surfaces $z=xy-2$ and $x^2+y^2+z^2=3$ have the same tangent plane at (1, 1, −1).

Example 14

Find equations of the normal line of the sphere $x^2+y^2+z^2=6$ at the point (a, b, c). Show that the normal line passes through the origin.

Solution

Let $f(x, y, z)=x^2+y^2+z^2-6$

$$\begin{aligned} f_x(x, y, z)&=2x, & f_x(a, b, c)&=2a \\ f_y(x, y, z)&=2y, & f_y(a, b, c)&=2b \\ f_z(x, y, z)&=2z, & f_z(a, b, c)&=2c \end{aligned}$$

Hence, the set of equations of the normal line at (a, b, c) is

$$\frac{x-a}{2a}=\frac{y-b}{2b}=\frac{z-c}{2c}$$

At the origin, $x=0$, $y=0$, $z=0$.

$$\frac{0-a}{2a}=\frac{0-b}{2b}=\frac{0-c}{2c}=-\frac{1}{2}$$

The point (0,0,0) satisfies the equation of the normal line.
Hence, the normal line passes through the origin (0, 0, 0).

EXERCISE 8.6

1. Find the equations of the tangent plane and the normal line to the following surfaces at indicated points.

(a) $z = \sqrt{4 - x^2 - 2y^2}$, $(1, -1, 1)$

(b) $z = \sqrt{1 - x^2 - y^2}$, $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

(c) $z = 1 - \frac{1}{10}\left(x^2 + 4y^2\right)$, $\left(1, 1, \frac{1}{2}\right)$

(d) $x^2y^2 + xz - 2y^3 = 10$, $(2, 1, 4)$

$$\left[\begin{array}{lll} \textbf{Ans.: } (a)\ x - 2y + z - 4 = 0 & , & \frac{x-1}{1} = \frac{y+1}{-2} = \frac{z-1}{1} \\ (b)\ 2x + 2y + z - 3 = 0 & , & \frac{3x-2}{2} = \frac{3y-2}{2} = \frac{3z-1}{1} \\ (c)\ 2x + 8y + 10z - 15 = 0 & , & \frac{x-1}{2} = \frac{y-1}{8} = \frac{z-\frac{1}{2}}{10} \\ (d)\ 4x + y + z = 13 = 0 & , & \frac{x-2}{4} = \frac{y-1}{1} = \frac{z-4}{1} \end{array}\right]$$

2. Show that the tangent plane to the surface $x^2 = y(x + z)$ at any point passes through the origin.

3. Show that the plane $3x + 12y - 6z - 17 = 0$ touches the surface $3x^2 - 6y^2 + 9z^2 + 17 = 0$. Find also the point of contact.

$$\left[\textbf{Ans.: } \left(-1, 2, \frac{2}{3}\right)\right]$$

4. Show that the plane $ax + by + cz + d = 0$ touches the surface

$$px^2 + 9y^2 + 2z = 0, \text{ if } \frac{a^2}{p} + \frac{b^2}{q} + 2cd = 0$$

8.10 LOCAL EXTREME VALUES (MAXIMUM AND MINIMUM VALUES)

Let $u = f(x, y)$ be a continuous function of x and y. u will be maximum at $x = a$, $y = b$, if $f(a, b) > f(a + h, b + k)$ and will be minimum at $x = a$, $y = b$, if $f(a, b) < f(a + h, b + k)$ for small positive or negative values of h and k.

The point at which function $f(x, y)$ is either maximum or minimum is known as *stationary point.* The value of the function at stationary point is known as extreme (maximum or minimum) value of the function $f(x, y)$.

Working Rule to determine extreme values of a function f(x, y)

Step I Solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ simultaneously for x and y.

Step II Obtain the values of $r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$.

Step III

(i) If $rt - s^2 > 0$ and $r < 0$ (or $t < 0$) at (a, b) then $f(x, y)$ is maximum at (a, b) and the maximum value of the function is $f(a, b)$.

(ii) If $rt - s^2 > 0$ and $r > 0$ (or $t > 0$) at (a, b) then $f(x, y)$ is minimum at (a, b) and the minimum value of the function is $f(a, b)$.

(iii) If $rt - s^2 < 0$ at (a, b) then $f(x, y)$ is neither maximum nor minimum at (a, b). Such a point is known as *saddle point*.

(iv) If $rt - s^2 = 0$ at (a, b) then no conclusion can be made about the extreme values of $f(x, y)$ and further investigation is required.

Example 1

Discuss the maxima and minima of the function $x^2 + y^2 + 6x + 12$.

Solution

Let $$f(x, y) = x^2 + y^2 + 6x + 12$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$
$$2x + 6 = 0$$
$$2(x + 3) = 0$$
$$x + 3 = 0$$
$$x = -3$$

and

$$\frac{\partial f}{\partial y} = 0$$
$$2y = 0$$
$$y = 0$$

Stationary point is (– 3, 0).

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 2$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2$$

Step III At (–3, 0)

$$rt - s^2 = 2(2) - 0 = 4 > 0 \quad \text{and } r > 0$$

Hence, $f(x, y)$ is minimum at (–3, 0).

$$f_{min} = (-3)^2 + 0 + 6(-3) + 12 = 3$$

Example 2

Show that the minimum value of $f(x, y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$ *is* $3a^2$.

Solution

$$f(x, y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$y - \frac{a^3}{x^2} = 0$$

$$x^2 y = a^3 \quad \ldots(1)$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$x - \frac{a^3}{y^2} = 0$$

$$xy^2 = a^3 \quad \ldots(2)$$

Solving Eqs (1) and (2),

$$x = y$$

Substituting in Eq. (1),

$$x^3 = a^3$$

$$x = a$$

$$\therefore \quad y = a$$

Stationary point is (a, a).

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{2a^3}{x^3}$$

$$s = \frac{\partial^2 f}{\partial x\,\partial y} = 1$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{2a^3}{y^3}$$

Step III At (a, a), $r = 2$, $s = 1$, $t = 2$

$$rt - s^2 = (2)(2) - (1)^2 = 3 > 0$$

Also, $r = 2 > 0$

Hence, $f(x, y)$ is minimum at (a, a).

$$f_{min} = a^2 + a^3\left(\frac{1}{a} + \frac{1}{a}\right) = 3a^2.$$

Example 3

Discuss the maxima and minima of the function $3x^2 - y^2 + x^3$.

Solution

Let $f(x, y) = 3x^2 - y^2 + x^3$

Step I For extreme values, $\frac{\partial f}{\partial x} = 0$

$$6x + 3x^2 = 0$$

$$3x(x + 2) = 0$$

$$x = 0, -2$$

and $\frac{\partial f}{\partial y} = 0$

$$-2y = 0$$

$$y = 0$$

Stationary points are $(0, 0)$, $(-2, 0)$.

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 6 + 6x$$

$$s = \frac{\partial^2 f}{\partial x\partial y} = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = -2$$

Step III

(x,y)	r	s	t	$rt - s^2$	Conclusion
(0, 0)	6	0	–2	$-12 < 0$	neither maximum nor minimum
(–2, 0)	–6	0	–2	$12 > 0$ and $r < 0$	maximum

Hence, $f(x, y)$ is maximum at (–2, 0)

$$f_{max} = 3(-2)^2 - 0 + (-2)^3 = 4$$

Example 4

Find the stationary value of $x^3 + y^3 - 3axy,\ a > 0$.

Solution

Let $$f(x, y) = x^3 + y^3 - 3axy$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$3x^2 - 3ay = 0$$

$$x^2 - ay = 0 \qquad \ldots(1)$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$3y^2 - 3ax = 0$$

$$y^2 - ax = 0 \qquad \ldots(2)$$

From Eq. (1),

$$y = \frac{x^2}{a}$$

Substituting in Eq. (2),

$$x^4 - a^3 x = 0$$

$$x(x - a)(x^2 + ax + a^2) = 0$$

$$x = 0,\ x = a$$

$$\therefore \quad y = 0,\ y = a.$$

Hence, stationary points are (0, 0) and (a, a).

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 6x$$

$$s = \frac{\partial^2 f}{\partial x\,\partial y} = -3a$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6y$$

Step III At (0, 0), $r = 0, \quad s = -3a, \quad t = 0$

$$rt - s^2 = (0)(0) - (-3a)^2 = -9a^2 < 0$$

Hence, $f(x, y)$ is neither maximum nor minimum at (0, 0).

At (a, a), $r = 6a, \quad s = -3a, \quad t = 6a$

$$rt - s^2 = (6a)(6a) - (-3a)^2 = 27a^2 > 0$$

Also, $\quad r = 6a > 0$

Hence, $f(x, y)$ is minimum at (a, a).

$$f_{\min} = a^3 + a^3 - 3a^3 = -a^3.$$

Example 5

Find the extreme values of the function $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$, *if any.*

[Summer 2017]

Solution

Let $\quad f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$3x^2 + 3y^2 - 6x = 0$$

$$x^2 + y^2 - 2x = 0 \qquad \ldots(1)$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$6xy - 6y = 0$$

$$6y(x - 1) = 0$$

$$y = 0, x = 1$$

Putting $y = 0$ in Eq. (1),

$$x^2 - 2x = 0,$$

$$x = 0, 2$$

Stationary points are (0, 0), (2, 0).

Putting $x = 1$ in Eq. (1),

$$1 + y^2 - 2 = 0,$$

$$y^2 = 1, y = \pm 1$$

Stationary points are (1, 1), (1, –1).

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 6x - 6 = 6(x - 1)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6y$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6x - 6 = 6(x - 1)$$

Step III

(x, y)	r	s	t	$rt - s^2$	*Conclusion*
(0, 0)	$-6 < 0$	0	-6	$36 > 0$ and $r < 0$	maximum
(2, 0)	$6 > 0$	0	6	$36 > 0$ and $r > 0$	minimum
(1, 1)	0	6	0	$-36 < 0$	neither maximum nor minimum
(1, –1)	0	-6	0	$-36 < 0$	neither maximum nor minimum

Hence, $f(x, y)$ is maximum at (0, 0) and minimum at (2, 0).

$$f_{max} = 0 + 4 = 4$$

and

$$\begin{aligned} f_{min} &= 2^3 + 3(2)(0)^2 - 3(2)^2 - 3(0)^2 + 4 \\ &= 8 + 0 - 12 + 4 \\ &= 0 \end{aligned}$$

Example 6

Find the extreme values of the function $x^3 + y^3 - 63\,(x + y) + 12xy$.

Solution

Let $$f(x, y) = x^3 + y^3 - 63x - 63y + 12xy$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$3x^2 - 63 + 12y = 0,$$

$$3x^2 + 12y = 63$$

$$x^2 + 4y = 21 \qquad \text{...(1)}$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$3y^2 - 63 + 12x = 0,$$

$$12x + 3y^2 = 63$$

$$4x + y^2 = 21 \qquad \text{...(2)}$$

Equating Eqs (1) and (2),

$$x^2 + 4y = 4x + y^2$$

$$x^2 - y^2 - 4(x - y) = 0$$

$$(x + y)(x - y) - 4(x - y) = 0$$

$$(x - y)(x + y - 4) = 0$$

$$x - y = 0,\ x + y - 4 = 0$$

$$y = x,\ y = 4 - x$$

Putting $y = x$ in Eq. (1),

$$x^2 + 4x - 21 = 0,$$
$$(x+7)(x-3) = 0$$
$$x = -7, 3$$
$$\therefore y = -7, 3$$

Stationary points are (–7, –7), (3, 3).
Putting $y = 4 - x$ in Eq. (1),

$$x^2 + 4(4-x) = 21$$
$$x^2 - 4x - 5 = 0,$$
$$(x+1)(x-5) = 0$$
$$x = -1, 5$$
$$\therefore y = 5, -1$$

Stationary points are (–1, 5), (5, –1).

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 6x$$
$$s = \frac{\partial^2 f}{\partial x\, \partial y} = 12$$
$$t = \frac{\partial^2 f}{\partial y^2} = 6y$$

Step III

(x, y)	*r*	*s*	*t*	$rt - s^2$	*Conclusion*
(–7, –7)	–42	12	–42	1620 > 0 and $r < 0$	maximum
(3, 3)	18	12	18	180 > 0 and $r > 0$	minimum
(–1, 5)	–6	12	30	–324 < 0	neither maximum nor minimum
(5, –1)	30	12	–6	–324 < 0	neither maximum nor minimum

Hence, $f(x, y)$ is maximum at (–7, –7).

$$f_{max} = (-7)^3 + (-7)^3 - 63(-7-7) + 12(-7)(-7) = 784.$$

and $f(x, y)$ is minimum at (3, 3).

$$f_{min} = 3^3 + 3^3 - 63(3+3) + 12(3)(3) = -216.$$

Example 7

Find the extreme value of xy $(a - x - y)$.

Solution

Let
$$f(x, y) = xy(a - x - y)$$
$$= axy - x^2y - xy^2$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$
$$ay - 2xy - y^2 = 0 \quad \ldots(1)$$
$$y(a - 2x - y) = 0$$
$$y = 0,\ a - 2x - y = 0$$

and
$$\frac{\partial f}{\partial y} = 0 \quad \ldots(2)$$
$$ax - x^2 - 2xy = 0$$
$$x(a - x - 2y) = 0$$
$$x = 0,\ a - x - 2y = 0$$

Considering four pairs of equations of Eqs (1) and (2),

$$y = 0 \qquad x = 0$$
$$y = 0 \qquad a - x - 2y = 0$$
$$a - 2x - y = 0 \qquad x = 0$$
$$a - 2x - y = 0 \qquad a - x - 2y = 0$$

Solving these equations, the following pairs of values of stationary points are obtained: $(0, 0), (a, 0), (0, a), \left(\frac{a}{3}, \frac{a}{3}\right)$

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = -2y$$
$$s = \frac{\partial^2 f}{\partial x\, \partial y} = a - 2x - 2y$$
$$t = \frac{\partial^2 f}{\partial y^2} = -2x$$

Step III

(x, y)	r	s	t	$rt - s^2$	*Conclusion*
$(0, 0)$	0	a	0	$-a^2 < 0$	neither maximum nor minimum
$(a, 0)$	0	$-a$	$-2a$	$-a^2 < 0$	neither maximum nor minimum
$(0, a)$	$-2a$	$-a$	0	$-a^2 < 0$	neither maximum nor minimum
$\left(\frac{a}{3}, \frac{a}{3}\right)$	$\frac{-2a}{3}$	$\frac{-a}{3}$	$\frac{-2a}{3}$	$\frac{a^2}{3} > 0$	maximum or minimum

Hence, $f(x, y)$ is maximum or minimum at $\left(\frac{a}{3}, \frac{a}{3}\right)$ depending on whether $a > 0$ or $a < 0$.

$$f_{\text{extreme}} = \frac{a}{3} \cdot \frac{a}{3}\left(a - \frac{a}{3} - \frac{a}{3}\right) = \frac{a^3}{27}.$$

Example 8

Examine the function $x^3 y^2(12 - 3x - 4y)$ for extreme values.

Solution

Let $f(x, y) = 12x^3y^2 - 3x^4y^2 - 4x^3y^3$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$36x^2y^2 - 12x^3y^2 - 12x^2y^3 = 0$$

$$12x^2y^2(3 - x - y) = 0$$

$$x = 0,\ y = 0,\ x + y = 3 \qquad \ldots(1)$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$24x^3y - 6x^4y - 12x^3y^2 = 0$$

$$6x^3y(4 - x - 2y) = 0$$

$$x = 0,\ y = 0,\ x + 2y = 4 \qquad \ldots(2)$$

Considering six pairs of equations of Eqs (1) and (2),

$x = 0$	$y = 0$
$x = 0$	$x + 2y = 4$
$y = 0$	$x + 2y = 4$
$x + y = 3$	$x = 0$
$x + y = 3$	$y = 0$
$x + y = 3$	$x + 2y = 4$

Solving these equations, the following pairs of stationary points are obtained:

(0, 0), (0, 2), (4, 0), (0, 3), (3, 0), (2, 1)

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 72xy^2 - 36x^2y^2 - 24xy^3 = 12xy^2(6 - 3x - 2y)$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = 72x^2y - 24x^3y - 36x^2y^2 = 12x^2y(6 - 2x - 3y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = 24x^3 - 6x^4 - 24x^3y = 6x^3(4 - x - 4y)$$

Step III

(x, y)	r	s	t	$rt - s^2$	*Conclusion*
(0, 0)	0	0	0	0	no conclusion
(0, 2)	0	0	0	0	no conclusion
(4, 0)	0	0	0	0	no conclusion
(0, 3)	0	0	0	0	no conclusion
(3, 0)	0	0	162	0	no conclusion
(2, 1)	–48	–48	–96	2304 > 0 and $r < 0$	maximum

Hence, $f(x, y)$ is maximum at (2, 1).

$$f_{max} = (2^3)(1^2)(12-6-4) = 16.$$

Example 9

Find the maxima and minima of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

Solution

Let $$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$4x^3 - 4x + 4y = 0$$

$$4(x^3 - x + y) = 0$$

$$x^3 - x + y = 0 \quad \ldots(1)$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$4y^3 + 4x - 4y = 0$$

$$4(y^3 + x - y) = 0$$

$$y^3 + x - y = 0 \quad \ldots(2)$$

Adding Eqs (1) and (2),

$$x^3 + y^3 = 0$$

$$y = -x$$

Substituting $y = -x$ in Eq. (2),

$$-x^3 + x + x = 0$$

$$x^3 - 2x = 0$$

$$x(x^2 - 2) = 0$$

$$x = 0, \quad \pm\sqrt{2}$$

$$y = 0, \quad \mp\sqrt{2}$$

Stationary points are $(0, 0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$.

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = 4$$

$$t = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$$

Step III

(x, y)	*r*	*s*	*t*	*rt – s²*	*Conclusion*
(0, 0)	–4	4	–4	0	no conclusion
$(\sqrt{2}, -\sqrt{2})$	20	4	20	384 > 0 and *r* > 0	minimum
$(-\sqrt{2}, \sqrt{2})$	20	4	20	384 > 0 and *r* > 0	minimum

Hence, $f(x, y)$ is minimum at the point $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.

At $(\sqrt{2}, -\sqrt{2})$, $f_{\min} = (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4\sqrt{2}(-\sqrt{2}) - 2(-\sqrt{2})^2 = -8$

Example 10

Find the extreme values of the function

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20.$$ **[Summer 2014]**

Solution

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$3x^2 - 3 = 0$$

$$3(x^2 - 1) = 0$$

$$x^2 - 1 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

and
$$\frac{\partial f}{\partial y} = 0$$
$$3y^2 - 12 = 0$$
$$3(y^2 - 4) = 0$$
$$y^2 - 4 = 0$$
$$y^2 = 4$$
$$y = \pm 2$$

Stationary points are (1, 2), (1, –2), (–1, 2), (–1, –2).

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 6x$$
$$s = \frac{\partial^2 f}{\partial x\, \partial y} = 0$$
$$t = \frac{\partial^2 f}{\partial y^2} = 6y$$

Step III

(x, y)	*r*	*s*	*t*	*rt – s²*	*Conclusion*
(1, 2)	6	0	12	$72 > 0$ and $r > 0$	minimum
(1, –2)	6	0	–12	$-72 < 0$	neither maximum nor minimum
(–1, 2)	–6	0	12	$-72 < 0$	neither maximum nor minimum
(–1, –2)	–6	0	–12	$72 > 0$ and $r < 0$	maximum

Hence, $f(x, y)$ is maximum at (–1, –2) and minimum at (1, 2).

$$f_{max} = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20 = 38$$
$$f_{min} = (1)^3 + (2)^3 - 3(1) - 12(2) + 20 = 2$$

Example 11

Find the extreme values of the function $x^3 + xy^2 + 21x - 12x^2 - 2y^2$.

Solution

Let
$$f(x, y) = x^3 + xy^2 + 21x - 12x^2 - 2y^2$$

Step I For extreme values,
$$\frac{\partial f}{\partial x} = 0,$$

$$3x^2 + y^2 - 24x + 21 = 0 \qquad \ldots(1)$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$2xy - 4y = 0$$

$$2y(x-2) = 0$$

$$y = 0,\ x = 2$$

Putting $y = 0$ in Eq. (1),

$$3x^2 - 24x + 21 = 0,\ x^2 - 8x + 7 = 0$$

$$(x-1)(x-7) = 0$$

$$x = 1, 7$$

Stationary points are (1, 0), (7, 0).

Putting $x = 2$ in Eq. (1),

$$12 + y^2 - 48 + 21 = 0$$

$$y^2 = 15,\ y = \pm\sqrt{15}$$

Stationary points are $\left(2, \sqrt{15}\right), \left(2, -\sqrt{15}\right)$.

Hence, all stationary points are (1, 0), (7, 0), $\left(2, \sqrt{15}\right), \left(2, -\sqrt{15}\right)$.

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 6x - 24 = 6(x-4)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 2y$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2x - 4 = 2(x-2)$$

Step III

(x, y)	*r*	*s*	*t*	$rt - s^2$	*Conclusion*
(1, 0)	–18	0	–2	36 > 0 and $r < 0$	maximum
(7, 0)	18	0	10	180 > 0 and $r > 0$	minimum
$\left(2, \sqrt{15}\right)$	–12	$2\sqrt{15}$	0	–60 < 0	neither maximum nor minimum
$\left(2, -\sqrt{15}\right)$	–12	$-2\sqrt{15}$	0	–60 < 0	neither maximum nor minimum

Hence, $f(x, y)$ is maximum at (1, 0) and minimum at (7, 0).

$$f_{max} = 1^3 + (1 \times 0^2) + 21 - (12 \times 1^2) - (2 \times 0^2) = 10,$$

$$f_{min} = 7^3 + (7 \times 0^2) + (21 \times 7) - (12 \times 7^2) - (2 \times 0^2) = -98$$

Example 12

Find all the stationary points of the function

$x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

after examining whether the function is maximum or minimum at those points. **[Summer 2016]**

Solution

Let $$f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$3x^2 + 3y^2 - 30x + 72 = 0 \qquad \ldots(1)$$

$$\frac{\partial f}{\partial y} = 0$$

$$6xy - 30y = 0 \qquad \ldots(2)$$

$$6y\,(x - 5) = 0$$

$$y = 0,\ x = 5$$

Putting $y = 0$ in Eq. (1),

$$3x^2 - 30x + 72 = 0$$

$$x = 4,\ x = 6$$

Stationary points are (4, 0), (6, 0).

Putting $x = 5$ in Eq. (1),

$$3y^2 - 3 = 0$$

$$y = \pm 1$$

Stationary points are (5, 1) and (5, –1).

Hence, all stationary points are (5, 1) (5, –1), (4, 0) and (6, 0).

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 6x - 30$$

$$s = \frac{\partial^2 f}{\partial x\,\partial y} = 6y$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6x - 30$$

Step III

(x, y)	r	s	t	$rt - s^2$	Conclusion
(4, 0)	–6	0	–6	$36 > 0$ and $r < 0$	maximum
(6, 0)	6	0	6	$36 > 0$ and $r > 0$	minimum
(5, 1)	0	6	0	$-36 < 0$	neither maximum nor minimum
(5, –1)	0	–6	0	$-36 < 0$	neither maximum nor minimum

Hence, $f(x, y)$ is maximum at (4, 0) and $f(x, y)$ is minimum at (6, 0).

$$f_{max} = (4)^3 + 0 - 15\,(4)^2 - 0 + 72\,(4) = 112$$
$$f_{min} = (6)^3 + 0 - 15\,(6)^2 - 0 + 72\,(6) = 108$$

Example 13

Find the extreme values of $\sin x + \sin y + \sin(x + y)$.

Solution

Let $$f(x, y) = \sin x + \sin y + \sin(x + y)$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$
$$\cos x + \cos(x + y) = 0 \qquad \ldots(1)$$

and
$$\frac{\partial f}{\partial y} = 0$$
$$\cos y + \cos(x + y) = 0 \qquad \ldots(2)$$

Equating Eqs (1) and (2),

$$\cos x + \cos(x + y) = \cos y + \cos(x + y)$$
$$\cos x = \cos y$$
$$x = y$$

Substituting $y = x$ in Eq. (1),

$$\cos x + \cos 2x = 0,$$
$$\cos x = -\cos 2x$$
$$= \cos(\pi - 2x) \text{ or } \cos(\pi + 2x)$$
$$x = \pi - 2x \text{ or } \pi + 2x$$
$$x = \frac{\pi}{3}, -\pi$$
$$y = \frac{\pi}{3}, -\pi$$

Stationary points are $\left(\frac{\pi}{3}, \frac{\pi}{3}\right), (-\pi, -\pi)$.

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$$

Step III

(x, y)	r	s	t	$rt - s^2$	*Conclusion*
$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$\frac{9}{4} > 0$ and $r < 0$	maximum
$(-\pi, -\pi)$	0	0	0	0	no conclusion

Hence, $f(x, y)$ is maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$$f_{\max} = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

Example 14

Find the extreme values of $\sin x \sin y \sin(x + y)$.

Solution

Let $f(x, y) = \sin x \sin y \sin(x + y)$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$\sin y\,[\cos x \sin(x+y) + \sin x \cos(x+y)] = 0$$

$$\sin y \sin(2x+y) = 0$$

$$\frac{1}{2}[\cos 2x - \cos(2x+2y)] = 0$$

$$\cos 2x - \cos(2x+2y) = 0 \qquad ...(1)$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$\sin x\,[\cos y \sin(x+y) + \sin y \cos(x+y)] = 0$$

$$\sin x \sin(x+2y) = 0$$
$$\frac{1}{2}[\cos 2y - \cos(2x+2y)] = 0$$
$$\cos 2y - \cos(2x+2y) = 0 \qquad \text{...(2)}$$

Equating Eqs (1) and (2),

$$\cos 2x = \cos 2y$$
$$x = y$$

Substituting $x = y$ in Eq. (1),

$$\cos 2x - \cos(2x+2x) = 0$$
$$\cos 2x = \cos 4x = 2\cos^2 2x - 1$$
$$2\cos^2 2x - \cos 2x - 1 = 0$$
$$\cos 2x = \frac{1 \pm \sqrt{1+8}}{4}$$
$$= 1, -\frac{1}{2}$$

$$\cos 2x = 1 = \cos 0, \qquad \cos 2x = -\frac{1}{2} = \cos\frac{2\pi}{3}$$
$$x = 0, \qquad x = \frac{\pi}{3}$$
$$y = 0, \qquad y = \frac{\pi}{3}$$

Stationary points are $(0,0), \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = -\sin 2x + \sin 2(x+y) = 2\sin y \cos(2x+y)$$
$$s = \frac{\partial^2 f}{\partial x\,\partial y} = \sin 2(x+y)$$
$$t = \frac{\partial^2 f}{\partial y^2} = -\sin 2y + \sin 2(x+y) = 2\sin x \cos(x+2y)$$

Step III

(x, y)	*r*	*s*	*t*	*rt* – *s*²	*Conclusion*
(0, 0)	0	0	0	0	no conclusion
$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$\frac{9}{4} > 0$ and $r < 0$	maximum

Hence, $f(x, y)$ is maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$$f_{max} = \sin\frac{\pi}{3}\sin\frac{\pi}{3}\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}$$

8.11 EXTREME VALUES WITH CONSTRAINED VARIABLES

Sometimes we have to find the extreme values of a function of three (or more) variables, say $f(x, y, z)$, which are not independent but are connected by some given relation $\phi(x, y, z) = 0$. The extreme values of $f(x, y, z)$ in such a situation are called *constrained extreme values*.

In such situations, we use $\phi(x, y, z) = 0$ to eliminate one of the variables, say z, from the given function, thus converting the function as a function of only two variables and then find the extreme values of the function.

Example 1

Find the minimum value of $x^2 + y^2 + z^2$ *with the constraint* $x + y + z = 3a$.

Solution

Let $f = x^2 + y^2 + z^2$

$$x + y + z = 3a$$

$$z = 3a - x - y \quad \ldots(1)$$

Substituting the value of z in Eq. (1),

$$f = x^2 + y^2 + (3a - x - y)^2$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$2x - 2(3a - x - y) = 0$$

$$4x - 6a + 2y = 0$$

$$2x + y = 3a \quad \ldots(2)$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$2y - 2(3a - x - y) = 0$$

$$2y - 6a + 2x + 2y = 0$$

$$x + 2y = 3a \quad \ldots(3)$$

Solving Eqs (2) and (3),

$$x = y = a$$

The stationary point is (a, a).

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 4$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 2$$

$$t = \frac{\partial^2 f}{\partial y^2} = 4$$

Step III At (a, a), $r = 4$, $s = 2$, $t = 4$

$$rt - s^2 = (4)(4) - (2)^2 = 12 > 0$$

Also, $r = 4 > 0$
Hence, $f(x, y)$ is minimum at (a, a)

$$f_{min} = a^2 + a^2 + (3a - a - a)^2$$
$$= 3a^2$$

Example 2

Find the minimum value of x^3y^2z subject to the condition $x + y + z = 1$.

Solution

Let

$$f = x^3y^2z$$
$$x + y + z = 1$$
$$z = 1 - x - y \quad \text{...(1)}$$

Substituting the value of z in Eq. (1),

$$f = x^3y^2(1 - x - y)$$
$$= x^3y^2 - x^4y^2 - x^3y^3$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$
$$x^2y^2(3 - 4x - 3y) = 0$$
$$x = 0, y = 0, 4x + 3y = 3 \quad \text{...(2)}$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$2x^3y - 2x^4y - 3x^3y^2 = 0$$
$$x^3y(2 - 2x - 3y) = 0$$
$$x = 0, y = 0, 2x + 3y = 2 \quad \text{...(3)}$$

Considering six pairs of Eqs (2) and (3),

$$\begin{array}{rr} x = 0 & y = 0 \\ x = 0 & 2x + 3y = 2 \\ y = 0 & 2x + 3y = 2 \\ 4x + 3y = 3 & x = 0 \\ 4x + 3y = 3 & y = 0 \\ 4x + 3y = 3 & 2x + 3y = 2 \end{array}$$

Solving these equations, the following pairs of stationary points are formed:

$$(0, 0), \left(0, \frac{2}{3}\right), (1, 0), (0, 1), \left(\frac{3}{4}, 0\right), \left(\frac{1}{2}, \frac{1}{3}\right)$$

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3 = 6xy^2(1 - 2x - y)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6x^2y - 8x^3y - 9x^2y^2 = x^2y(6 - 8x - 9y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y = 2x^3(1 - x - 3y)$$

Step III

(x, y)	*r*	*s*	*t*	*rt – s²*	*Conclusion*
(0, 0)	0	0	0	0	no conclusion
$\left(0, \frac{2}{3}\right)$	0	0	0	0	no conclusion
(1, 0)	0	0	0	0	no conclusion
(0, 1)	0	0	0	0	no conclusion
$\left(\frac{3}{4}, 0\right)$	0	0	$\frac{27}{128}$	0	no conclusion
$\left(\frac{1}{2}, \frac{1}{3}\right)$	$-\frac{1}{9}$	$-\frac{1}{12}$	$-\frac{1}{8}$	$\frac{1}{144} > 0$ and $r < 0$	maximum

Hence, $f(x, y)$ is maximum at $\left(\frac{1}{2}, \frac{1}{3}\right)$.

$$f_{max} = \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{1}{432}$$

Example 3

Divide 120 *into three parts so that the sum of their products taken two at a time shall be maximum.*

Solution

Step I Let x, y, z be three numbers.

Let
$$x + y + z = 120$$
$$\begin{aligned} f &= xy + yz + zx \\ &= xy + y(120 - x - y) + (120 - x - y)x \\ &= xy + 120y - xy - y^2 + 120x - x^2 - xy \\ &= 120x + 120y - xy - x^2 - y^2 \end{aligned}$$

For extreme values,
$$\frac{\partial f}{\partial x} = 0$$
$$120 - y - 2x = 0 \qquad \text{...(1)}$$

and
$$\frac{\partial f}{\partial y} = 0$$
$$120 - x - 2y = 0 \qquad \text{...(2)}$$

Solving Eqs (1) and (2),
$$x = 40$$
$$y = 40$$

Stationary point is (40, 40).

Step II
$$r = \frac{\partial^2 f}{\partial x^2} = -2$$
$$s = \frac{\partial^2 f}{\partial x \partial y} = -1$$
$$t = \frac{\partial^2 f}{\partial y^2} = -2$$

Step III At (40, 40).
$$rt - s^2 = (-2)(-2) - (-1)^2 = 3 > 0$$

and
$$r = -2 < 0$$

$f(x, y)$ is maximum at (40, 40).

Hence, three parts are 40, 40 and 40.

Example 4

Find a point on the plane $2x + 3y - z = 5$ *which is nearest to the origin.*

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Solution

Let $P(x, y, z)$ be any point on the plane $2x + 3y - z = 5$.
Its distance from the origin is given by

$$d = \sqrt{x^2 + y^2 + z^2}$$
$$d^2 = x^2 + y^2 + z^2$$

Since P lies on the plane $z = 2x + 3y - 5$,

$$d^2 = x^2 + y^2 + (2x + 3y - 5)^2$$

Let
$$f(x, y) = x^2 + y^2 + 4x^2 + 9y^2 + 25 + 12xy - 30y - 20x$$
$$= 5x^2 + 10y^2 + 12xy - 30y - 20x + 25$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$
$$10x + 12y = 20$$
$$5x + 6y = 10 \quad \text{...(1)}$$

and
$$\frac{\partial f}{\partial y} = 0$$
$$20y + 12x = 30$$
$$10y + 6x = 15 \quad \text{...(2)}$$

Solving Eqs (1) and (2),

$$x = \frac{5}{7}, y = \frac{15}{14}$$

Stationary point is $\left(\frac{5}{7}, \frac{15}{14}\right)$.

Step II
$$r = \frac{\partial^2 f}{\partial x^2} = 10$$
$$s = \frac{\partial^2 f}{\partial x \partial y} = 6$$
$$t = \frac{\partial^2 f}{\partial y^2} = 10$$

Step III At $\left(\frac{5}{7}, \frac{15}{14}\right)$, $r = 10$, $s = 6$, $t = 10$

$$rt - s^2 = 100 - 36 = 64 > 0$$

Also, $r = 10 > 0$

$f(x, y)$ i.e. d^2 is minimum at $\left(\frac{5}{7}, \frac{15}{14}\right)$ and hence d is minimum at $\left(\frac{5}{7}, \frac{15}{14}\right)$.

At $\left(\frac{5}{7}, \frac{15}{14}\right)$,

$$z = 2x + 3y - 5 = 2\left(\frac{5}{7}\right) + 3\left(\frac{15}{14}\right) - 5 = -\frac{5}{14}$$

Hence, d is minimum at $\left(\frac{5}{7}, \frac{15}{14}, -\frac{5}{14}\right)$.

The point $\left(\frac{5}{7}, \frac{15}{14}, -\frac{5}{14}\right)$ on the plane $2x + 3y - z = 5$ is nearest to the origin.

Example 5

Find the points on the surface $z^2 = xy + 1$ nearest to the origin. Also find that distance.

Solution

Let $P(x, y, z)$ be any point on the surface $z^2 = xy + 1$.

Its distance from the origin is given by

$$d = \sqrt{(x^2 + y^2 + z^2)}$$
$$d^2 = x^2 + y^2 + z^2$$

Since P lies on the surface $z^2 = xy + 1$,

$$d^2 = x^2 + y^2 + xy + 1$$

Let
$$f(x, y) = x^2 + y^2 + xy + 1$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$
$$2x + y = 0 \qquad \ldots(1)$$

and
$$\frac{\partial f}{\partial y} = 0$$
$$2y + x = 0 \qquad \ldots(2)$$

Solving Eqs (1) and (2),

$$x = 0, y = 0$$

Stationary point is (0, 0).

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 2$$
$$s = \frac{\partial^2 f}{\partial x \partial y} = 1$$
$$t = \frac{\partial^2 f}{\partial y^2} = 2$$

Step III At (0, 0), $r = 2$, $t = 2$, $s = 1$

$$rt - s^2 = (2)(2) - (1)^2 = 3 > 0$$

Also, $r = 2 > 0$

$f(x, y)$, i.e., d^2 is minimum at (0, 0) and hence d is minimum at (0, 0).

At (0, 0),

$$z^2 = xy + 1 = 1$$

$$z = \pm 1$$

Hence, d is minimum at (0, 0, 1) and (0, 0, –1).

The points (0, 0, 1) and (0, 0, –1) on the surface $z^2 = xy + 1$ are nearest to the origin.

Minimum distance $= \sqrt{0+0+1} = 1$.

Example 6

A rectangular box open at the top is to have a volume of 108 *cubic metres. Find the dimensions of the box if its total surface area is minimum.*

Solution

Let x, y and z be the dimensions of the box. Let V and S be its volume and surface area respectively.

$$V = xyz$$

$$S = xy + 2xz + 2yz$$

Substituting $z = \frac{V}{xy}$,

$$S = xy + 2x \cdot \frac{V}{xy} + 2y \cdot \frac{V}{xy}$$

$$= xy + \frac{2V}{y} + \frac{2V}{x}$$

Step I For extreme values,

$$\frac{\partial S}{\partial x} = 0$$

$$y - \frac{2V}{x^2} = 0 \qquad \text{...(1)}$$

and

$$\frac{\partial S}{\partial y} = 0$$

$$x - \frac{2V}{y^2} = 0 \qquad \text{...(2)}$$

Substituting $y = \frac{2V}{x^2}$ from Eq. (1) in Eq. (2),

$$x - 2V\left(\frac{x^4}{4V^2}\right) = 0$$

$$x\left(1 - \frac{x^3}{2V}\right) = 0$$

$$x = (2V)^{\frac{1}{3}}$$

$$\therefore \quad y = \frac{2V}{x^2} = \frac{2V}{(2V)^{\frac{2}{3}}} = (2V)^{\frac{1}{3}} \quad [\text{Since } x \neq 0 \text{ being the side of the box}]$$

Stationary point is $\left[(2V)^{\frac{1}{3}}, (2V)^{\frac{1}{3}}\right]$.

Step II

$$r = \frac{\partial^2 S}{\partial x^2} = \frac{4V}{x^3}$$

$$s = \frac{\partial^2 S}{\partial x\, \partial y} = 1$$

$$t = \frac{\partial^2 S}{\partial y^2} = \frac{4V}{y^3}$$

Step III At $\left[(2V)^{\frac{1}{3}}, (2V)^{\frac{1}{3}}\right]$, $r = \dfrac{4V}{2V} = 2 > 0$, $s = 1$, $t = \dfrac{4V}{2V} = 2$

$$rt - s^2 = (2)(2) - (1)^2 = 3 > 0 \text{ and } r = 2 > 0$$

Hence, S is minimum at $x = y = (2V)^{\frac{1}{3}}$.

Putting $\quad V = 108 \text{ m}^3,$

$$x = y = (2 \times 108)^{\frac{1}{3}} = 6$$

and $\quad z = \dfrac{V}{xy} = \dfrac{108}{6 \times 6} = 3$

Hence, dimensions of the box which make its total surface area S minimum are $x = 6$, $y = 6$, $z = 3$.

Example 7

Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.

Solution

Let x, y, z be the length, breadth and height of the rectangular solid and V be its volume.

$$V = xyz \qquad \ldots(1)$$

Let the given sphere be

$$x^2 + y^2 + z^2 = a^2$$

$$z^2 = a^2 - x^2 - y^2$$

Substituting in Eq. (1),

$$V = xy\sqrt{a^2 - x^2 - y^2}$$

$$V^2 = x^2y^2(a^2 - x^2 - y^2)$$

Let

$$f(x, y) = V^2 = x^2y^2(a^2 - x^2 - y^2) \qquad \dots(2)$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$y^2[2x(a^2 - x^2 - y^2) + x^2(-2x)] = 0$$

$$2xy^2(a^2 - 2x^2 - y^2) = 0$$

$$x = 0,\ y = 0,\ 2x^2 + y^2 = a^2 \qquad \dots(3)$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$x^2[2y(a^2 - x^2 - y^2) + y^2(-2y)] = 0$$

$$2x^2y(a^2 - x^2 - 2y^2) = 0$$

$$x = 0,\ y = 0,\ x^2 + 2y^2 = a^2 \qquad \dots(4)$$

But x and y are the sides of the rectangular solid, and therefore cannot be zero.
Solving $2x^2 + y^2 = a^2$ and $x^2 + 2y^2 = a^2$,

$$x^2 = \frac{a^2}{3},\ y^2 = \frac{a^2}{3}$$

$$x = \frac{a}{\sqrt{3}},\ y = \frac{a}{\sqrt{3}} \qquad [\because \text{side cannot be negative}]$$

$$z = \sqrt{a^2 - \frac{a^2}{3} - \frac{a^2}{3}} = \frac{a}{\sqrt{3}}$$

Stationary points are $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$.

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 2a^2y^2 - 12x^2y^2 - 2y^4$$

$$s = \frac{\partial^2 f}{\partial x\,\partial y} = 4a^2xy - 8x^3y - 8xy^3$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2a^2x^2 - 2x^4 - 12x^2y^2$$

Step III At $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$, $r = \frac{2a^4}{3} - \frac{4a^4}{3} - \frac{2a^4}{9} = -\frac{8a^4}{9}$

$$s = \frac{4a^4}{3} - \frac{8a^4}{9} - \frac{8a^4}{9} = -\frac{4a^4}{9}$$

$$t = \frac{2a^4}{3} - \frac{2a^4}{9} - \frac{12a^4}{9} = -\frac{8a^4}{9}$$

$$rt - s^2 = \frac{64a^8}{81} - \frac{16a^8}{81} = \frac{48a^8}{81} > 0$$

$$rt - s^2 > 0 \text{ and } r < 0$$

$f(x, y)$, i.e. V^2 is maximum at $x = y = z$ and hence, V is maximum when $x = y = z$, i.e. the rectangular solid is a cube.

EXERCISE 8.7

1. Examine maxima and minima of the following functions and find their extreme values:

(i) $2 + 2x + 2y - x^2 - y^2$ (ii) $x^2y^2 - 5x^2 - 8xy - 5y^2$

(iii) $x^2 + y^2 + xy + x - 4y + 5$ (iv) $x^2 + y^2 + 6x = 12$

(v) $x^3y^2(1 - x - y)$ (vi) $xy(3a - x - y)$

(vii) $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$ (viii) $x^4 + y^4 - 2(x - y)^2$

(ix) $x^4 + x^2y + y^2$ (x) $x^4 + y^4 - 4a^2xy$

(xi) $y^4 - x^4 + 2(x^2 - y^2)$ (xii) $x^3 + 3x^2 + y^2 + 4xy$

(xiii) $x^2y - 3x^2 - 2y^2 - 4y + 3$ (xiv) $x^4 - y^4 - x^2 - y^2 + 1$.

[**Ans.:** (i) Max. at $(1, 1)$; 4 (ii) Max. at $(0, 0)$; 0
(iii) Min. at $(-2, 3)$; -2 (iv) Min. at $(-3, 0)$; 3
(v) Max. at $\left(\frac{1}{2}, \frac{1}{3}\right)$; $\frac{1}{432}$ (vi) Max. at (a, a); a^3
(vii) Max. at $(0, 0)$; 4 (viii) Min. at $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$; -8
(ix) Min. at $(0, 0)$; 0 (x) Min. at (a, a) and $(-a, a)$; a^4
(xi) No extreme values (xii) No extreme values
(xiii) Max. at $(0, -1)$; 5 (xiv) Max. at $(0, 0)$; 1, min at $\left(\pm\frac{1}{\sqrt{2}}, \pm\sqrt{\frac{1}{\sqrt{2}}}\right)$; $\frac{1}{2}$]

2. A rectangular box, open at the top, is to have a volume of 32 cc. Find the dimensions of the box requiring least materials for its construction.

[**Ans.:** 4, 4, 2]

3. Divide 120 into three parts so that the sum of their products taken two at a time shall be maximum.

[**Hint:** $f = xy + yz + zx$ where $x + y + z = 120$]

[**Ans.:** 40, 40, 40]

4. The sum of three positive numbers is 'a'. Determine the maximum value of their product.

$$\left[\text{Ans.: } \frac{a^3}{27} \text{ at } \left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)\right]$$

5. Find the volume of the largest rectangular parallelepiped that can be inscribed in an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

$$\left[\begin{array}{l}\text{Hint: Let } 2x, 2y, 2z \text{ be the sides of the parallelepiped; then its volume} \\ \quad v = 8xyz = 8xy\sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}}\end{array}\right]$$

$$\left[\text{Ans.: } \frac{8abc}{3\sqrt{3}}\right]$$

6. Prove that area of a triangle with constant perimeter is maximum when the triangle is equilateral.

$$\left[\begin{array}{l}\text{Hint: Area} = \sqrt{s(s-a)(s-b)(s-c)} \\ \qquad \text{where } 2s = a + b + c, c = 2s - a - b, s \text{ is constant}\end{array}\right]$$

7. Find the shortest distance from the origin to the surface $xyz^2 = 2$.

[**Ans.:** 2]

8. Find the shortest distance from the origin to the plane $x - 2y - 2z = 3$.

[**Ans.:** 1]

9. Find the shortest distance between the lines $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$ and $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$.

$$\left[\text{Ans.: } 2\sqrt{29}\right]$$

10. Find the maximum value of $\cos A \cos B \cos C$, where A, B, C are angles of a triangle.

$$\left[\text{Ans.: max. at } \left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right); \frac{1}{8}\right]$$

8.12 METHOD OF LAGRANGE MULTIPLIERS

Let $f(x, y, z)$ be a function of three variables x, y, z, and the variables be connected by the relation

$$\phi(x, y, z) = 0 \qquad \ldots(1)$$

Suppose we wish to find the values of x, y, z, for which $f(x, y, z)$ is stationary (maximum and minimum).

For this purpose, we construct an auxiliary equation

$$f(x, y, z) + \lambda\phi(x, y, z) = 0 \qquad \ldots(2)$$

Differentiating Eq. (2) partially w.r.t. x, y, z,

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \qquad \ldots(3)$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \qquad \ldots(4)$$

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \qquad \ldots(5)$$

Eliminating λ from Eqs (3), (4) and (5), the values of x, y, and z are obtained for which $f(x, y, z)$ has stationary value. This method of obtaining stationary values of $f(x, y, z)$ is called Lagrange's method of undetermined multipliers, and equations (3), (4) and (5) are called *Lagrange's equations*. The term λ is called *undetermined multiplier.*

Example 1

Find the minimum value of $x^2 + y^2$, subject to the condition $ax + by = c$.

Solution

Let $f(x, y) = x^2 + y^2 \qquad \ldots(1)$

$ax + by = c \qquad \ldots(2)$

Let $\phi(x, y) = ax + by - c = 0$

Let the auxiliary equation be

$$(x^2 + y^2) + \lambda(ax + by - c) = 0 \qquad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t x,

$$2x + \lambda a = 0$$

$$\lambda = -\frac{2x}{a} \qquad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t y,

$$2y + \lambda b = 0$$

$$\lambda = -\frac{2y}{b} \qquad \ldots(5)$$

From Eqs (4) and (5),

$$\frac{2x}{a} = \frac{2y}{b}$$

$$y = \frac{b}{a}x$$

Substituting y in Eq. (2),

$$ax + b\left(\frac{b}{a}x\right) = c$$

$$ax + \frac{b^2}{a}x = c$$

$$(a^2 + b^2)\,x = ac$$

$$x = \frac{ac}{a^2 + b^2}$$

$$\therefore\ y = \frac{b}{a}\left(\frac{ac}{a^2 + b^2}\right) = \frac{bc}{a^2 + b^2}$$

Minimum value of $x^2 + y^2 = \dfrac{a^2c^2}{(a^2 + b^2)^2} + \dfrac{b^2c^2}{(a^2 + b^2)^2}$

Example 2

Find the minimum values of x^2yz^3, subject to the condition $2x + y + 3z = a$.

[Summer 2014]

Solution

Let $$f(x, y, z) = x^2yz^3 \qquad \ldots(1)$$

$$2x + y + 3z = a \qquad \ldots(2)$$

Let $$\phi(x, y, z) = 2x + y + 3z - a = 0$$

Let the auxiliary equation be

$$x^2yz^3 + \lambda(2x + y + 3z - a) = 0 \qquad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t x,

$$2xyz^3 + 2\lambda = 0$$

$$\lambda = -xyz^3 \qquad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t y,

$$x^2z^3 + \lambda = 0$$

$$\lambda = -x^2z^3 \qquad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t z,

$$3x^2yz^2 + 3\lambda = 0$$

$$\lambda = -x^2yz^2 \qquad \ldots(6)$$

From Eqs (4), (5), and (6),

$$xyz^3 = x^2z^3 = x^2yz^2$$
$$yz = xz = xy$$
$$\therefore\ y = x \text{ and } z = y$$

Substituting $y = z = x$ in Eq. (2),

$$2x + x + 3x = a$$
$$6x = a$$
$$x = \frac{a}{6}$$
$$\therefore\ y = \frac{a}{6}, z = \frac{a}{6}$$

Minimum value of $x^2yz^3 = \left(\frac{a}{6}\right)^2\left(\frac{a}{6}\right)\left(\frac{a}{6}\right)^3 = \left(\frac{a}{6}\right)^6$

Example 3

Find the maximum value of $f = x^2y^3z^4$, subject to the condition $x + y + z = 5$.

Solution

Let $f(x, y, z) = x^2y^3z^4 \qquad \ldots(1)$

$x + y + z = 5 \qquad \ldots(2)$

Let $\phi(x, y, z) = x + y + z - 5 = 0$

Let the auxiliary equation be

$$x^2y^3z^4 + \lambda(x + y + z - 5) = 0 \qquad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t x,

$$2xy^3z^4 + \lambda = 0$$
$$\lambda = -2xy^3z^4 \qquad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t y,

$$3x^2y^2z^4 + \lambda = 0$$
$$\lambda = -3x^2y^2z^4 \qquad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t z,

$$4x^2y^3z^3 + \lambda = 0$$
$$\lambda = -4x^2y^3z^3 \qquad \ldots(6)$$

From Eqs (4), (5), and (6),

$$2xy^3z^4 = 3x^2y^2z^4 = 4x^2y^3z^3$$

$$2yz = 3xz = 4xy$$

$$\therefore \; y = \frac{3}{2}x \quad \text{and} \quad z = 2x$$

Substituting y and z in Eq. (2),

$$x + \frac{3}{2}x + 2x = 5$$

$$9x = 10$$

$$x = \frac{10}{9}$$

$$\therefore \; y = \frac{3}{2}x = \frac{3}{2}\left(\frac{10}{9}\right) = \frac{5}{3}$$

and

$$z = 2x = 2\left(\frac{10}{9}\right) = \frac{20}{9}$$

Maximum value of $x^2y^3z^4 = \left(\frac{10}{9}\right)^2\left(\frac{5}{3}\right)^3\left(\frac{20}{9}\right)^4 = \frac{(2^{10})(5^9)}{3^{15}}$

Example 4

Find the maximum value of $x^m y^n z^p$ when $x + y + z = a$.

Solution

Let $$f(x, y, z) = x^m y^n z^p \qquad \ldots(1)$$

$$x + y + z = a \qquad \ldots(2)$$

Let $$\phi(x, y, z) = x + y + z - a = 0$$

Let the auxiliary equation be

$$x^m y^n z^p + \lambda(x + y + z - a) = 0 \qquad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t x,

$$mx^{m-1}y^n z^p + \lambda = 0$$

$$\lambda = -mx^{m-1}y^n z^p \qquad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t y,

$$ny^{n-1}x^m z^p + \lambda = 0$$

$$\lambda = -ny^{n-1}x^m z^p \qquad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t z,

$$px^m y^n z^{p-1} + \lambda = 0$$

$$\lambda = -px^m y^n z^{p-1} \qquad \ldots(6)$$

From Eqs (4), (5) and (6),

$$mx^{m-1}y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}$$

$$\frac{m}{x} = \frac{n}{y} = \frac{p}{z}$$

$$\therefore \quad y = \frac{n}{m}x \quad \text{and} \quad z = \frac{p}{m}x$$

Substituting y and z in Eq. (2),

$$x + \frac{n}{m}x + \frac{p}{m}x = a$$

$$x = \frac{am}{m+n+p}$$

$$\therefore \quad y = \frac{n}{m}x = \frac{an}{m+n+p}$$

and
$$z = \frac{p}{m}x = \frac{ap}{m+n+p}$$

Maximum value of $x^m y^n z^p = \dfrac{a^{m+n+p} m^m n^n p^p}{(m+n+p)^{m+n+p}}$

Example 5

Find the minimum value of $x^2 + y^2 + z^2$ with the constraint $xy + yz + zx = 3a^2$.

Solution

Let
$$f(x, y, z) = x^2 + y^2 + z^2 \qquad \ldots(1)$$

$$xy + yz + zx = 3a^2 \qquad \ldots(2)$$

Let
$$\phi(x, y, z) = xy + yz + zx - 3a^2$$

Let the auxiliary equation be

$$(x^2 + y^2 + z^2) + \lambda(xy + yz + zx - 3a^2) = 0 \qquad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t x,

$$2x + \lambda(y + z) = 0$$

$$\lambda = -\frac{2x}{y+z} \qquad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t. y,

$$2y + \lambda(x + z) = 0$$

$$\lambda = -\frac{2y}{z+x} \qquad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t. z,

$$2z + \lambda(y + x) = 0$$

$$\lambda = -\frac{2z}{x+y} \qquad \ldots(6)$$

From Eqs (4), (5) and (6),

$$\frac{2x}{y+z} = \frac{2y}{z+x} = \frac{2z}{x+y} = \frac{2x+2y+2z}{y+z+z+x+x+y} = 1$$

$$2x - y - z = 0$$

$$-x + 2y - z = 0$$

$$-x - y + 2z = 0$$

Solving these equations,

$$x = y = z$$

Substituting $y = z = x$ in Eq. (2),

$$3x^2 = 3a^2$$

$$x = \pm a$$

$$x = y = z = \pm a$$

Minimum value of $x^2 + y^2 + z^2 = 3a^2$

Example 6

Using Lagrange's method of multipliers, show that the stationary value of $a^3x^2 + b^3y^2 + c^3z^2$ where $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ occur at $x=\frac{a+b+c}{a}, y=\frac{a+b+c}{b}, z=\frac{a+b+c}{c}$.

Solution

Let
$$f(x, y, z) = a^3x^2 + b^3y^2 + c^3z^2 \qquad \ldots(1)$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1 \qquad \ldots(2)$$

Let
$$\phi(x, y, z) = \frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1=0$$

Let the auxiliary equation be

$$\left(a^3x^2+b^3y^2+c^3z^2\right)+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)=0 \qquad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t. x,

$$2a^3x - \frac{\lambda}{x^2} = 0$$

$$2a^3x^3 - \lambda = 0$$

$$\lambda = 2a^3x^3 \quad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t. y,

$$2b^3y - \frac{\lambda}{y^2} = 0$$

$$2b^3y^3 - \lambda = 0$$

$$\lambda = 2b^3y^3 \quad \text{...(5)}$$

Differentiating Eq. (3) partially w.r.t. z,

$$2c^3z - \frac{\lambda}{z^2} = 0$$

$$2c^3z^3 - \lambda = 0$$

$$\lambda = 2c^3z^3 \quad \text{...(6)}$$

From Eqs (3), (4) and (5),

$$2a^3x^3 = 2b^3y^3 = 2c^3z^3$$

$$ax = by = cz$$

$$\therefore y = \frac{ax}{b} \quad \text{and} \quad z = \frac{ax}{c}$$

Substituting y and z in Eq. (2),

$$\frac{1}{x} + \frac{b}{ax} + \frac{c}{ax} = 1$$

$$\frac{a+b+c}{ax} = 1$$

$$x = \frac{a+b+c}{a}$$

$$\therefore y = \frac{ax}{b} = \frac{a+b+c}{b}$$

and

$$z = \frac{ax}{c} = \frac{a+b+c}{c}$$

Hence, the stationary value of $a^3x^2 + b^3y^2 + c^3z^2$ occurs at

$$x = \frac{a+b+c}{a}, y = \frac{a+b+c}{b}, z = \frac{a+b+c}{c}$$

Example 7

Find the point on the plane $ax + by + cz = p$ at which the function $f = x^2 + y^2 + z^2$ has a minimum value and find this minimum value of f.

[Summer 2015]

Solution

Let $$f(x, y, z) = x^2 + y^2 + z^2 \quad \ldots(1)$$

$$ax + by + cz = p \quad \ldots(2)$$

Let $$\phi(x, y, z) = ax + by + cz - p = 0$$

Let the auxiliary equation be

$$(x^2 + y^2 + z^2) + \lambda(ax + by + cz - p) = 0 \quad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t x,

$$2x + \lambda a = 0$$

$$\lambda = -\frac{2x}{a} \quad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t y,

$$2y + \lambda b = 0$$

$$\lambda = -\frac{2y}{b} \quad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t z,

$$2z + \lambda c = 0$$

$$\lambda = -\frac{2z}{c} \quad \ldots(6)$$

From Eqs (4), (5) and (6),

$$\frac{2x}{a} = \frac{2y}{b} = \frac{2z}{c}$$

$$y = \frac{bx}{a} \text{ and } z = \frac{cx}{a}$$

Substituting y and z in Eq. (2),

$$ax + \frac{b^2 x}{a} + \frac{c^2 x}{a} = p$$

$$x = \frac{ap}{a^2 + b^2 + c^2}$$

$$\therefore y = \frac{bp}{a^2 + b^2 + c^2}$$

and $$z = \frac{cp}{a^2 + b^2 + c^2}$$

Thus, $(x^2 + y^2 + z^2)$ is minimum at $\left(\frac{ap}{a^2 + b^2 + c^2}, \frac{bp}{a^2 + b^2 + c^2}, \frac{cp}{a^2 + b^2 + c^2}\right)$

Minimum value of $x^2 + y^2 + z^2 = \frac{a^2 p^2}{(a^2 + b^2 + c^2)^2} + \frac{b^2 p^2}{(a^2 + b^2 + c^2)^2} + \frac{c^2 p^2}{(a^2 + b^2 + c^2)^2}$

$$= \frac{p^2(a^2 + b^2 + c^2)}{(a^2 + b^2 + c^2)^2} = \frac{p^2}{a^2 + b^2 + c^2}$$

Example 8

Find the length of the shortest line from the point $\left(0, 0, \frac{25}{9}\right)$ *to the surface* $z = xy$.

Solution

Let (x, y, z) be a point on the surface $z = xy$.

The distance d between (x, y, z) and $\left(0, 0, \frac{25}{9}\right)$ is

$$d = \sqrt{x^2 + y^2 + \left(z - \frac{25}{9}\right)^2}$$

$$d^2 = x^2 + y^2 + \left(z - \frac{25}{9}\right)^2$$

Let $$f(x, y, z) = d^2 = x^2 + y^2 + \left(z - \frac{25}{9}\right)^2 \quad \text{...(1)}$$

$$z = xy \quad \text{...(2)}$$

Let $$\phi(x, y, z) = z - xy = 0$$

Let the auxiliary equation be

$$\left[x^2 + y^2 + \left(z - \frac{25}{9}\right)^2\right] + \lambda(z - xy) = 0 \quad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t. x,

$$2x + \lambda(-y) = 0$$

$$\lambda = \frac{2x}{y} \quad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t. y,

$$2y + \lambda(-x) = 0$$

$$\lambda = \frac{2y}{x} \quad \text{...(5)}$$

Differentiating Eq. (3) partially w.r.t. z,

$$2\left(z - \frac{25}{9}\right) + \lambda = 0$$

$$\lambda = -2\left(z - \frac{25}{9}\right) \quad \text{...(6)}$$

From Eqs (4) and (5),

$$\frac{2x}{y}=\frac{2y}{x}$$
$$x^2=y^2$$
$$x=\pm y \qquad \text{...(6)}$$

Substituting $x = y$, in Eq. (4),

$$\lambda = 2$$

Substituting $\lambda = 2$ in Eq. (6),

$$2=-2\left(z-\frac{25}{9}\right)$$
$$-1=z-\frac{25}{9}$$
$$z=-1+\frac{25}{9}=\frac{16}{9}$$

Substituting $y = x$ and $z=\frac{16}{9}$ in Eq. (2),

$$\frac{16}{9}=x^2$$
$$x=\pm\frac{4}{3}$$
$$\therefore y=\pm\frac{4}{3}$$

Similarly, when $x = -y$, $z=\frac{34}{9}$. But this gives a complex value of x and y.

Thus $f(x, y, z)$, i.e., d^2 is minimum when $x=\pm\frac{4}{3}, y=\pm\frac{4}{3}, z=\frac{16}{9}$.

Minimum distance $$d=\sqrt{\left(\frac{4}{3}\right)^2+\left(\frac{4}{3}\right)^2+\left(\frac{16}{9}-\frac{25}{9}\right)^2}=\frac{\sqrt{41}}{3}$$

Hence, the length of the shortest line from $\left(0, 0, \frac{25}{9}\right)$ to the surface $z = xy$ is $\frac{\sqrt{41}}{3}$.

Example 9

Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube. **[Summer 2016]**

Solution

Let $2x$, $2y$, $2z$ be the length, breadth and height of the rectangular solid.

Let r be the radius of the sphere.

Volume of solid, $$V = 8xyz \quad \text{...(1)}$$

Equation of the sphere, $$x^2 + y^2 + z^2 = r^2 \quad \text{...(2)}$$

Let $$\phi(x, y, z) = x^2 + y^2 + z^2 - r^2 = 0$$

Let the auxiliary equation be

$$8xyz + \lambda(x^2 + y^2 + z^2 - r^2) = 0 \quad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t. x,

$$8yz + \lambda \cdot 2x = 0$$
$$\lambda = -\frac{4yz}{x} \quad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t y,

$$8xz + \lambda \cdot 2y = 0$$
$$\lambda = -\frac{4xz}{y} \quad \text{...(5)}$$

Differentiating Eq. (3) partially w.r.t z,

$$8xy + \lambda \cdot 2z = 0$$
$$\lambda = -\frac{4xy}{z} \quad \text{...(6)}$$

From Eqs (4), (5) and (6),

$$\frac{4yz}{x} = \frac{4xz}{y} = \frac{4xy}{z}$$
$$y^2 = x^2 \quad \text{and} \quad z^2 = y^2$$
$$x^2 = y^2 = z^2$$
$$x = y = z$$

Hence, the rectangular solid is a cube.

Example 10

Find the minimum and maximum distance from the point (1, 2, 2) *to the sphere* $x^2 + y^2 + z^2 = 36$.

Solution

Let (x, y, z) be any point on the sphere. Its distance D from the point (1, 2, 2) is

$$D = \sqrt{(x-1)^2 + (y-2)^2 + (z-2)^2}$$

Let $$D^2 = f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-2)^2 \quad \text{...(1)}$$

$$x^2 + y^2 + z^2 = 36 \quad \text{...(2)}$$

Let $$\phi(x, y, z) = x^2 + y^2 + z^2 - 36$$

Let the auxiliary equation be

$$\left[(x-1)^2+(y-2)^2+(z-2)^2\right]+\lambda(x^2+y^2+z^2-36)=0 \quad ...(3)$$

Differentiating Eq. (3) partially w.r.t x,

$$2(x-1)+\lambda(2x)=0$$

$$\lambda=-\frac{x-1}{x}=-1+\frac{1}{x} \quad ...(4)$$

Differentiating Eq. (3) partially w.r.t y,

$$2(y-2)+\lambda(2y)=0$$

$$\lambda=-\frac{y-2}{y}=-1+\frac{2}{y} \quad ...(5)$$

Differentiating Eq. (3) partially, w.r.t. z,

$$2(z-2)+\lambda(2z)=0$$

$$\lambda=-\frac{z-2}{z}=-1+\frac{2}{z} \quad ...(6)$$

From Eqs (4), (5) and (6),

$$-1+\frac{1}{x}=-1+\frac{2}{y}=-1+\frac{2}{z}$$

$$\frac{1}{x}=\frac{2}{y}=\frac{2}{z}$$

$$y=2x \quad \text{and} \quad z=2x$$

Substituting y and z in Eq. (2),

$$x^2+4x^2+4x^2=36$$

$$9x^2=36$$

$$x^2=4$$

$$x=\pm 2$$

$$\therefore y=\pm 4$$

and $$z=\pm 4$$

Minimum distance $=\sqrt{(2-1)^2+(4-2)^2+(4-2)^2}=\sqrt{1+4+4}=3$

Maximum distance $=\sqrt{(-2-1)^2+(-4-2)^2+(-4-2)^2}=\sqrt{9+36+36}=9$

Example 11

A rectangular box open at the top is to have a volume of 32 cubic units. Find the dimensions of the box requiring least material for its construction.

[Winter 2016, 2014]

Solution

Let x, y, z be the dimensions of the box.
The box is open at the top.

Surface area $S = xy + 2xz + 2yz$ ...(1)

Volume $V = xyz = 32$ ...(2)

Let $\phi(x, y, z) = xyz - 32$

Let the auxiliary equation be

$$(xy + 2xz + 2yz) + \lambda(xyz - 32) = 0 \quad ...(3)$$

Differentiating Eq. (3) partially w.r.t. x,

$$y + 2z + \lambda yz = 0$$

$$\lambda = -\frac{y+2z}{yz} = -\frac{1}{z} - \frac{2}{y} \quad ...(4)$$

Differentiating Eq. (3) partially w.r.t. y,

$$x + 2z + \lambda xz = 0$$

$$\lambda = -\frac{x+2z}{xz} = -\frac{1}{z} - \frac{2}{x} \quad ...(5)$$

Differentiating Eq. (3) partially w.r.t. z,

$$2x + 2y + \lambda xy = 0$$

$$\lambda = -\frac{2x+2y}{xy} = -\frac{2}{y} - \frac{2}{x} \quad ...(6)$$

From Eqs (4), (5) and (6),

$$-\frac{1}{z} - \frac{2}{y} = -\frac{1}{z} - \frac{2}{x} = -\frac{2}{y} - \frac{2}{x}$$

$$\frac{2}{y} = \frac{2}{x} \quad \text{and} \quad \frac{1}{z} = \frac{2}{x}$$

$$y = x \quad \text{and} \quad z = \frac{x}{2}$$

Substituting y and z in Eq. (2),

$$x(x)\left(\frac{x}{2}\right) = 32$$

$$x^3 = 64$$

$$x = 4$$

$$\therefore y = 4, \quad z = 2$$

Hence, dimensions of the box requiring least material for its construction are 4, 4, 2.

Example 12

A rectangular box without a lid is to be made from 12 m² *of cardboard. Find the maximum volume of such a box.* **[Winter 2013, 2015]**

Solution

Let x, y, z be the dimensions of the box.
The box is open at the top.

Volume $$V = xyz \quad \text{...(1)}$$

Surface area $$S = xy + 2xz + 2yz = 12 \quad \text{...(2)}$$

Let $$\phi(x, y, z) = xy + 2xz + 2yz - 12$$

Let the auxiliary equation be

$$xyz + \lambda(xy + 2xz + 2yz - 12) = 0 \quad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t. x,

$$yz + \lambda(y + 2z) = 0$$

$$\lambda = -\frac{yz}{y+2z}$$

$$\frac{1}{\lambda} = -\frac{y+2z}{yz} = -\frac{1}{z} - \frac{2}{y} \quad \text{...(4)}$$

Differentiating Eq. (4) partially w.r.t. y,

$$xz + \lambda(x + 2z) = 0$$

$$\lambda = -\frac{xz}{x+2z}$$

$$\frac{1}{\lambda} = -\frac{x+2z}{xz} = -\frac{1}{z} - \frac{2}{x} \quad \text{...(5)}$$

Differentiating Eq. (3) partially w.r.t. z,

$$xy + \lambda(2x + 2y) = 0$$

$$\lambda = -\frac{xy}{2(x+y)}$$

$$\frac{1}{\lambda} = -\frac{2x+2y}{xy} = -\frac{2}{y} - \frac{2}{x} \quad \text{...(6)}$$

From Eqs (4), (5) and (6),

$$-\frac{1}{z} - \frac{2}{y} = -\frac{1}{z} - \frac{2}{x} = -\frac{2}{y} - \frac{2}{x}$$

$$\frac{2}{y} = \frac{2}{x} \quad \text{and} \quad \frac{1}{z} = \frac{2}{x}$$

$$y = x \quad \text{and} \quad z = \frac{x}{2}$$

Substituting y and z in Eq. (2),

$$x(x) + 2x\left(\frac{x}{2}\right) + 2x\left(\frac{x}{2}\right) = 12$$

$$3x^2 = 12$$

$$x^2 = 4$$

$$x = 2$$

$$\therefore \; y = 2, \; z = 1$$

Hence, maximum volume = $xyz = 2(2)(1) = 4$ m^3

Example 13

A wire of length b is cut into two parts which are bent in the form of a square and circle respectively. Find the least value of the sum of the areas so found.

Solution

Let the piece of length x be bent in the form of a square so that each side is $\frac{x}{4}$.

The area of the square, $A_1 = \frac{x}{4} \cdot \frac{x}{4} = \frac{x^2}{16}$.

Suppose a piece of length y is bent in the form of a circle of radius r; so perimeter of the circle is y.

$$2\pi r = y$$

$$r = \frac{y}{2\pi}$$

The area of the circle, $A_2 = \pi\left(\frac{y}{2\pi}\right)^2 = \frac{y^2}{4\pi}$.

Let sum of the areas be given as

$$f(x, y) = \frac{x^2}{16} + \frac{y^2}{4\pi} \qquad \text{...(1)}$$

Also, $\quad x + y = b \qquad$ ...(2)

Let $\quad \phi(x, y) = x + y - b$

Let the auxiliary equation be

$$\left(\frac{x^2}{16} + \frac{y^2}{4\pi}\right) + \lambda(x + y - b) = 0 \qquad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t. x,

$$\frac{2x}{16} + \lambda = 0$$

$$\lambda = -\frac{x}{8} \qquad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t. y,

$$\frac{2y}{4\pi} + \lambda = 0$$

$$\lambda = -\frac{y}{2\pi} \qquad \text{...(5)}$$

From Eqs (4) and (5),

$$\frac{x}{8} = \frac{y}{2\pi}$$

$$y = \frac{\pi}{4}x$$

Substituting y in Eq. (2),

$$x + \frac{\pi}{4}x = b$$

$$x = \frac{4b}{4+\pi}$$

$$\therefore y = \frac{\pi b}{4+\pi}$$

Hence, the least value of the sum of the areas is

$$\frac{x^2}{16} + \frac{y^2}{4\pi} = \frac{1}{16}\left(\frac{4b}{4+\pi}\right)^2 + \frac{1}{4\pi}\left(\frac{\pi b}{4+\pi}\right)^2$$

$$= \frac{b^2}{(4+\pi)^2}\left(1 + \frac{\pi^2}{4\pi}\right)$$

$$= \frac{b^2\pi(4+\pi)}{4\pi(4+\pi)^2}$$

$$= \frac{b^2}{4(\pi+4)}$$

Example 14

A closed rectangular box has length twice its breadth and has constant volume V. Determine the dimensions of the box requiring least surface area.

Solution

Let x be the breadth and y be the height of the rectangular box so length of the box will be $2x$.

Surface area of the box

$$S = 2(2x \cdot x + x \cdot y + y \cdot 2x) = 4x^2 + 6xy$$

Let $f(x, y) = 4x^2 + 6xy$ ...(1)

Volume of the box $V = x \cdot 2x \cdot y = 2x^2y$ ...(2)

Let $\phi(x, y) = 2x^2y - V$

Let the auxiliary equation be

$$\left(4x^2 + 6xy\right) + \lambda(2x^2y - V) = 0 \quad ...(3)$$

Differentiating Eq. (3) partially w.r.t. x,

$$8x + 6y + \lambda(4xy) = 0$$

$$\lambda = -\frac{4x + 3y}{2xy} \quad ...(4)$$

Differentiating Eq. (3) partially w.r.t. y,

$$6x + \lambda(2x^2) = 0$$

$$\lambda = -\frac{3}{x} \quad ...(5)$$

From Eqs (4) and (5),

$$\frac{4x + 3y}{2xy} = \frac{3}{x}$$

$$4x + 3y = 6y$$

$$y = \frac{4x}{3}$$

Substituting y in Eq. (2),

$$2x^2 \cdot \frac{4x}{3} = V$$

$$x^3 = \frac{3V}{8}$$

$$x = \left(\frac{3V}{8}\right)^{\frac{1}{3}}$$

$$\therefore y = \frac{4}{3}\left(\frac{3V}{8}\right)^{\frac{1}{3}} = \left(\frac{8V}{9}\right)^{\frac{1}{3}}$$

Hence, the dimensions of the box requiring least surface area are $2\left(\frac{3V}{8}\right)^{\frac{1}{3}}, \left(\frac{3V}{8}\right)^{\frac{1}{3}}, \left(\frac{8V}{9}\right)^{\frac{1}{3}}$.

Example 15

Show that if the perimeter of a triangle is a constant, the triangle has maximum area when it is equilateral.

Solution

Let x, y and z be the sides of the triangle.

Perimeter of the triangle $s = \dfrac{x+y+z}{2}$

Area of the triangle $A = \sqrt{s(s-x)(s-y)(s-z)}$

Let $$f(x, y, z) = A^2 = s(s-x)(s-y)(s-z) \quad \ldots(1)$$

Also, $$x + y + z = 2s \quad \ldots(2)$$

Let $$\phi(x, y, z) = x + y + z - 2s$$

Let the auxiliary equation be

$$[s(s-x)(s-y)(s-z)] + \lambda(x+y+z-2s) = 0 \quad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t x,

$$-s(s-y)(s-z) + \lambda = 0$$

$$\lambda = s(s-y)(s-z) \quad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t y,

$$-s(s-x)(s-z) + \lambda = 0$$

$$\lambda = s(s-x)(s-z) \quad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t z,

$$-s(s-x)(s-y) + \lambda = 0$$

$$\lambda = s(s-x)(s-y) \quad \ldots(6)$$

From Eqs (4), (5) and (6),

$$s(s-y)(s-z) = s(s-x)(s-z) = s(s-x)(s-y)$$

$$s-y = s-x \quad \text{and} \quad s-z = s-y$$

$$y = x \quad \text{and} \quad z = y$$

$$\therefore x = y = z$$

Hence, the triangle is equilateral.

Example 16

The temperature $u(x, y, z)$ at any point in space is $u = 400\,xyz^2$. Find the highest temperature on surface of the sphere $x^2 + y^2 + z^2 = 1$.

Solution

Let $$f(x, y, z) = u = 400\, xyz^2 \quad \ldots(1)$$

$$x^2 + y^2 + z^2 = 1 \quad \ldots(2)$$

Let $$\phi(x, y, z) = x^2+y^2+z^2-1=0$$

Let the auxiliary equation be

$$400\, xyz^2 + \lambda(x^2 + y^2 + z^2 - 1) = 0 \quad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t. x,

$$\lambda = -\frac{200yz^2}{x} \quad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t. y,

$$400xz^2 + \lambda(2y) = 0$$

$$\lambda = -\frac{200xz^2}{y} \quad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t. z,

$$800xyz + \lambda(2z) = 0$$

$$\lambda = -400xy$$

$$\lambda = -400xy \quad \ldots(6)$$

From Eqs (4), (5) and (6),

$$\frac{200yz^2}{x} = \frac{200xz^2}{y} = 400xy$$

$$200y^2z^2 = 200x^2z^2 = 400x^2y^2$$

$$\frac{1}{x^2} = \frac{1}{y^2} = \frac{2}{z^2}$$

$$x^2 = y^2 = \frac{z^2}{2}$$

Substituting y^2 and z^2 in Eq. (2),

$$x^2 + x^2 + 2x^2 = 1$$

$$4x^2 = 1$$

$$x^2 = \frac{1}{4}$$

$$x = \pm\frac{1}{2}$$

$$\therefore y = \pm\frac{1}{2}, \quad z = \pm\frac{1}{\sqrt{2}}$$

Considering positive sign,

$$x = \frac{1}{2}, \quad y = \frac{1}{2}, \quad z = \frac{1}{\sqrt{2}}$$

Highest temperature on the surface of the sphere

$$u = 400xyz^2 = 400\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 50.$$

Example 17

Use the method of the Lagrange's multipliers to find volume of largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

Solution

Let $2x$, $2y$, $2z$ be the length, breadth and height of the rectangular parallelepiped inscribed in the ellipsoid.

Volume of the parallelepiped, $V = (2x)\,(2y)\,(2z) = 8xyz$.

Let $$f(x, y, z) = 8\,xyz \qquad \text{...(1)}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad \text{...(2)}$$

Let $$\phi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

Let the auxiliary equation be

$$8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) = 0 \qquad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t. x,

$$8yz + \lambda\frac{2x}{a^2} = 0$$

$$\lambda = -\frac{4yza^2}{x} \qquad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t. y,

$$8xz + \lambda \frac{2y}{b^2} = 0$$

$$\lambda = -\frac{4xzb^2}{y} \qquad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t. z,

$$8xy + \lambda \frac{2z}{c^2} = 0$$

$$\lambda = -\frac{4xyc^2}{z} \qquad \ldots(6)$$

From Eqs (4), (5) and (6),

$$\frac{4yza^2}{x} = \frac{4xzb^2}{y} = \frac{4xyc^2}{z}$$

$$\frac{a^2}{x^2} = \frac{b^2}{y^2} = \frac{c^2}{z^2}$$

$$\therefore y^2 = \frac{b^2}{a^2}x^2 \quad \text{and} \quad z^2 = \frac{c^2}{a^2}x^2$$

Substituting y^2, z^2 in Eq. (2),

$$\frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1$$

$$\frac{3x^2}{a^2} = 1$$

$$x^2 = \frac{a^2}{3}$$

$$x = \pm \frac{a}{\sqrt{3}}$$

$$\therefore y = \pm \frac{b}{\sqrt{3}}, \quad z = \pm \frac{c}{\sqrt{3}}$$

Since sides cannot be negative,

$$x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}}, \quad z = \frac{c}{\sqrt{3}}$$

Volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid

$$V = 8xyz = 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right) = \frac{8abc}{3\sqrt{3}}$$

Example 18

Find the minimum distance from origin to the plane $3x + 2y + z = 12$.

Solution

Let (x, y, z) be a point on the plane $3x + 2y + z = 12$.
The distance d between (x, y, z) and origin $(0, 0, 0)$ is

$$d = \sqrt{x^2 + y^2 + z^2}$$
$$d^2 = x^2 + y^2 + z^2$$

Let $$f(x, y, z) = d^2 = x^2 + y^2 + z^2 \qquad ...(1)$$

$$3x + 2y + z = 12 \qquad ...(2)$$

Let $$\phi(x, y, z) = 3x + 2y + z - 12 = 0$$

Let the auxiliary equation be

$$(x^2 + y^2 + z^2) + \lambda(3x + 2y + z - 12) = 0 \qquad ...(3)$$

Differentiating Eq. (3) partially w.r.t. x,

$$2x + \lambda(3) = 0$$
$$\lambda = -\frac{2x}{3} \qquad ...(4)$$

Differentiating Eq. (3) partially w.r.t. y,

$$2y + \lambda(2) = 0$$
$$\lambda = -\frac{2y}{2} = -y \qquad ...(5)$$

Differentiating Eq. (3) partially w.r.t. z,

$$2z + \lambda(1) = 0$$
$$\lambda = -2z \qquad ...(6)$$

From Eqs (3), (4) and (5),

$$\frac{2x}{3} = y = 2z$$
$$y = \frac{2x}{3}$$
$$z = \frac{x}{3}$$

Substituting y, z in Eq. (2),

$$3x + \frac{4x}{3} + \frac{x}{3} = 12$$

$$\frac{14x}{3} = 12$$

$$\therefore \quad x = \frac{18}{7},\ y = \frac{12}{7},\ z = \frac{6}{7}$$

The minimum distance is

$$d = \sqrt{\left(\frac{18}{7}\right)^2 + \left(\frac{12}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \frac{\sqrt{504}}{7}$$

Example 19

Divide 24 into three parts such that the continued product of the first, square of the second and cube of the third may be maximum.

Solution

Let x, y and z be three parts of 24.

Let $$f(x, y, z) = xy^2 z^3 \qquad \text{...(1)}$$

Let $$x + y + z = 24 \qquad \text{...(2)}$$

$$\phi(x, y, z) = x + y + z - 24 = 0$$

Let the auxiliary equation be

$$xy^2 z^3 + \lambda (x + y + z - 24) = 0 \qquad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t. x,

$$y^2z^3 + \lambda = 0$$
$$\lambda = -y^2z^3 \qquad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t. y,

$$2xyz^3 + \lambda = 0$$
$$\lambda = -2xyz^3 \qquad \text{...(5)}$$

Differentiating Eq. (3) partially w.r.t. z,

$$3xy^2z^2 + \lambda = 0$$
$$\lambda = -3xy^2z^2 \qquad \text{...(6)}$$

From Eqs (4), (5), (6),

$$y^2z^3 = 2xyz^3 = 3xy^2z^2$$

Dividing by xy^2z^3,

$$\frac{1}{x} = \frac{2}{y} = \frac{3}{z}$$
$$y = 2x,\ z = 3x$$

Substituting *y*, *z* in Eq. (2)

$$x + 2x + 3x = 24$$
$$6x = 24$$
$$x = 4$$
$$\therefore \quad y = 8, z = 12$$

Hence, 4, 8 and 12 are three parts of 24.

Example 20

A space probe in the shape of the ellipsoid $4x^2 + y^2 + 4z^2 = 16$ *enters the earth's atmosphere and its surface begins to heat. After* 1 *hour, the temperature at the point* (*x*, *y*, *z*) *on the surface of the probe* $T(x, y, z) = 8x^2 + 4yz - 16z + 600$. *Find the hottest points on the probe's surface.*

Solution

Let $$f(x, y, z) = 8x^2 + 4yz - 16z + 600 \qquad \text{...(1)}$$

$$4x^2 + y^2 + 4z^2 = 16 \qquad \text{...(2)}$$

Let $$\phi(x, y, z) = 4x^2 + y^2 + 4z^2 - 16 = 0$$

Let the auxiliary equation be

$$(8x^2 + 4yz - 16z + 600) + \lambda(4x^2 + y^2 + 4z^2 - 16) = 0 \qquad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t. *x*,

$$16x + \lambda(8x) = 0$$
$$\lambda = -2 \qquad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t. *y*,

$$4z + \lambda(2y) = 0$$
$$\lambda = -\frac{2z}{y} \qquad \text{...(5)}$$

Differentiating Eq. (3) partially w.r.t. *z*,

$$4y - 16 + \lambda(8z) = 0$$
$$\lambda = \frac{16 - 4y}{8z} = \frac{4 - y}{2z} \qquad \text{...(6)}$$

From Eqs (4) and (5),

$$-2 = -\frac{2z}{y}$$
$$y = z \qquad \text{...(7)}$$

From Eqs (4) and (6),

$$-2 = \frac{4-y}{2y}$$

$$-4y = 4 - y$$

$$-3y = 4$$

$$y = -\frac{4}{3}$$

$$z = -\frac{4}{3}$$

Substituting in Eq. (2),

$$4x^2 + \frac{16}{9} + \frac{64}{9} = 16$$

$$4x^2 = \frac{64}{9}$$

$$x^2 = \frac{16}{9}$$

$$x = \pm\frac{4}{3}$$

The hottest points on the probe's surface are $\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$

Example 21

Find the points on the surface $z^2 = xy + 1$ *nearest to the origin.*

Solution

Let (x, y, z) be any point on the surface $z^2 = xy + 1$.
The distance d between (x, y, z) and origin $(0, 0, 0)$ is

$$d = \sqrt{x^2 + y^2 + z^2}$$

$$d^2 = x^2 + y^2 + z^2$$

Let $$f(x, y, z) = d^2 = x^2 + y^2 + z^2 \qquad \ldots(1)$$

$$z^2 = xy + 1 \qquad \ldots(2)$$

Let $$\phi(x, y, z) = z^2 - xy - 1 = 0$$

Let the auxiliary equation be

$$(x^2 + y^2 + z^2) + \lambda(z^2 - xy - 1) = 0 \qquad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t. x,

$$2x - \lambda y = 0$$

$$\lambda = \frac{y}{2x} \qquad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t. y,

$$2y - \lambda x = 0$$

$$\lambda = \frac{x}{2y} \qquad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t. z,

$$2z + \lambda(2z) = 0$$

$$\lambda = -1 \qquad \ldots(6)$$

From Eqs (4) and (6),

$$\frac{y}{2x} = -1$$

$$y = -2x$$

From Eqs (5) and (6),

$$\frac{x}{2y} = -1$$

$$x = -2y = -2(-2x) = 4x$$

$$x = 0$$

$$\therefore \quad y = 0$$

Substituting in Eq. (2)

$$z^2 = 1$$

$$z = \pm 1$$

The nearest points on the surface from the origin are (0, 0, ±1).

Example 22

If $u = \frac{x^2}{a^3} + \frac{y^2}{b^3} + \frac{z^2}{c^3}$ *where* $x + y + z = 1$ *then prove that stationary value of* u *is given by* $x = \frac{a^3}{a^3 + b^3 + c^3}$, $y = \frac{b^3}{a^3 + b^3 + c^3}$, $z = \frac{c^3}{a^3 + b^3 + c^3}$.

Solution

Let

$$f(x, y, z) = u = \frac{x^2}{a^3} + \frac{y^2}{b^3} + \frac{z^2}{c^3} \qquad \ldots(1)$$

$$x + y + z = 1 \qquad \ldots(2)$$

Let

$$\phi(x, y, z) = x + y + z - 1 = 0$$

Let the auxiliary equation be

$$\left(\frac{x^2}{a^3}+\frac{y^2}{b^3}+\frac{z^2}{c^3}\right)+\lambda(x+y+z-1)=0 \qquad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t. x,

$$\frac{2x}{a^3}+\lambda=0$$

$$\lambda=-\frac{2x}{a^3} \qquad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t. y,

$$\frac{2y}{b^3}+\lambda=0$$

$$\lambda=-\frac{2y}{b^3} \qquad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t. z,

$$\frac{2z}{c^3}+\lambda=0$$

$$\lambda=-\frac{2z}{c^3} \qquad \ldots(6)$$

From Eqs (4), (5) and (6),

$$\frac{2x}{a^3}=\frac{2y}{b^3}=\frac{2z}{c^3}$$

$$\frac{x}{a^3}=\frac{y}{b^3}=\frac{z}{c^3}$$

$$y=\frac{b^3}{a^3}x, \quad z=\frac{c^3}{a^3}x$$

Substituting y, z in Eq. (2),

$$x+\frac{b^3}{a^3}x+\frac{c^3}{a^3}x=1$$

$$\frac{(a^3+b^3+c^3)x}{a^3}=1$$

$$x=\frac{a^3}{a^3+b^3+c^3}$$

$$\therefore \qquad y=\frac{b^3}{a^3+b^3+c^3}, \quad z=\frac{c^3}{a^3+b^3+c^3}$$

Example 23

If $u = \frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2}$ *where* $x + y + z = 1$ *then find the stationary values.*

Solution

Let $$f(x, y, z) = u = \frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2} \quad \ldots(1)$$

$$x + y + z = 1 \quad \ldots(2)$$

Let $$\phi(x, y, z) = x + y + z - 1 = 0$$

Let the auxiliary equation be

$$\left(\frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2}\right) + \lambda(x + y + z - 1) = 0 \quad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t. x,

$$-\frac{2a^3}{x^3} + \lambda = 0$$

$$\lambda = \frac{2a^3}{x^3} \quad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t. y,

$$-\frac{2b^3}{y^3} + \lambda = 0$$

$$\lambda = \frac{2b^3}{y^3} \quad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t. z,

$$-\frac{2c^3}{z^3} + \lambda = 0$$

$$\lambda = \frac{2c^3}{z^3} \quad \ldots(6)$$

From Eqs (4), (5) and (6),

$$\frac{2a^3}{x^3} = \frac{2b^3}{y^3} = \frac{2c^3}{z^3}$$

$$\frac{a^3}{x^3} = \frac{b^3}{y^3} = \frac{c^3}{z^3}$$

$$\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$$

$$y = \frac{b}{a}x$$
$$z = \frac{c}{a}x$$

Substituting y, z in Eq. (2),

$$x + \frac{b}{a}x + \frac{c}{a}x = 1$$
$$\frac{(a+b+c)}{a}x = 1$$
$$x = \frac{a}{a+b+c}$$
$$\therefore \qquad y = \frac{b}{a+b+c}, z = \frac{c}{a+b+c}$$

Example 24

Prove that the stationary values of $u = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$ *where* $lx + my + nz = 0$ *and* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ *are given by the roots of the equation* $\frac{l^2a^4}{1-a^2u} + \frac{m^2b^4}{1-b^2u} + \frac{n^2c^4}{1-c^2u} = 0$.

Solution

Let
$$f(x, y, z) = u = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \qquad \ldots(1)$$
$$lx + my + nz = 0 \qquad \ldots(2)$$
$$\phi(x, y, z) = lx + my + nz = 0$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad \ldots(3)$$
$$\psi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

Let the auxiliary equation be

$$\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) + \lambda_1(lx + my + nz) + \lambda_2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) = 0 \qquad \ldots(4)$$

Differentiating Eq. (4) partially w.r.t. x,

$$\frac{2x}{a^4}+\lambda_1 l+\lambda_2\left(\frac{2x}{a^2}\right)=0 \qquad \ldots(5)$$

Differentiating Eq. (4) partially w.r.t. y,

$$\frac{2y}{b^4}+\lambda_1 m+\lambda_2\left(\frac{2y}{b^2}\right)=0 \qquad \ldots(6)$$

Differentiating Eq. (4) partially w.r.t. z,

$$\frac{2z}{c^4}+\lambda_1 n+\lambda_2\left(\frac{2z}{c^2}\right)=0 \qquad \ldots(7)$$

Multiplying Eq. (5) by x, Eq. (6) by y, Eq. (7) by z and then adding,

$$2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)+\lambda_1(lx+my+nz)+2\lambda_2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=0$$

$$2u+\lambda_1(0)+2\lambda_2(1)=0$$

$$\lambda_2=-u$$

Substituting $\lambda_2=-u$ in Eq. (5),

$$\frac{2x}{a^4}+\lambda_1 l-\frac{2xu}{a^2}=0$$

$$2x\left(\frac{1-a^2u}{a^4}\right)+\lambda_1 l=0$$

$$x=-\frac{a^4\lambda_1 l}{2(1-a^2u)}$$

Similarly

$$y=-\frac{b^4\lambda_1 m}{2(1-b^2u)}$$

$$z=-\frac{c^4\lambda_1 n}{2(1-c^2u)}$$

Substituting x, y, z in Eq. (2),

$$-\frac{l^2a^4\lambda_1}{2(1-a^2u)}-\frac{m^2b^4\lambda_1}{2(1-b^2u)}-\frac{n^2c^4\lambda_1}{2(1-c^2u)}=0$$

$$\left(\frac{l^2a^4}{1-a^2u}+\frac{m^2b^4}{1-b^2u}+\frac{n^2c^4}{1-c^2u}\right)\lambda_1=0$$

But $\lambda_1\neq 0$

$$\therefore \qquad \frac{l^2a^4}{1-a^2u}+\frac{m^2b^4}{1-b^2u}+\frac{n^2c^4}{1-c^2u}=0$$

Example 25

In a plane triangle ABC, find the extreme values of cos *A* cos *B* cos *C*.

[Summer 2015]

Solution

Let $$f(A, B, C) = \cos A \cos B \cos C \qquad \ldots(1)$$

In a triangle *ABC*,

$$A + B + C = 180° \qquad \ldots(2)$$

Let $$\phi(A, B, C) = A + B + C - 180°$$

Let the auxiliary equation be

$$\cos A \cos B \cos C + \lambda\,(A + B + C - 180°) = 0 \qquad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t. *A*,

$$-\sin A \cos B \cos C + \lambda = 0$$

$$\lambda = \sin A \cos B \cos C \qquad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t. *B*,

$$-\cos A \sin B \cos C + \lambda = 0$$

$$\lambda = \cos A \sin B \cos C \qquad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t. *C*,

$$-\cos A \cos B \sin C + \lambda = 0$$

$$\lambda = \cos A \cos B \sin C \qquad \ldots(6)$$

From Eqs (4), (5) and (6),

$$\sin A \cos B \cos C = \cos A \sin B \cos C = \cos A \cos B \sin C$$

Dividing by cos *A* cos *B* cos *C*,

$$\tan A = \tan B = \tan C$$

$$A = B = C = \frac{\pi}{3}$$

Hence, $$f_{\max} = \cos A \cos B \cos C = \cos\frac{\pi}{3}\cos\frac{\pi}{3}\cos\frac{\pi}{3}$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$$

EXERCISE 8.8

1. Find stationary values of the function $f(x, y, z) = x^2 + y^2 + z^2$, given that $z^2 = xy + 1$.

   [Ans.: (0, 0, –1), (0, 0, 1)]

2. Find the stationary value of $a^3x^2 + b^3y^2 + c^3z^2$ subject to the fulfillment of the condition $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$, given a, b, c are not zero.

   $$\left[\text{Ans.: } x=\frac{1}{a}(a+b+c),\ y=\frac{1}{b}(a+b+c),\ z=\frac{1}{c}(a+b+c)\right]$$

3. Find the largest product of the numbers x, y and z when $x + y + z^2 = 16$.

   $$\left[\text{Ans.: } \frac{4096}{25\sqrt{5}}\right]$$

4. Find the largest product of the numbers x, y and z when $x^2 + y^2 + z^2 = 9$.

   $$\left[\text{Ans.: } 3\sqrt{3}\right]$$

5. Find a point in the plane $x + 2y + 3z = 13$ nearest to the point (1, 1, 1).

   $$\left[\text{Ans.: } \left(\frac{3}{2}, 2, \frac{5}{2}\right)\right]$$

6. Find the shortest distance from the point (1, 2, 2) to the sphere $x^2 + y^2 + z^2 = 36$. **[Ans.:** 3]

7. Find the maximum distance from the origin (0, 0) to the curve $3x^2 + 3y^2 + 4xy - 2 = 0$.

   $$\left[\text{Ans.: } \sqrt{2}\right]$$

8. Decompose a positive number a into three parts so that their product is maximum.

   $$\left[\text{Ans.: } \left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)\right]$$

9. Find the maximum value of $x^m y^n z^p$ when $x + y + z = a$.

   $$\left[\text{Ans.: } \frac{a^{m+n+p} m^m n^n p^p}{(m+n+p)^{m+n+p}}\right]$$

10. Find the dimensions of a rectangular box of maximum capacity whose surface area is given when

    (i) box is open at the top (ii) box is closed

    $$\left[\text{Ans.: (i) } \sqrt{\frac{s}{3}}, \sqrt{\frac{s}{3}}, \frac{1}{2}\sqrt{\frac{s}{3}} \quad \text{(ii) } \sqrt{\frac{s}{6}}, \sqrt{\frac{s}{6}}, \sqrt{\frac{s}{6}}\right]$$

11. Determine the perpendicular distance of the point (a, b, c) from the plane $lx + my + nz = 0$.

$$\left[\text{Ans. : minimum distance } \frac{la + mb + nc}{\sqrt{l^2 + m^2 + n^2}}\right]$$

12. Find the length and breadth of a rectangle of maximum area that can be inscribed in the ellipse $4x^2 + y^2 = 36$.

$$\left[\text{Ans. : } \frac{3\sqrt{2}}{2}, \sqrt{2}, \text{ Area} = 12\right]$$

13. Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid of revolution $4x^2 + 4y^2 + 9z^2 = 36$.

$$\left[\text{Ans. : } 16\sqrt{3}\right]$$

14. Find the extreme volume of $x^2 + y^2 + z^2 + xy + xz + yz$ subject to the conditions $x + y + z = 1$ and $x + 2y + 3z = 3$.

$$\left[\text{Ans. : } \frac{1}{6}, \frac{1}{3}, \frac{5}{6}\right]$$

Points to Remember

Chain Rule

$$\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} \quad \text{or} \quad \frac{df}{du}\frac{\partial u}{\partial x} \quad \text{or} \quad f'(u)\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} \quad \text{or} \quad \frac{df}{du}\frac{\partial u}{\partial y} \quad \text{or} \quad f'(u)\frac{\partial u}{\partial y}$$

Composite Function of One Variable

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

Composite Function of Two Variables

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Implicit Differentiation

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}$$

Gradient and Directional Derivative

Gradient

$$\text{grad } f = \nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$df = |\nabla f||d\bar{r}|\cos\theta$$

Directional Derivative

$$D_a f = \nabla f \cdot \hat{a} = \nabla f \cdot \frac{a}{|a|}$$

Tangent Plane and Normal to a Surface

The equation of the tangent plane at $P(x_0, y_0, z_0)$ to the surface $f(x, y, z) = 0$ is

$$(x - x_0) f_x(x_0, y_0, z_0) + (y - y_0) f_y(x_0, y_0, z_0) + (z - z_0) f_z(x_0, y_0, z_0) = 0$$

The equations of the normal line to the surface at $P(x_0, y_0, z_0)$ are

$$\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)} = \frac{z - z_0}{f_z(x_0, y_0, z_0)}$$

Local Extreme Values (Maximum and Minimum Values)

To determine the maxima and minima (extreme values) of a function $f(x, y)$

Step 1: Solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ simultaneously for x and y.

Step 2: Obtain the values of $r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$.

Step 3: (i) If $rt - s^2 > 0$ and $r < 0$ (or $t < 0$) at (a, b) then $f(x, y)$ is maximum at (a, b) and the maximum value of the function is $f(a, b)$.

(ii) If $rt - s^2 > 0$ and $r > 0$ (or $t > 0$) at (a, b) then $f(x, y)$ is minimum at (a, b) and the minimum value of the function is $f(a, b)$.

(iii) If $rt - s^2 < 0$ at (a, b) then $f(x, y)$ is neither maximum nor minimum at (a, b).

(iv) If $rt - s^2 = 0$ at (a, b) then no conclusion can be made about the extreme values of $f(x, y)$.

Method of Lagrange Multipliers

Let $f(x, y, z)$ be a function of three variables x, y, z, and the variables be connected by the relation

$$\phi(x, y, z) = 0$$

Let $f(x, y, z) + \lambda\phi(x, y, z) = 0$ be an auxiliary equation.
Differentiating this equation partially w.r.t x, y and z

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

Eliminating λ from these equations, the values of x, y and z are obtained for which $f(x, y, z)$ has a stationary (maximum and minimum) value.

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. The value of $\lim\limits_{(x,y)\to(0,0)} \dfrac{x-y}{x+y}$ is

(a) limit does not exist (b) 0
(c) 1 (d) –1

2. The value of $\lim\limits_{(x,y)\to(0,0)} \dfrac{x+\sqrt{y}}{\sqrt{x^2+y}}$, $x \neq 0, y \neq 0$ is

(a) limit does not exist (b) 0
(c) 1 (d) –1

3. The value of $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^4+y^2}$ is

(a) 0 (b) $\dfrac{1}{2}$ (c) 1 (d) does not exist

4. The value of $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{y^2-x^2}$ is

(a) 0 (b) 1 (c) –1 (d) does not exist

5. The value of $\lim\limits_{\substack{x\to 1\\ y\to 1}} \dfrac{8x^2y}{x^2+y^2+5}$ is

(a) $\frac{3}{7}$ (b) $\frac{8}{5}$ (c) $\frac{8}{7}$ (d) $\frac{3}{5}$

6. The value of $\lim\limits_{\substack{x\to 1\\ y\to 2}} \dfrac{4xy}{6x^2+y^2}$ is

(a) $\frac{4}{5}$ (b) $\frac{2}{3}$ (c) $\frac{3}{10}$ (d) $\frac{2}{5}$

7. The value of $\lim\limits_{\substack{x\to 1\\ y\to 0}} \dfrac{2x^2+y}{4x-y}$ is

(a) $\frac{3}{2}$ (b) $\frac{1}{2}$ (c) 1 (d) $\frac{5}{2}$

8. The value of $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{2x^2+y}{4x^2-y}$ is

(a) –1 (b) $\frac{1}{2}$ (c) 1 (d) does not exist

9. The value of $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$ is **[Winter 2015]**

(a) 1 (b) 0 (c) –1 (d) does not exist

10. The value of $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-yx}{x+y}$ is **[Winter 2014]**

(a) 2 (b) 1 (c) 0 (d) –1

11. If $x = r\cos\theta$, $y = r\sin\theta$ then $\dfrac{\partial r}{\partial x} =$ — and $\dfrac{\partial r}{\partial y} =$ —

(a) $\frac{x}{r}, \tan\theta$ (b) $\frac{x}{r}, \frac{y}{r}$ (c) $\tan\theta, \sin\theta$ (d) $\frac{x}{r}, \sin\theta$

12. If $u = \sin(ax+by+cz)$ then $\dfrac{\partial u}{\partial x} =$

(a) $a\cos(ax+by+cz)$ (b) $a\sin(ax+by+cz)$
(c) $b\cos(ax+by+cz)$ (d) $b\sin(ax+by+cz)$

13. If $u = x^2y + y^2z + z^2x$ then $\dfrac{\partial u}{\partial x} + \dfrac{\partial x}{\partial y} + \dfrac{\partial u}{\partial z} =$

(a) $x+y+z$ (b) $(x+y+z)^2$ (c) $\dfrac{1}{x+y+z}$ (d) $\dfrac{1}{(x+y+z)^2}$

14. If $f = x^2 + y^2$ then $\dfrac{\partial^2 f}{\partial x \partial y} =$

(a) 1 (b) 0 (c) –1 (d) 2

15. If $u = \log(x^2 + y^2)$ then $\dfrac{\partial u}{\partial x} =$

(a) $\dfrac{2y}{x^2 + y^2}$ (b) $\dfrac{2}{x^2 + y^2}$ (c) $\dfrac{2x}{x^2 + y^2}$ (b) $\dfrac{y}{x^2 + y^2}$

16. If $u = \sin(x + y)$ then $\dfrac{\partial u}{\partial y} =$

(a) $\sin x$ (b) $\cos(x + y)$ (c) $\tan(x + y)$ (d) $\cos x$

17. If $u = e^x(x\cos y - y\sin y)$ then $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} =$

(a) 1 (b) –1 (c) 0 (d) 2

18. If $u = x^y$ then $\dfrac{\partial u}{\partial x}$ is

(a) 0 (b) yx^{y-1} (c) $x^y \log x$ (d) yx^y

19. If $u = y^x$ then $\dfrac{\partial u}{\partial x} =$ ——— and $\dfrac{\partial u}{\partial y} =$ ———

(a) $\log y, y^{x-1}$ (b) $y^x \log x, x$ (c) y^x, xy^{x-1} (d) $y^x \log y, xy^{x-1}$

20. If $u = x^3 + y^3$ then $\dfrac{\partial^2 u}{\partial x \partial y} =$

(a) –3 (b) 3 (c) 0 (d) 1

21. If $u = f(x + ay) + g(x - ay)$ then $\dfrac{\partial^2 u}{\partial y^2}$ is

(a) $\dfrac{\partial^2 u}{\partial x^2}$ (b) $a\dfrac{\partial^2 u}{\partial x^2}$ (c) $a^2\dfrac{\partial^2 u}{\partial x^2}$ (d) $\dfrac{\partial^2 u}{\partial x \partial y}$

22. If $u = (x - y)^4 + (y - z)^4 + (z - x)^4$ then $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z}$ is

(a) 1 (b) u (c) $4u$ (d) 0

23. If $u = \tan^{-1}(x + y)$ then $u_x - u_y$ is

(a) 0 (b) 1 (c) –1 (d) $\sin x \cos y$

24. If $f = \log(x \tan^{-1} y)$ then f_{xy} is equal to

(a) $-\dfrac{1}{x^2}$ (b) 0 (c) $\dfrac{1}{x^2}$ (d) $\dfrac{1}{x}$

25. If $z = x^2 - y^2$ then $\frac{\partial z}{\partial x} =$ **[Winter 2014]**

(a) $2y$ (b) 0 (c) $2z$ (d) $2x$

26. If $f(x, y, z, w) = \dfrac{3\cos(xw)\sin y^5}{e^y + \dfrac{(1+y^2)}{xyw}} + 5xzw$ then $\frac{\partial f}{\partial z}$ at $(1,2,3,4)$ is **[Winter 2015]**

(a) 20 (b) 200 (c) 0 (d) 1

27. If $z = f(x,y)$, dz is equal to

(a) $\left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy$ (b) $\left(\frac{\partial f}{\partial y}\right)dx + \left(\frac{\partial f}{\partial x}\right)dy$

(c) $\left(\frac{\partial f}{\partial x}\right)dx - \left(\frac{\partial f}{\partial y}\right)dy$ (d) $\left(\frac{\partial f}{\partial y}\right)dx - \left(\frac{\partial f}{\partial x}\right)dy$

28. For an implicit function $f(x,y) = c$, the value of $\frac{dy}{dx}$ is **[Summer 2017, 2015]**

(a) $\frac{f_x}{f_y}$ (b) $\frac{f_y}{f_x}$ (c) $-\frac{f_x}{f_y}$ (d) $-\frac{f_y}{f_x}$

29. If $x^3 + y^3 + 3xy = 0$ then $\frac{dy}{dx} =$

(a) $\frac{x^2 - y}{y^2 - x}$ (b) $-\frac{x^2 + y}{y^2 + x}$ (c) $\frac{x^2 + y}{x^2 - y}$ (d) $\frac{x^2 + y}{x - y}$

30. If $u = \sin(xy^2)$, $x = \log t$, $y = e^t$ then $\frac{du}{dt} =$

(a) $y^2\left(\frac{1}{t} + 2x\right)\cos xy^2$ (b) 0

(c) 1 (d) -1

31. If f is a function of u, v, w and u, v, w are functions of x, y, z then $\frac{\partial f}{\partial y}$ is

(a) $\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial z}$ (b) $\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u}$

(c) $\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial y}$ (d) $\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial v}$

32. If $u = x^2 - y^2$, $v = xy$ then $\frac{\partial x}{\partial u}$ is

(a) $\frac{x}{2(x^2 + y^2)}$ (b) $\frac{y}{2(x^2 + y^2)}$ (c) $\frac{y}{x^2 + y^2}$ (d) $\frac{x}{x^2 + y^2}$

33. If $f(x,y) = e^{xy^2}$, the total differential of the function at the point (1,2) is

(a) $e(dx + dy)$ (b) $e^4 (dx + dy)$ (c) $e^4 (4dx + dy)$ (d) $4e^4 (dx + dy)$

34. If $f(x, y) = x^2 + y^2 + 3$, the minimum value of $f(x, y)$ is

(a) 3 (b) ∞ (c) 0 (d) 1

35. The stationary points of $x^3 + y^3 - 3axy$ are

(a) $(0, 0), (a, a)$ (b) $(0, 0), (a, 0)$ (c) $(a, 0), (0, -a)$ (d) $(0, a), (a, 0)$

36. If $f(x,y) = xy + (x - y)$, the stationary points are

(a) (0, 0) (b) (1, –1) (c) (1, 2) (d) (1, –2)

37. The stationary points of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ are

(a) $(\sqrt{2}, \sqrt{2})$ (b) $(-\sqrt{2}, -\sqrt{2})$ (c) $(\sqrt{2}, -\sqrt{2})$ (d) $(0, \sqrt{2})$

38. In a plane triangle ABC, the maximum value of $\cos A \cos B \cos C$ is

(a) 0 (b) $\frac{1}{8}$ (c) $\frac{\sqrt{3}}{8}$ (d) $\frac{3}{8}$

39. For the function $f(x,y) = x^2 + y^2 + 6x + 12$, minima occurs at

(a) (0, 3) (b) (–3, 0) (c) (3, 0) (d) (0, –3)

40. The function $f(x, y) = 2x^2 + 2xy - y^3$ has

(a) only one stationary point at (0, 0)

(b) two stationary points at (0, 0) and $\left(\frac{1}{6}, -\frac{1}{3}\right)$

(c) two stationary points at (0, 0) and (1, –1)

(d) no stationary points

41. The function $z = 5xy - 4x^2 + y^2 - 2x - y + 5$ has at $x = \frac{1}{41}, y = \frac{18}{41}$

(a) maxima (b) saddle point (c) minima (d) no conclusion

42. With usual notations, the properties of maxima and minima under various conditions are

I		II	
(P)	Maxima	(i)	$rt - s^2 = 0$
(Q)	Minima	(ii)	$rt - s^2 < 0$
(R)	Saddle Point	(iii)	$rt - s^2 > 0, r > 0$
(S)	Failure Case	(iv)	$rt - s^2 > 0, r < 0$

(a) P – i, Q – iii, R – iv, S – ii (b) P – ii Q – i, R – iii, S – iv

(c) P–iv, Q – iii, R–ii, S – i (d) P – iv, Q – ii, R – i, S – iii

43. The minimum value of $f(x,y) = x^2y^2$ is **[Winter 2015]**

(a) 1 (b) 2 (c) 4 (d) no conclusion

44. The sum of the squares of two positive numbers is 200, their minimum product is

(a) 200 (b) $25\sqrt{7}$ (c) 28 (d) 0

45. The minimum value of $x^2 + y^2 + z^2$ given that $xy + yz + zx = 3a^2$ is

(a) $3a$ (b) $4a^2$ (c) $\frac{1}{3}a^2$ (d) $3a^2$

46. For the auxiliary equation $F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z) = 0$, the Lagrange's equations are **[Winter 2014]**

(a) $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$ (b) $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0$

(c) $\frac{\partial \phi}{\partial x} = 0, \frac{\partial \phi}{\partial y} = 0, \frac{\partial \phi}{\partial z} = 0$ (d) $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial \phi} = 0, \frac{\partial f}{\partial z} = 0$

47. The equation of the tangent plane of $z = x$ at (2, 0, 2) is

(a) $z = x$ (b) $x + y + z = 2$ (c) $z + x = 0$ (d) $x + y = 2$

48. A point (a, b) is said to be saddle point if at (a, b) **[Summer 2016]**

(a) $rt - s^2 > 0$ (b) $rt - s^2 = 0$ (c) $rt - s^2 < 0$ (d) $rt - s^2 \geq 0$

49. The minimum value of $f(x, y) = x^2 + y^2$ is **[Winter 2016]**

(a) 1 (b) 2 (c) 4 (d) 0

Answers

1. (a)	2. (c)	3. (d)	4. (d)	5. (c)	6. (a)	7. (b)	8. (d)	9. (d)
10. (c)	11. (b)	12. (a)	13. (b)	14. (b)	15. (c)	16. (b)	17. (c)	18. (b)
19. (d)	20. (c)	21. (c)	22. (d)	23. (a)	24. (b)	25. (d)	26. (a)	27. (a)
28. (c)	29. (b)	30. (a)	31. (c)	32. (a)	33. (d)	34. (a)	35. (a)	36. (b)
37. (c)	38. (b)	39. (b)	40. (b)	41. (b)	42. (c)	43. (d)	44. (d)	45. (d)
46. (a)	47. (a)	48. (b)	49. (d)					

UNIT-5

Chapter 9. Multiple Integrals

CHAPTER 9

Multiple Integrals

Chapter Outline

9.1 INTRODUCTION

Integration of functions of two or more variables is normally called multiple integration. The particular case of integration of functions of two variables is called *double integration* and that of three variables is called *triple integration*. Sometimes, we have to change the variables to simplify the integrand while evaluating the multiple integrals. Variables can be changed by substitution or by changing the coordinate system (polar, spherical or cylindrical coordinates). Multiple integrals are useful in evaluating plane area, mass of a lamina, mass and volume of solid regions, etc.

9.2 DOUBLE INTEGRALS

Let $f(x, y)$ be a continuous function defined in a closed and bounded region R in the xy-plane. Divide the region R into small elementary rectangles by drawing lines parallel to coordinate axes. Let the total number of complete rectangles which lie inside the region R be n. Let δA_r be the area of r^{th} rectangle and (x_r, y_r) be any point in this rectangle.

Consider the sum $$S = \sum_{r=1}^{n} f(x_r, y_r)\delta A_r \qquad \ldots(1)$$

where $\delta A_r = \delta x_r \cdot \delta y_r$

If we increase the number of elementary rectangles then the area of each rectangle decreases. Hence, as $n \to \infty$, $\delta A_r \to 0$. The limit of the sum given by the Eq. (1), if it exists, is called the double integral of $f(x, y)$ over the region R and is denoted by $\iint_R f(x, y)\mathrm{d}A$.

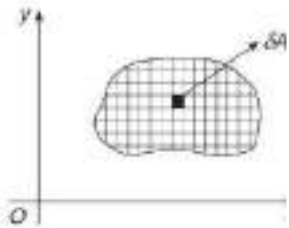

Fig. 9.1

Hence, $$\iint_R f(x, y)\mathrm{d}A = \lim_{\substack{n\to\infty \\ \delta A_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\delta A_r$$

where $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$

9.2.1 Double Integrals over Rectangles and General Regions

Double integral of a function $f(x, y)$ over a region R can be evaluated by two successive integrations. There are two different methods to evaluate a double integral.

Method-I Let the region R, i.e., $PQRS$ be bounded by the curves $y = y_1(x)$, $y = y_2(x)$ and the lines $x = a$, $x = b$.

In the region $PQRS$, draw a vertical strip AB. Along the strip AB, y varies from y_1 to y_2 and x is fixed. Therefore, the double integral is integrated first w.r.t. y between the limits y_1 and y_2 treating x as constant.

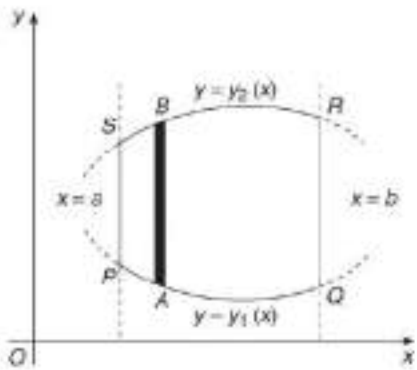

Fig. 9.2

Now, move the strip AB horizontally from PS (i.e., $x = a$) to QR (i.e., $x = b$) to cover the entire region $PQRS$. The result of the first integral is integrated w.r.t. x between the limits a and b. Hence,

$$\iint_R f(x, y)\mathrm{d}x\,\mathrm{d}y = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x, y)\mathrm{d}y\right]\mathrm{d}x$$

Method-II Let the region R, i.e., $PQRS$ be bounded by the curves $x = x_1(y)$, $x = x_2(y)$ and the lines $y = c$, $y = d$.

In the region $PQRS$, draw a horizontal strip AB. Along the strip AB, x varies from x_1 to x_2 and y is fixed. Therefore, the double integral is integrated first w.r.t. x between the limits x_1 and x_2 treating y as constant.

Now, move the strip AB vertically from PQ (i.e., $y = c$) to RS (i.e., $y = d$) to cover the entire region $PQRS$. The result of the first integral is integrated w.r.t. y between the limits c and d.

Hence,

$$\iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left[\int_{x_1(y)}^{x_2(y)} f(x, y)\,\mathrm{d}x\right]\mathrm{d}y$$

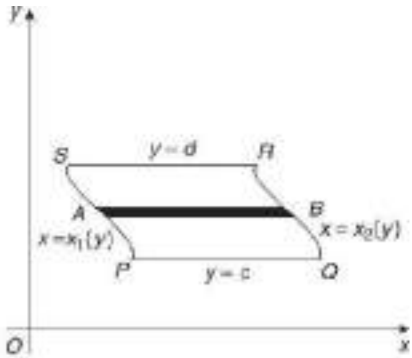

Fig. 9.3

Note:

(i) If all the four limits are constant then region of integration R is a rectangle. In this case, the function $f(x,y)$ can be integrated w.r.t. any variable first.

(ii) If all the four limits are constant and $f(x, y)$ is explicit then double integral can be written as product of two single integrals.

(iii) If inner limits depends on x then the function $f(x, y)$ is integrated first w.r.t. y and vice-versa.

9.2.2 Properties of Double Integrals

Various properties of double integrals are analogous to those for single integrals. For f and g continuous in region R with k as rational number,

(i) $\iint_R (f+g)\,\mathrm{d}x\mathrm{d}y = \iint_R f\,\mathrm{d}x\mathrm{d}y + \iint_R g\,\mathrm{d}x\mathrm{d}y$

(ii) $\iint_R k\,f\,\mathrm{d}x\mathrm{d}y = k\iint_R f\,\mathrm{d}x\mathrm{d}y$, where k is a constant.

For f continuous in region R, where $R = R_1 \cup R_2$ where R_1 and R_2 are non-overlapping regions whose union is R:

(iii) $\iint_R f\,\mathrm{d}x\mathrm{d}y = \iint_{R_1} f\,\mathrm{d}x\mathrm{d}y + \iint_{R_2} f\,\mathrm{d}x\mathrm{d}y$

9.2.3 Double Integrals as Volumes

If $z = f(x, y) \geq 0$ represents a surface and R is a rectangle in the xy-plane, then the double integral of $f(x, y)$ over R,

$$\iint_R f(x, y)\,\mathrm{d}x\mathrm{d}y$$

represents the volume of the solid under the surface $z = f(x, y)$ and above the region R.

Example 1

Evaluate $\int_0^3 \int_0^1 (x^2 + 3y^2)\,\mathrm{d}y\,\mathrm{d}x.$

Solution

$$\begin{aligned}\int_0^3 \int_0^1 (x^2 + 3y^2)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^3 \left| x^2 y + y^3 \right|_0^1 \mathrm{d}x \\ &= \int_0^3 (x^2 + 1)\,\mathrm{d}x \\ &= \left| \frac{x^3}{3} + x \right|_0^3 \\ &= 12\end{aligned}$$

Example 2

Evaluate $\int_0^1 \int_0^2 (x^2 + y^2)\,\mathrm{d}y\,\mathrm{d}x.$

Solution

$$\begin{aligned}\int_0^1 \int_0^2 (x^2 + y^2)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^1 \left| x^2 y + \frac{y^3}{3} \right|_0^2 \mathrm{d}x \\ &= \int_0^1 \left(2x^2 + \frac{8}{3} \right) \mathrm{d}x \\ &= \left| \frac{2x^3}{3} + \frac{8x}{3} \right|_0^1 \\ &= \frac{10}{3}\end{aligned}$$

Example 3

Evaluate $\int_{-1}^1 \int_0^2 (1 - 6x^2 y)\,\mathrm{d}x\,\mathrm{d}y$. **[Winter 2016]**

Solution

$$\begin{aligned}\int_{-1}^1 \int_0^2 (1 - 6x^2 y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{-1}^1 \left[\int_0^2 (1 - 6x^2 y)\,\mathrm{d}x \right] \mathrm{d}y \\ &= \int_{-1}^1 \left| x - 6y \cdot \frac{x^3}{3} \right|_0^2 \mathrm{d}y\end{aligned}$$

$$= \int_{-1}^{1} \left| x - 2yx^3 \right|_0^2 \mathrm{d}y$$
$$= \int_{-1}^{1} (2 - 16y)\,\mathrm{d}y$$
$$= 2\int_0^1 2\,\mathrm{d}y$$
$$= 4$$

Example 4

Evaluate $\int_2^a \int_2^b \frac{\mathrm{d}x\,\mathrm{d}y}{xy}$.

Solution

$$\int_2^a \int_2^b \frac{\mathrm{d}x\,\mathrm{d}y}{xy} = \int_2^a \left(\int_2^b \frac{\mathrm{d}x}{x} \right) \frac{\mathrm{d}y}{y} = \int_2^a \left| \log x \right|_2^b \frac{1}{y}\,\mathrm{d}y$$
$$= (\log b - \log 2) \int_2^a \frac{1}{y}\,\mathrm{d}y$$
$$= \log\left(\frac{b}{2}\right) \left| \log y \right|_2^a$$
$$= \log\left(\frac{b}{2}\right) (\log a - \log 2)$$
$$= \log\left(\frac{b}{2}\right) \log\left(\frac{a}{2}\right)$$

Another method: Since both the limits are constant and integrand (function) is explicit in x and y, the integral can be written as

$$\int_2^a \int_2^b \frac{\mathrm{d}x\,\mathrm{d}y}{xy} = \int_2^a \frac{\mathrm{d}y}{y} \int_2^b \frac{\mathrm{d}x}{x}$$
$$= \left| \log y \right|_2^a \left| \log x \right|_2^b$$
$$= (\log a - \log 2)(\log b - \log 2)$$
$$= \log\left(\frac{a}{2}\right) \cdot \log\left(\frac{b}{2}\right)$$
$$= \log\left(\frac{b}{2}\right) \cdot \log\left(\frac{a}{2}\right)$$

Example 5

Evaluate $\int_0^1\int_1^2 xy\,dy\,dx$.

Solution

$$\begin{aligned}\int_0^1\int_1^2 xy\,dy\,dx &= \int_0^1\left\{\int_1^2 y\,dy\right\}x\,dx\\ &= \int_0^1\left|\frac{y^2}{2}\right|_1^2 x\,dx\\ &= \int_0^1\left(\frac{4}{2}-\frac{1}{2}\right)x\,dx\\ &= \frac{3}{2}\left|\frac{x^2}{2}\right|_0^1\\ &= \frac{3}{2}\cdot\frac{1}{2}\\ &= \frac{3}{4}\end{aligned}$$

Another method: Since both the limits are constant and integrand (function) is explicit in x and y, the integral can be written as

$$\begin{aligned}\int_0^1\int_1^2 xy\,dy\,dx &= \int_0^1 x\,dx\cdot\int_1^2 y\,dy\\ &= \left|\frac{x^2}{2}\right|_0^1\left|\frac{y^2}{2}\right|_1^2\\ &= \frac{1}{2}\left(\frac{4}{2}-\frac{1}{2}\right)\\ &= \frac{3}{4}\end{aligned}$$

Example 6

Evaluate $\int_0^1\int_0^x dy\,dx$.

Solution

$$\int_0^1\int_0^x dy\,dx = \int_0^1 |y|_0^x\,dx$$

$$= \int_0^1 x\,\mathrm{d}x$$

$$= \left|\frac{x^2}{2}\right|_0^1$$

$$= \frac{1}{2}$$

Example 7

Evaluate $\int_0^1\int_0^x e^{\frac{y}{x}}\,\mathrm{d}y\,\mathrm{d}x.$

Solution

$$\int_0^1\int_0^x e^{\frac{y}{x}}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \left|xe^{\frac{y}{x}}\right|_0^x \mathrm{d}x$$

$$= \int_0^1 x(e-1)\,\mathrm{d}x$$

$$= \left|\frac{x^2}{2}(e-1)\right|_0^1$$

$$= \frac{1}{2}(e-1)$$

Example 8

Evaluate $\int_0^1\int_x^{x^2} xy\,\mathrm{d}y\,\mathrm{d}x.$

Solution

$$\int_0^1\int_x^{x^2} xy\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\left\{\int_x^{x^2} y\,\mathrm{d}y\right\}x\,\mathrm{d}x$$

$$= \int_0^1\left|\frac{y^2}{2}\right|_x^{x^2} x\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^1\left[(x^2)^2 - x^2\right]x\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^1 (x^5 - x^3)\,\mathrm{d}x$$

$$= \frac{1}{2}\left|\frac{x^6}{6} - \frac{x^4}{4}\right|_0^1$$

$$= \frac{1}{2}\left(\frac{1}{6} - \frac{1}{4}\right)$$

$$= -\frac{1}{24}$$

Example 9

Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$.

Solution

$$\int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{dy}{\left(\sqrt{1+x^2}\right)^2+y^2}\right]dx = \int_0^1 \frac{1}{\sqrt{1+x^2}}\left|\tan^{-1}\frac{y}{\sqrt{1+x^2}}\right|_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}}(\tan^{-1}1-\tan^{-1}0)dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}}\cdot\frac{\pi}{4}dx$$

$$= \frac{\pi}{4}\left|\log\left(x+\sqrt{1+x^2}\right)\right|_0^1$$

$$= \frac{\pi}{4}\log(1+\sqrt{2})$$

Example 10

Evaluate $\int_0^1 \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$.

Solution

$$\int_0^1 \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx\,dy}{\sqrt{1-x^2-y^2}} = \int_0^1 \left[\int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx}{\sqrt{(1-y^2)-x^2}}\right]dy$$

$$= \int_0^1 \left|\sin^{-1}\frac{x}{\sqrt{1-y^2}}\right|_0^{\sqrt{\frac{1-y^2}{2}}} dy$$

$$= \int_0^1 \left(\sin^{-1}\frac{1}{\sqrt{2}}-\sin^{-1}0\right)dy$$

$$= \frac{\pi}{4}|y|_0^1$$

$$= \frac{\pi}{4}$$

Example 11

Sketch the region of integration and evaluate $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{\frac{y}{\sqrt{x}}} \, dy \, dx$.

[Summer 2017]

Solution

1. Since the inner limits depend on x, the function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of y: $y = 0$ to $y = \sqrt{x}$

   Limits of x: $x = 1$ to $x = 4$
3. The region is bounded by the line $x = 1$, $y = 0$, $x = 4$ and parabola $y^2 = x$.
4. The points of intersection of $y^2 = x$ and $x = 1$, $x = 4$ are (1, 1) (4, 2).

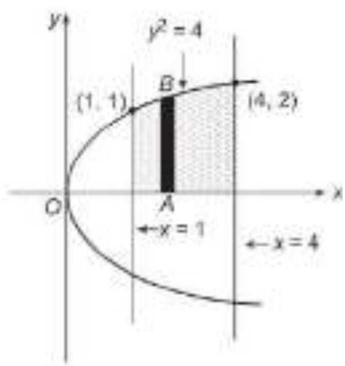

Fig. 9.4

$$\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{\frac{y}{\sqrt{x}}} \, dy \, dx = \int_1^4 \frac{3}{2} \left[\int_0^{\sqrt{x}} e^{\frac{y}{\sqrt{x}}} \, dy \right] dx$$

$$= \int_1^4 \frac{3}{2} \left| \frac{e^{\frac{y}{\sqrt{x}}}}{1/\sqrt{x}} \right|_0^{\sqrt{x}} dx$$

$$= \int_1^4 \frac{3}{2} \sqrt{x} \left| e^{\frac{4}{\sqrt{x}}} \right|_0^{\sqrt{x}} dx$$

$$= \int_1^4 \frac{3}{2} \sqrt{x} \, (e - 1) dx$$

$$= \frac{3}{2}(e - 1) \int_1^4 \sqrt{x} \, dx$$

$$= \frac{3}{2}(e - 1) \left| \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right|_1^4$$

$$= \frac{3}{2}(e - 1) \cdot \frac{2}{3} \left| x^{\frac{3}{2}} \right|_1^4$$

$$= (e-1)\left[4^{\frac{3}{2}} - 1\right]$$

$$= (e-1)\,[2^{2\cdot\frac{3}{2}} - 1]$$

$$= [e-1]\,[2^3 - 1]$$

$$= (e-1)\,[8-1]$$

$$= 7(e-1)$$

EXERCISE 9.1

Evaluate the following integrals:

1. $\int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} 2x^2y^2\,dx\,dy$

$\left[\textbf{Ans.:} \frac{856}{945}\right]$

2. $\int_0^1 \int_0^y xy\,e^{-x^2}\,dx\,dy$

$\left[\textbf{Ans.:} \frac{1}{4e}\right]$

3. $\int_0^1 \int_0^x e^{x+y}\,dx\,dy$

$\left[\textbf{Ans.:} \frac{1}{2}(e-1)^2\right]$

4. $\int_{10}^{1} \int_0^{\frac{1}{y}} y e^{xy}\,dx\,dy$

[**Ans.:** 9(1 – *e*)]

5. $\int_1^{\log 8} \int_0^{\log y} e^{x+y}\,dx\,dy$

[**Ans.:** 8(log 8 – 1)]

6. $\int_0^1 \int_{y^2}^{y} (1+xy^2)\,dx\,dy$

$\left[\textbf{Ans.:} \frac{41}{210}\right]$

7. $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} xy\,dy\,dx$

$\left[\textbf{Ans.:} \frac{2a^4}{3}\right]$

9.2.4 Working Rule for Evaluation of Double Integrals Over a General Region

1. If the region is bounded by more than one curve then find the points of intersection of all the curves.
2. Draw all the curves and mark their point of intersection.
3. Identify the region of integration.
4. Draw a vertical or horizontal strip in the region whichever makes the integration easier.
5. The vertical strip starts from the lowest part of the region and terminates on the highest part of the region.
6. For vertical strip: (i) The lower limit of y is obtained from the curve where the vertical strip starts and the upper limit of y is obtained from the curve where it terminates.
   (ii) The lower limit of x is obtained from the leftmost point of the region and the upper limit of x is obtained from the rightmost point of the region.
7. The horizontal strip starts from the left part of the region and terminates on the right part of the region.
8. For horizontal strip: (i) The lower limit of x is obtained from the curve where the horizontal strip starts and upper limit is obtained from the curve where it terminates.
   (ii) The lower limit of y is obtained from the lowest point of the region and the upper limit of y is obtained from the highest point of the region.
9. If variation along the strip changes within the region then the region is divided into parts.

Example 1

Evaluate $\iint e^{ax+by}\,dx\,dy$, *over the triangle bounded by* $x = 0$, $y = 0$, $ax + by = 1$.

Solution

1. The region of integration is the ΔOPQ.
2. The integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the line $ax + by = 1$.
3. Limits of y: $y = 0$ to $y = \dfrac{1-ax}{b}$

   Limits of x: $x = 0$ to $x = \dfrac{1}{a}$

$$I = \int_0^{\frac{1}{a}} \int_0^{\frac{1-ax}{b}} e^{ax+by}\,dx\,dy$$

$$= \int_0^{\frac{1}{a}} e^{ax} \int_0^{\frac{1-ax}{b}} e^{by}\,dy\,dx$$

$$= \int_0^{\frac{1}{a}} e^{ax} \left| \frac{e^{by}}{b} \right|_0^{\frac{1-ax}{b}}$$

$$= \frac{1}{b} \int_0^{\frac{1}{a}} e^{ax} \left[e^{(1-ax)} - 1 \right] dx$$

$$= \frac{1}{b} \int_0^{\frac{1}{a}} (e - e^{ax})\,dx$$

$$= \frac{1}{b} \left| ex - \frac{e^{ax}}{a} \right|_0^{\frac{1}{a}}$$

$$= \frac{1}{b} \left(\frac{e}{a} - \frac{e}{a} + \frac{1}{a} \right)$$

$$= \frac{1}{ab}$$

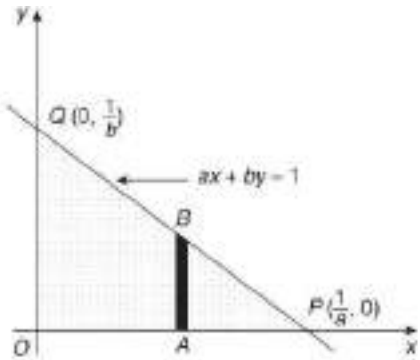

Fig. 9.5

Example 2

Evaluate $\iint \frac{xy}{\sqrt{1-y^2}}\,dx\,dy$ *over the first quadrant of the circle* $x^2 + y^2 = 1$.

Solution

1. The region of integration is *OPQ*.
2. The integration can be done w.r.t any variable first. Draw a vertical strip *AB* parallel to *y*-axis which starts from *x*-axis and terminates on the circle $x^2 + y^2 = 1$.
3. Limits of y: $y = 0$ to $y = \sqrt{1-x^2}$

   Limits of x: $x = 0$ to $x = 1$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{xy}{\sqrt{1-y^2}}\,dx\,dy$$

$$= \int_0^1 x \int_0^{\sqrt{1-x^2}} -\frac{1}{2}(1-y^2)^{-\frac{1}{2}}(-2y)\,dy\,dx$$

$$= -\frac{1}{2} \int_0^1 x \left| 2(1-y^2)^{\frac{1}{2}} \right|_0^{\sqrt{1-x^2}} dx$$

$$= -\frac{1}{2} \int_0^1 2x(x-1)\,dx$$

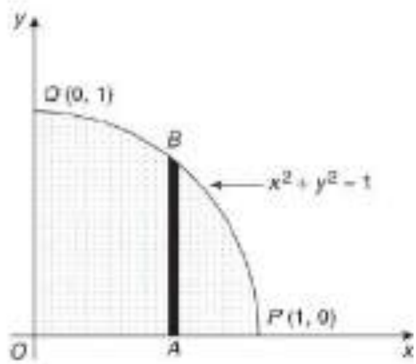

Fig. 9.6

$$\left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} \right]$$

$$= -\left|\frac{x^3}{3} - \frac{x^2}{2}\right|_0^1$$
$$= -\left(\frac{1}{3} - \frac{1}{2}\right)$$
$$= \frac{1}{6}$$

Example 3

Evaluate $\iint (a-x)^2 \, dx\, dy$, *over the right half of the circle* $x^2 + y^2 = a^2$.

Solution

1. The region of integration is *PQR*.
2. The integration can be done w.r.t. any variable first. Draw a vertical strip *AB* parallel to *y*-axis which starts from the part of the circle $x^2 + y^2 = a^2$ below *x*-axis and terminates on the part of the circle $x^2 + y^2 = a^2$ above *x*-axis.
3. Limits of

$y: y = -\sqrt{a^2 - x^2}$ to $y = \sqrt{a^2 - x^2}$

Limits of $x: x = 0$ to $x = a$

$$I = \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (a-x)^2 \, dx\, dy$$
$$= \int_0^a (a-x)^2 \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\, dx$$
$$= \int_0^a (a-x)^2 \left| y \right|_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$
$$= \int_0^a (a^2 + x^2 - 2ax) \cdot 2\sqrt{a^2 - x^2}\, dx$$
$$= 2\int_0^a (a^2 + x^2 - 2ax)\sqrt{a^2 - x^2}\, dx$$

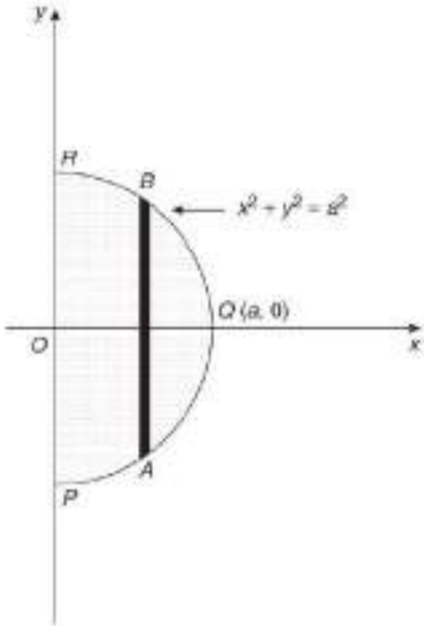

Fig. 9.7

Putting $x = a\sin\theta$, $dx = a\cos\theta \, d\theta$

When $x = 0$, $\theta = 0$

When $x = a$, $\theta = \frac{\pi}{2}$

$$I = 2\int_0^{\frac{\pi}{2}} (a^2 + a^2\sin^2\theta - 2a^2\sin\theta) \cdot a\cos\theta \cdot a\cos\theta \, d\theta$$
$$= 2a^4 \int_0^{\frac{\pi}{2}} (\cos^2\theta + \sin^2\theta\cos^2\theta - 2\sin\theta\cos^2\theta) d\theta$$

$$= a^4\left[\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} + \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)} - 2\frac{\Gamma(1)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}\right)}\right] \qquad \left[\because 2\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\,\mathrm{d}\theta = B\left(\frac{p+1}{2},\frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}\right]$$

$$= a^4\left[\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{1} + \frac{\left(\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right)^2}{2!} - 2\frac{\Gamma\left(\frac{3}{2}\right)}{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}\right]$$

$$= a^4\left[\frac{\pi}{2} + \frac{\pi}{8} - \frac{4}{3}\right]$$

$$= a^4\left[\frac{5\pi}{8} - \frac{4}{3}\right]$$

Example 4

Evaluate $\iint xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}}\mathrm{d}x\,\mathrm{d}y$, *over the first quadrant of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

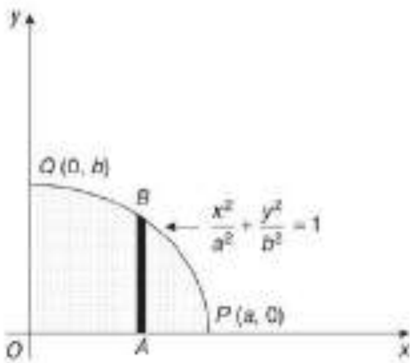

Fig. 9.8

Solution

1. The region of integration is OPQ.
2. The integration can be done w.r.t any variable first. Draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
3. Limits of y: $y = 0$ to $y = b\sqrt{1-\frac{x^2}{a^2}}$

   Limits of x: $x = 0$ to $x = a$

$$I = \int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}}\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^a x\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \frac{b^2}{2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}}\frac{2y}{b^2}\,\mathrm{d}y\,\mathrm{d}x$$

$$= \frac{b^2}{2}\int_0^a x\left|\frac{1}{\left(\frac{n}{2}+1\right)}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}+1}\right|_0^{b\sqrt{1-\frac{x^2}{a^2}}}\mathrm{d}x \qquad \left[\because \int [f(y)]^n f'(y)\,\mathrm{d}y = \frac{[f(y)]^{n+1}}{n+1}\right]$$

$$= \frac{b^2}{n+2}\left|\frac{x^2}{2} - \frac{1}{a^{n+2}}\cdot\frac{x^{n+4}}{n+4}\right|_0^a$$

$$= \frac{b^2}{n+2}\left|\frac{a^2}{2} - \frac{1}{a^{n+2}}\cdot\frac{a^{n+4}}{n+4}\right|$$

$$= \frac{a^2b^2}{(n+2)}\cdot\frac{(n+2)}{2(n+4)}$$

$$= \frac{a^2b^2}{2(n+4)}$$

Example 5

Evaluate $\iint (x^2 + y^2)\,dx\,dy$ *over the ellipse* $2x^2 + y^2 = 1$.

Solution

1. The region of integration is *PQRS*, the ellipse $2x^2 + y^2 = 1$ or $\frac{x^2}{\left(\frac{1}{\sqrt{2}}\right)^2} + \frac{y^2}{1^2} = 1$ with $\frac{1}{\sqrt{2}}$ and 1 as its axes.
2. The integration can be done w.r.t any variable first. Draw a vertical strip *AB* parallel to *y*-axis which starts from the part of the ellipse $2x^2 + y^2 = 1$ below *x*-axis and terminates on the part of the ellipse $2x^2 + y^2 = 1$ above *x*-axis.
3. Limits of

   $y: y = -\sqrt{1-2x^2}$ to $y = \sqrt{1-2x^2}$

   Limits of $x: x = -\frac{1}{\sqrt{2}}$ to $x = \frac{1}{\sqrt{2}}$

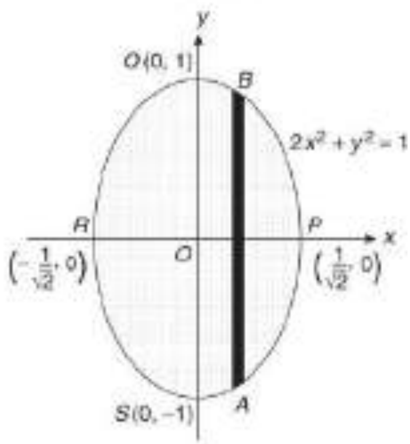

Fig. 9.9

$$I = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} (x^2 + y^2)\,dy\,dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\left|x^2y + \frac{y^3}{3}\right|_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} 2\left[x^2\sqrt{1-2x^2} + \frac{1}{3}(1-2x^2)^{\frac{3}{2}}\right]dx$$

$$= 4\int_0^{\frac{1}{\sqrt{2}}}\left[x^2\sqrt{1-2x^2} + \frac{1}{3}(1-2x^2)^{\frac{3}{2}}\right]dx$$

Putting $2x^2 = t,\ x = \sqrt{\frac{t}{2}},\ \mathrm{d}x = \frac{1}{2\sqrt{2}\sqrt{t}}\mathrm{d}t$

When $x = 0,\ t = 0$

$$x = \frac{1}{\sqrt{2}},\ t = 1$$

$$\begin{aligned}
\mathrm{I} &= 4\int_0^1 \left[\frac{t}{2}\sqrt{1-t} + \frac{1}{3}(1-t)^{\frac{3}{2}}\right]\frac{1}{2\sqrt{2}\sqrt{t}}\mathrm{d}t \\
&= \sqrt{2}\int_0^1 \left[\frac{1}{2}t^{\frac{1}{2}}(1-t)^{\frac{1}{2}} + \frac{1}{3}t^{-\frac{1}{2}}(1-t)^{\frac{3}{2}}\right]\mathrm{d}t \\
&= \sqrt{2}\left[\frac{1}{2}B\left(\frac{3}{2},\frac{3}{2}\right) + \frac{1}{3}B\left(\frac{1}{2},\frac{5}{2}\right)\right] \\
&= \sqrt{2}\left[\frac{1}{2}\cdot\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)} + \frac{1}{3}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(3)}\right] \\
&= \sqrt{2}\left[\frac{1}{2}\cdot\frac{\left(\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right)^2}{2} + \frac{1}{3}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2}\right] \\
&= \sqrt{2}\left[\frac{1}{4}\cdot\frac{\pi}{4} + \frac{\pi}{8}\right] \\
&= \frac{3\sqrt{2}\pi}{16}
\end{aligned}$$

Example 6

Evaluate $\iint (x^2 - y^2)\,\mathrm{d}x\,\mathrm{d}y$ *over the triangle with the vertices* (0, 1), (1, 1), (1, 2).

Solution

1. The region of integration is ΔPQR.
2. Equation of the line PQ is $y = 1$.

   Equation of the line PR is

$$y - 1 = \frac{2-1}{1-0}(x-0) = x$$
$$y = x + 1$$

3. The integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to y-axis which starts from the line $y = 1$ and terminates on the line $y = x + 1$.
4. Limits of y: $y = 1$ to $y = x + 1$
   Limits of x: $x = 0$ to $x = 1$

$$I = \int_0^1 \int_1^{x+1} (x^2 - y^2)\,dy\,dx$$
$$= \int_0^1 \left| x^2 y - \frac{y^3}{3} \right|_1^{x+1} dx$$
$$= \int_0^1 \left[x^2(x+1) - \frac{(x+1)^3}{3} - x^2 + \frac{1}{3} \right] dx$$
$$= \left| \frac{x^4}{4} + \frac{x^3}{3} - \frac{(x+1)^4}{12} - \frac{x^3}{3} + \frac{x}{3} \right|_0^1$$
$$= \frac{1}{4} + \frac{1}{3} - \frac{16}{12} + \frac{1}{12}$$
$$= -\frac{2}{3}$$

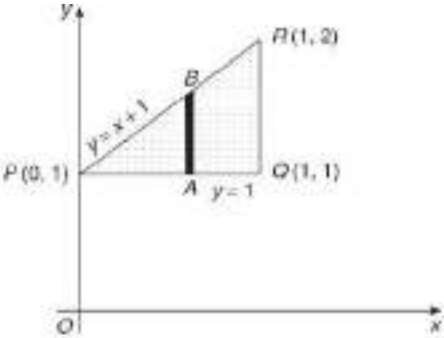

Fig. 9.10

Example 7

Evaluate $\iint e^{y^2}\,dx\,dy$ *over the region bounded by the triangle with vertices* (0, 0), (2, 1), (0, 1).

Solution

1. The region of integration is ΔOPQ.
2. Equation of the line OQ is $y = \frac{x}{2}$ or $x = 2y$.
3. Here, it is easier to integrate w.r.t. x first than y. Draw a horizontal strip AB parallel to x-axis which starts from y-axis and terminates on the line $x = 2y$.
4. Limits of x: $x = 0$ to $x = 2y$
   Limits of y: $y = 0$ to $y = 1$

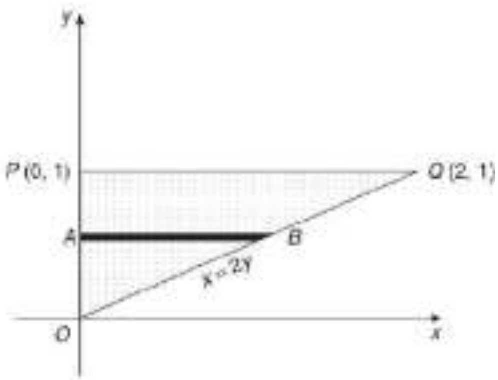

Fig. 9.11

$$I = \int_0^1 \int_0^{2y} e^{y^2}\,dx\,dy$$
$$= \int_0^1 e^{y^2} \int_0^{2y} dx\,dy$$
$$= \int_0^1 e^{y^2} \left| x \right|_0^{2y} dy$$
$$= \int_0^1 e^{y^2} \cdot 2y\,dy$$

$$= \left| e^{y^2} \right|_0^1 \qquad \left[\because \int e^{f(y)} f'(y)\mathrm{d}y = e^{f(y)} \right]$$

$$= e - 1$$

Example 8

Evaluate $\iint \frac{2xy^5}{\sqrt{1+x^2y^2-y^4}}\,\mathrm{d}x\,\mathrm{d}y$ *over the triangle having vertices* (0, 0), (1, 1) *and* (0, 1).

Solution

1. The region of integration is the ΔOPQ.
2. Equation of the line OP is $y = x$.
3. Here, it is easier to integrate w.r.t. x first than with y. Draw a horizontal strip AB parallel to x-axis which starts from y-axis and terminates on the line $y = x$.
4. Limits of $x : x = 0$ to $x = y$
   Limits of $y : y = 0$ to $y = 1$

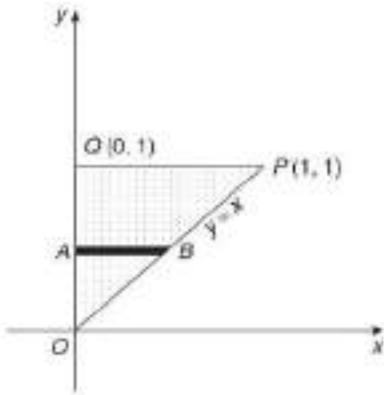

Fig. 9.12

$$I = \int_0^1 \int_0^y \frac{2xy^5}{\sqrt{1+x^2y^2-y^4}}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 y^3 \int_0^y (1+x^2y^2-y^4)^{-\frac{1}{2}} \cdot 2xy^2\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 y^3 \left| 2(1+x^2y^2-y^4)^{\frac{1}{2}} \right|_0^y \mathrm{d}y \qquad \left[\because \int [f(x)]^n f'(x)\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1} \right]$$

$$= \int_0^1 y^3 \cdot 2\left[1-(1-y^4)^{\frac{1}{2}} \right] \mathrm{d}y$$

$$= \int_0^1 2y^3\,\mathrm{d}y - 2\int_0^1 (1-y^4)^{\frac{1}{2}} y^3\,\mathrm{d}y$$

$$= 2\left| \frac{y^4}{4} \right|_0^1 - 2\int_0^1 (1-y^4)^{\frac{1}{2}} \frac{(-4y^3)}{-4}\,\mathrm{d}y$$

$$= \frac{1}{2} + \frac{1}{2}\left| \frac{2}{3}(1-y^4)^{\frac{3}{2}} \right|_0^1 \qquad \left[\because \int [f(x)]^n f'(x)\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1} \right]$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}$$

Example 9

Evaluate $\iint (x^2 + y^2)\,dx\,dy$ *over the region bounded by the lines* $y = 4x$, $x + y = 3$, $y = 0$, $y = 2$.

Solution

1. The region of integration is *OPQR*.
2. The integration can be done w.r.t. any variable first. But in case of vertical strip we need to divide the region into three parts. Therefore, draw a horizontal strip *AB* parallel to *x*-axis which starts from the line $y = 4x$ and terminates on the line $x + y = 3$.

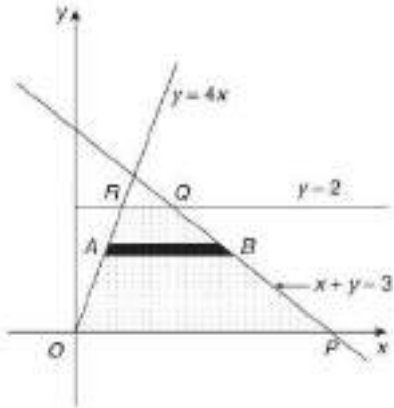

Fig. 9.13

3. Limits of $x: x = \frac{y}{4}$ to $x = 3 - y$

   Limits of $y: y = 0$ to $y = 2$

$$I = \int_0^2 \int_{\frac{y}{4}}^{3-y} (x^2 + y^2)\,dx\,dy$$

$$= \int_0^2 \left| \frac{x^3}{3} + xy^2 \right|_{\frac{y}{4}}^{3-y} dy$$

$$= \int_0^2 \left[\frac{(3-y)^3}{3} + (3-y)y^2 - \frac{1}{3}\cdot\frac{y^3}{64} - \frac{y^3}{4} \right] dy$$

$$= \int_0^2 \left[\frac{(3-y)^3}{3} + 3y^2 - \frac{241}{192}y^3 \right] dy$$

$$= \left| \frac{1}{3}\cdot\frac{(3-y)^4}{-4} + 3\cdot\frac{y^3}{3} - \frac{241}{192}\cdot\frac{y^4}{4} \right|_0^2$$

$$= -\frac{1}{12} + 8 - \frac{241}{192}\cdot 4 - \left(-\frac{27}{4}\right)$$

$$= \frac{463}{48}$$

Example 10

Evaluate $\iint\limits_R (x+y)\mathrm{d}y\,\mathrm{d}x$, *where R is the region bounded by* $x = 0, x = 2,$ $y = x, y = x + 2.$ **[Summer 2016]**

Solution

1. The region of integration is *OPQR*.
2. The point of intersection of $x = 2$ and $y = x + 2$ is obtained as $y = 4$.

   The point of intersection is (2, 4).
3. The integration can be done w.r.t. any variable first. Draw a vertical strip *AB* parallel to *y*-axis which starts from the line $y = x$ and terminates on the line $y = x + 2$.
4. Limits of y: $y = x$ to $y = x + 2$

   Limits of x: $x = 0$ to $x = 2$

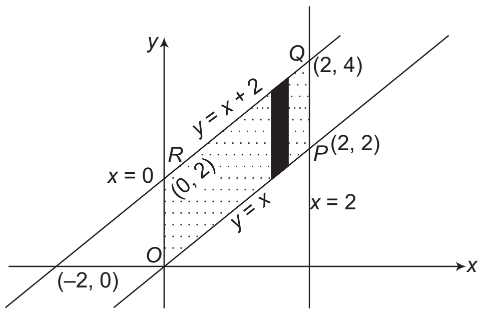

Fig. 9.14

$$I = \int_0^2\int_x^{x+2} (x+y)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^2\left[\int_x^{x+2} (x+y)\,\mathrm{d}y\right]\mathrm{d}x$$

$$= \int_0^2 \left| xy + \frac{y^2}{2}\right|_x^{x+2} \mathrm{d}x$$

$$= \int_0^2 \left[x(x+2) + \frac{1}{2}(x+2)^2 - x^2 - \frac{x^2}{2}\right]\mathrm{d}x$$

$$= \int_0^2 \left[x^2 + 2x + \frac{x^2}{2} + 2x + 2 - x^2 - \frac{x^2}{2}\right]\mathrm{d}x$$

$$= \int_0^2 [4x+2]\,\mathrm{d}x$$

$$= \left| 4\frac{x^2}{2} + 2x\right|_0^2$$

$$= \left| 2x^2 + 2x\right|_0^2$$

$$= 8 + 4 - 0$$

$$= 12$$

Example 11

$\iint\limits_R (2x - y^2)\,dA$ *over the triangular region R enclosed between the lines* $y = -x + 1$, $y = x + 1$ *and* $y = 3$. **[Summer 2015]**

Solution

1. The region of integration is ΔPQR.
2. The points of intersection of
   (i) $y = -x + 1$ and $y = x + 1$ is obtained as
   $-x + 1 = x + 1$
   $2x = 0, x = 0$
   $y = 1$
   The points of intersection is $P(0, 1)$.
   (ii) $y = x + 1$ and $y = 3$ is obtained as
   $3 = x + 1$
   $x = 2, y = 3$
   The points of intersection is $Q(2, 3)$.
   (iii) $y = -x + 1$ and $y = 3$ is obtained as
   $3 = -x + 1$
   $x = -2, y = 3$
   The points of intersection is $R(-2, 3)$.

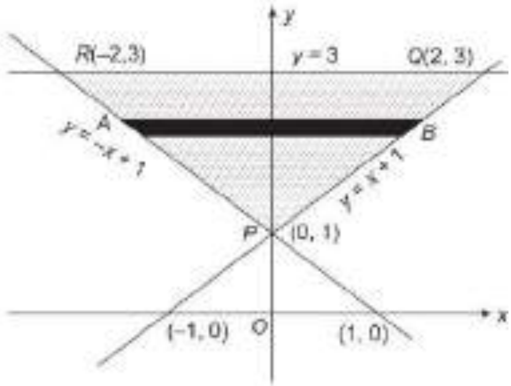

Fig. 9.15

3. The integration can be done w.r.t. any variable first. But in case of vertical strip, we need to divide the region into two parts. Therefore, draw a horizontal strip *AB* parallel to *x*-axis which starts from the line $y = -x + 1$ and terminates on the line $y = x + 1$.
4. Limits of $x : x = 1 - y$ to $x = y - 1$
   Limits of $y : y = 1$ to $y = 3$

$$I = \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\,dA$$

$$= \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\,dx\,dy$$

$$= \int_1^3 \left| x^2 - xy^2 \right|_{1-y}^{y-1} dy$$

$$= \int_1^3 \left[(y-1)^2 - y^2(y-1) - (1-y)^2 + y^2(1-y) \right] dy$$

$$= \int_1^3 \left[(y-1)^2 - (1-y)^2 - 2y^3 + 2y^2 \right] dy$$

$$= \left| \frac{(y-1)^3}{3} - \frac{(1-y)^3}{-3} - 2\frac{y^4}{4} + 2\frac{y^3}{3} \right|_1^3$$

$$= \left[\frac{8}{3} - \frac{8}{3} - \frac{81}{2} + \frac{54}{3} + \frac{1}{2} - \frac{2}{3} \right]$$

$$= -\frac{68}{3}$$

Example 12

Evaluate $\iint \frac{y\,dx\,dy}{(a-x)\sqrt{ax-y^2}}$ *over the region bounded by the parabola* $y^2 = x$ *and the line* $y = x$.

Solution

1. The region of integration is OPQ.
2. The points of intersection of $y^2 = x$ and $y = x$ are obtained as

$$x^2 = x$$
$$x(x-1) = 0$$
$$x = 0, 1$$
$$\therefore y = 0, 1$$

   The points of intersection are O (0, 0) and P (1, 1).

3. Here, it is easier to integrate w.r.t. y first than x. Draw a vertical strip AB parallel to y-axis, which starts from the line $y = x$ and terminates on the parabola $y^2 = x$.
4. Limits of y: $y = x$ to $y = \sqrt{x}$

   Limits of x: $x = 0$ to $x = 1$

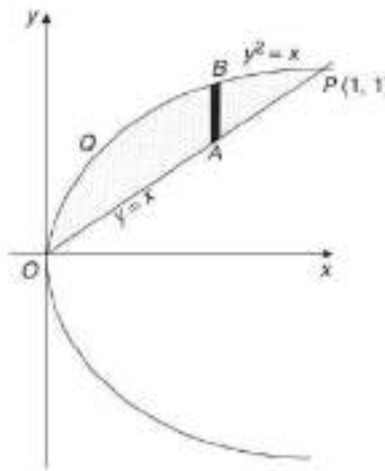

Fig. 9.16

$$I = \int_0^1 \int_x^{\sqrt{x}} \frac{y\,dx\,dy}{(a-x)\sqrt{ax-y^2}}$$

$$= \int_0^1 \frac{1}{(a-x)} \int_x^{\sqrt{x}} \left(-\frac{1}{2}\right)(ax-y^2)^{-\frac{1}{2}}(-2y)\,dy\,dx$$

$$= -\frac{1}{2}\int_0^1 \frac{1}{(a-x)}\left|2(ax-y^2)^{\frac{1}{2}}\right|_x^{\sqrt{x}} \mathrm{d}x \qquad \left[\because \int [f(y)]^n f'(y)\mathrm{d}y = \frac{[f(y)]^{n+1}}{n+1}\right]$$

$$= -\int_0^1 \frac{1}{(a-x)}\left[(ax-x)^{\frac{1}{2}} - (ax-x^2)^{\frac{1}{2}}\right]\mathrm{d}x$$

$$= -\int_0^1 \frac{\sqrt{x}}{a-x}\left(\sqrt{a-1}-\sqrt{a-x}\right)\mathrm{d}x$$

Putting $x = a\sin^2\theta$, $\mathrm{d}x = 2a\sin\theta\cos\theta\,\mathrm{d}\theta$

When $x = 0, \theta = 0$

When $x = 1, \theta = \sin^{-1}\frac{1}{\sqrt{a}}$

$$I = -\int_0^{\sin^{-1}\frac{1}{\sqrt{a}}} \frac{\sqrt{a}\sin\theta}{a\cos^2\theta}\left(\sqrt{a-1}-\sqrt{a}\cos\theta\right)2a\sin\theta\cos\theta\,\mathrm{d}\theta$$

$$= -2\sqrt{a}\int_0^{\sin^{-1}\frac{1}{\sqrt{a}}} \frac{\sin^2\theta}{\cos\theta}\left(\sqrt{a-1}-\sqrt{a}\cos\theta\right)\mathrm{d}\theta$$

$$= -2\sqrt{a}\int_0^{\sin^{-1}\frac{1}{\sqrt{a}}}\left[\left(\frac{1-\cos^2\theta}{\cos\theta}\right)\sqrt{a-1}-\sqrt{a}\sin^2\theta\right]\mathrm{d}\theta$$

$$= -2\sqrt{a}\int_0^{\sin^{-1}\frac{1}{\sqrt{a}}}\left[\sqrt{a-1}(\sec\theta-\cos\theta)-\frac{\sqrt{a}}{2}(1-\cos 2\theta)\right]\mathrm{d}\theta$$

$$= -2\sqrt{a}\left|\sqrt{a-1}[\log(\sec\theta+\tan\theta)-\sin\theta]-\frac{\sqrt{a}\theta}{2}+\frac{\sqrt{a}\sin 2\theta}{4}\right|_0^{\sin^{-1}\frac{1}{\sqrt{a}}}$$

$$= -2\sqrt{a}\left|\sqrt{a-1}\left[\log\left(\frac{1+\sin\theta}{\cos\theta}\right)-\sin\theta\right]-\frac{\sqrt{a}\theta}{2}+\frac{\sqrt{a}\sin\theta\cos\theta}{2}\right|_0^{\sin^{-1}\frac{1}{\sqrt{a}}}$$

$$= -2\sqrt{a}\left[\sqrt{a-1}\left(\log\frac{1+\frac{1}{\sqrt{a}}}{\sqrt{1-\frac{1}{a}}}-\frac{1}{\sqrt{a}}\right)-\frac{\sqrt{a}\sin^{-1}\frac{1}{\sqrt{a}}}{2}+\frac{\sqrt{a}}{2}\cdot\frac{1}{\sqrt{a}}\cdot\sqrt{1-\frac{1}{a}}\right]$$

$$= -2\sqrt{a(a-1)}\log\frac{\sqrt{a}+1}{\sqrt{a-1}}+\sqrt{a-1}+a\sin^{-1}\frac{1}{\sqrt{a}}$$

Example 13

Evaluate $\iint y\,dx\,dy$ *over the region enclosed by the parabola* $x^2 = y$ *and the line* $y = x + 2$.

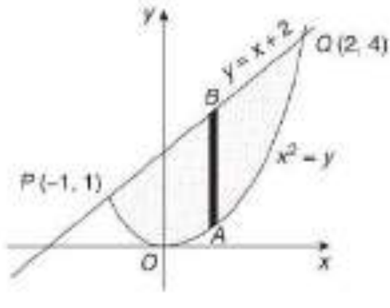

Fig. 9.17

Solution

1. The region of integration is *POQ*.
2. The points of intersection of $x^2 = y$ and $y = x + 2$ are obtained as

$$\begin{aligned} x^2 &= x + 2 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x &= 2, -1 \\ \therefore y &= 4, 1 \end{aligned}$$

The points of intersection are $P(-1, 1)$ and $Q(2, 4)$.

3. The integration can be done w.r.t. any variable first. Draw a vertical strip *AB* parallel to *y*-axis which starts from the parabola $x^2 = y$ and terminates on the line $y = x + 2$.
4. Limits of y : $y = x^2$ to $y = x + 2$
   Limits of x : $x = -1$ to $x = 2$

$$\begin{aligned} I &= \int_{-1}^{2}\int_{x^2}^{x+2} y\,dy\,dx \\ &= \int_{-1}^{2}\left|\frac{y^2}{2}\right|_{x^2}^{x+2} dx \\ &= \frac{1}{2}\int_{-1}^{2}\left[(x+2)^2 - x^4\right]dx \\ &= \frac{1}{2}\left|\frac{(x+2)^3}{3} - \frac{x^5}{5}\right|_{-1}^{2} \\ &= \frac{1}{2}\left(\frac{64}{3} - \frac{32}{5} - \frac{1}{3} - \frac{1}{5}\right) \\ &= \frac{36}{5} \end{aligned}$$

Example 14

Evaluate $\iint xy(x+y)\,dx\,dy$, *over the region enclosed by the parabolas* $x^2 = y$, $y^2 = -x$.

Solution

1. The region of integration is *OPQ*.

2. The points of intersection of the parabola $x^2 = y$, and $y^2 = -x$ are obtained as

$$y^4 = y$$
$$y = 0, 1$$
$$\therefore\; x = 0, -1.$$

   The points of intersection are O (0, 0) and Q (–1, 1).

3. Here, it is easier to integrate w.r.t. x first. Draw a horizontal strip AB parallel to x-axis, which starts from the parabola $x^2 = y$ and terminates on the parabola $y^2 = -x$.

4. Limits of x : $x = -\sqrt{y}$ to $x = -y^2$

   Limits of y : $y = 0$ to $y = 1$

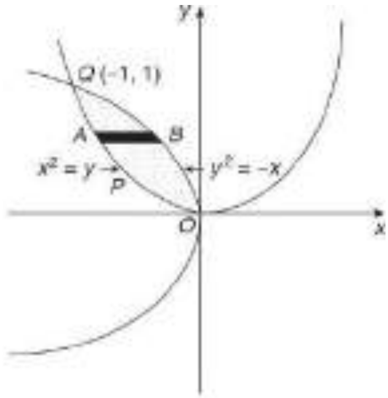

Fig. 9.18

$$I = \int_0^1 y \int_{-\sqrt{y}}^{-y^2} xy(x+y)\,dx\,dy$$

$$= \int_0^1 y \int_{-\sqrt{y}}^{-y^2} (x^2 + xy)\,dx\,dy$$

$$= \int_0^1 y \left| \frac{x^3}{3} + \frac{x^2 y}{2} \right|_{-\sqrt{y}}^{-y^2} dy$$

$$= \int_0^1 \left(\frac{-y^7}{3} + \frac{y^6}{2} + \frac{y^{\frac{5}{2}}}{3} - \frac{y^3}{2} \right) dy$$

$$= \left| -\frac{y^8}{24} + \frac{y^7}{14} + \frac{2y^{\frac{7}{2}}}{21} - \frac{y^4}{8} \right|_0^1$$

$$= 0$$

Example 15

Evaluate $\iint xy\,dx\,dy$ *over the region enclosed by the x-axis, the line* $x = 2a$ *and the parabola* $x^2 = 4ay$.

Solution

1. The region of integration is OPQ.
2. The point of intersection of the parabola $x^2 = 4ay$ and the line $x = 2a$ is obtained as

   $4a^2 = 4ay$

   $y = a$

   The point of intersection is Q ($2a$, a).
3. The integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to y-axis, which starts from x-axis and terminates on the parabola $x^2 = 4ay$.

4. Limits of y : $y = 0$ to $y = \frac{x^2}{4a}$

   Limits of x : $x = 0$ to $x = 2a$

$$
\begin{aligned}
\text{I} &= \int_0^{2a}\int_0^{\frac{x^2}{4a}} xy\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{2a} x\int_0^{\frac{x^2}{4a}} y\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^{2a} x\left|\frac{y^2}{2}\right|_0^{\frac{x^2}{4a}}\mathrm{d}x \\
&= \int_0^{2a} x\cdot\frac{x^4}{32a^2}\,\mathrm{d}x \\
&= \frac{1}{32a^2}\left|\frac{x^6}{6}\right|_0^{2a} \\
&= \frac{1}{32a^2}\cdot\frac{64a^6}{6} \\
&= \frac{a^4}{3}
\end{aligned}
$$

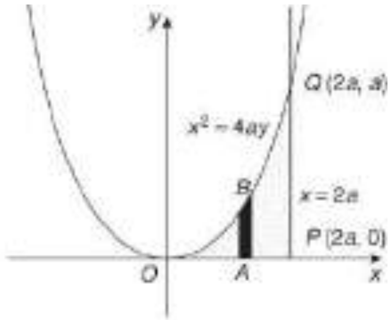

Fig. 9.19

Example 16

Evaluate $\iint xy\,\mathrm{d}x\,\mathrm{d}y$, *over the region enclosed by the circle* $x^2 + y^2 - 2x = 0$, *the parabola* $y^2 = 2x$ *and the line* $y = x$.

Solution

1. The region of integration is *OPQRO*.
2. (i) The points of intersection of the circle $x^2 + y^2 - 2x = 0$ and the line $y = x$ are obtained as

$$x^2 + x^2 - 2x = 0$$
$$x = 0, 1$$
$$\therefore\ y = 0, 1$$

   The points of intersection are O (0, 0) and P (1, 1).

   (ii) The point of intersection of the circle $x^2 + y^2 - 2x = 0$ and the parabola $y^2 = 2x$ is obtained as

$$x^2 + 2x - 2x = 0$$
$$x = 0$$
$$\therefore\ y = 0$$

   The point of intersection is O (0, 0).

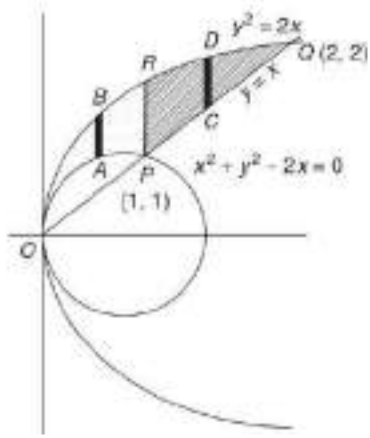

Fig. 9.20

(iii) The points of intersection of the parabola $y^2 = 2x$ and the line $y = x$ are obtained as $x^2 = 2x$

$x = 0, 2$

$\therefore\ y = 0, 2$

The points of intersection are O (0, 0) and Q (2, 2).

3. The integration can be done w.r.t. any variable first. To integrate w.r.t. y first we need to draw a vertical strip in the region. But one vertical strip does not cover the entire region, therefore, divide the region *OPQRO* into two subregions *OPR* and *RPQ* and draw one vertical strip in each subregion.
4. In the subregion *OPR,* strip starts from the circle $x^2 + y^2 - 2x = 0$ and terminates on the parabola $y^2 = 2x$.

   Limits of y : $y = \sqrt{2x - x^2}$ to $y = \sqrt{2x}$

   Limits of x : $x = 0$ to $x = 1$.
5. In the subregion *RPQ*, strip starts from the line $y = x$ and terminates on the parabola $y^2 = 2x$.

   Limits of y : $y = x$ to $y = \sqrt{2x}$

   Limits of x : $x = 1$ to $x = 2$

$$
\begin{aligned}
I &= \iint xy \,dx\,dy \\
&= \iint_{OPR} xy\,dx\,dy + \iint_{RPQ} xy\,dx\,dy \\
&= \int_0^1 \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} xy\,dy\,dx + \int_1^2 \int_x^{\sqrt{2x}} xy\,dy\,dx \\
&= \int_0^1 x \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} y\,dy\,dx + \int_1^2 x \int_x^{\sqrt{2x}} y\,dy\,dx \\
&= \int_0^1 x \left| \frac{y^2}{2} \right|_{\sqrt{2x-x^2}}^{\sqrt{2x}} dx + \int_1^2 x \left| \frac{y^2}{2} \right|_x^{\sqrt{2x}} dx \\
&= \frac{1}{2}\int_0^1 x(2x - 2x + x^2)\,dx + \frac{1}{2}\int_1^2 x(2x - x^2)\,dx \\
&= \frac{1}{2}\left| \frac{x^4}{4} \right|_0^1 + \frac{1}{2}\left| \frac{2x^3}{3} - \frac{x^4}{4} \right|_1^2 \\
&= \frac{1}{8} + \frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{8} \\
&= \frac{7}{12}
\end{aligned}
$$

Example 17

Evaluate $\iint x^2\,dx\,dy$, *over the region in the first quadrant enclosed by the rectangular hyperbola* $xy = 16$, *the lines* $y = x$, $y = 0$ *and* $x = 8$. **[Winter 2014]**

Solution

1. The region of integration is *OPQR*.
2. (i) The points of intersection of the hyperbola $xy = 16$ and the line $y = x$ are obtained as

   $x^2 = 16, x = \pm 4$

   $\therefore\ y = \pm 4$

   Hence, R (4, 4) is the point of intersection in the first quadrant.

   (ii) The point of intersection of the hyperbola $xy = 16$ and line $x = 8$ is obtained as

   $8y = 16$

   $y = 2$

   The point of intersection is Q (8, 2).

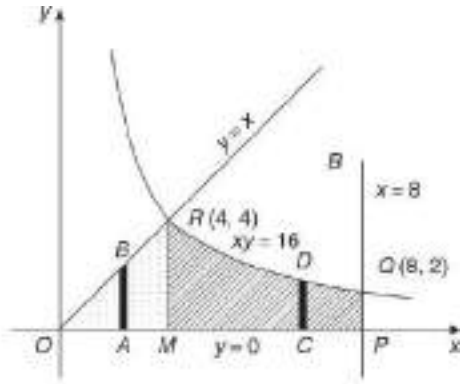

Fig. 9.21

3. The integration can be done w.r.t. any variable first. To integrate w.r.t. y first we need to draw a vertical strip in the region. But here one vertical strip cannot cover the entire region, and therefore divide the region *OPQR* into two subregions *OMR* and *RMPQ* and draw one vertical strip in each subregion.
4. In the subregion *OMR*, strip starts from x axis and terminates on the line $y = x$.

   Limits of y : $y = 0$ to $y = x$

   Limits of x : $x = 0$ to $x = 4$
5. In subregion *RMPQ*, strip starts from x axis and terminates on the rectangular hyperbola $xy = 16$

   Limits of y : $y = 0$ to $y = \dfrac{16}{x}$

   Limits of x : $x = 4$ to $x = 8$

$$
\begin{aligned}
I &= \iint x^2\,dx\,dy \\
&= \iint_{OMR} x^2\,dx\,dy + \iint_{RMPQ} x^2\,dx\,dy \\
&= \int_0^4\int_0^x x^2\,dy\,dx + \int_4^8\int_0^{\frac{16}{x}} x^2\,dy\,dx \\
&= \int_0^4 x^2\int_0^x dy\,dx + \int_4^8 x^2\int_0^{\frac{16}{x}} dy\,dx \\
&= \int_0^4 x^2\,|y|_0^x\,dx + \int_4^8 x^2\,|y|_0^{\frac{16}{x}}\,dx \\
&= \int_0^4 x^3\,dx + \int_4^8 x^2\cdot\frac{16}{x}\,dx \\
&= \left|\frac{x^4}{4}\right|_0^4 + 16\left|\frac{x^2}{2}\right|_4^8 \\
&= 64 + 8(64 - 16) \\
&= 448
\end{aligned}
$$

Example 18

Evaluate $\iint \frac{dx\,dy}{x^4+y^2}$, *over the region bounded by the* $y \geq x^2$, $x \geq 1$.

Solution

1. The region of integration is bounded by $y \geq x^2$ (the region inside the parabola $x^2 = y$) and $x \geq 1$ (the region on the right of line $x = 1$).
2. The point of intersection of $x^2 = y$ and $x = 1$ is obtained as $1 = y$.
   The point of intersection is $P(1, 1)$.
3. Here, it is easier to integrate w.r.t. y first than x. Draw a vertical strip AB parallel to y-axis in the region which starts from the parabola $x^2 = y$ and extends up to infinity.
4. Limits of $y : y = x^2$ to $y \to \infty$
   Limits of $x : x = 1$ to $x \to \infty$

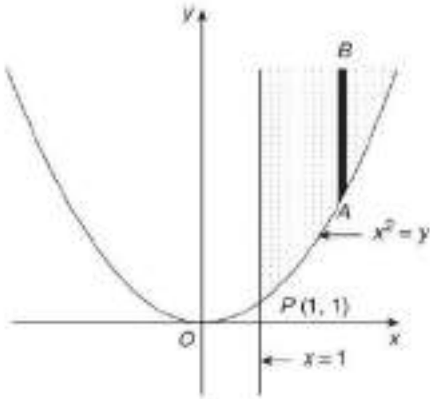

Fig. 9.22

$$I = \int_1^{\infty}\int_{x^2}^{\infty} \frac{1}{x^4+y^2}\,dy\,dx$$

$$= \int_1^{\infty} \left| \frac{1}{x^2}\tan^{-1}\frac{y}{x^2} \right|_{x^2}^{\infty} dx$$

$$= \int_1^{\infty} \frac{1}{x^2}(\tan^{-1}\infty - \tan^{-1}1)dx$$

$$= \int_1^{\infty} \frac{1}{x^2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)dx$$

$$= \frac{\pi}{4}\left| -\frac{1}{x} \right|_1^{\infty}$$

$$= \frac{\pi}{4}$$

EXERCISE 9.2

Evaluate the following integrals:

1. $\iint \frac{1}{xy}\,dx\,dy$, over the rectangle $1 \leq x \leq 2$, $1 \leq y \leq 2$.

[Ans.: $(\log 2)^2$]

2. $\iint \sin \pi(ax+by)\,dx\,dy$, over the triangle bounded by the lines $x = 0$, $y = 0$ and $ax + by = 1$.

$\left[\textbf{Ans.: } \frac{1}{\pi ab}\right]$

3. $\iint e^{3x+4y}\,dx\,dy$, over the triangle bounded by the lines $x = 0$, $y = 0$, and $x + y = 1$.

$$\left[\textbf{Ans.}: \frac{1}{12}(3e^4 - 4e^3 + 1)\right]$$

4. $\iint xy\sqrt{1-x-y}\,dx\,dy$, over the triangle bounded by $x = 0$, $y = 0$ and $x + y = 1$.

$$\left[\textbf{Ans.}: \frac{16}{945}\right]$$

5. $\iint \sqrt{xy - y^2}\,dx\,dy$, over the triangle having vertices (0, 0), (10, 1), (1, 1).

[Ans.: 6]

6. $\iint (x + y + a)\,dx\,dy$, over the region bounded by the circle $x^2 + y^2 = a^2$.

[Ans.: πa^3]

7. $\iint xy\,dx\,dy$, over the region bounded by the x-axis, the line $y = 2x$ and the parabola $y = \dfrac{x^2}{4a}$.

$$\left[\textbf{Ans.}: \frac{2048}{3}a^4\right]$$

8. $\iint 5 - 2x - y\,(dx)\,dy$, over the region bounded by x-axis, the line $x + 2y = 3$ and the parabola $y^2 = x$.

$$\left[\textbf{Ans.}: \frac{217}{60}\right]$$

9. $\iint (4x^2 - y^2)^{\frac{1}{2}}\,dx\,dy$, over the triangle bounded by x-axis, the line $y = x$ and $x = 1$.

$$\left[\textbf{Ans.}: \frac{1}{3}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)\right]$$

10. $\iint xy(x + y)\,dx\,dy$, over the region bounded by the parabola $y^2 = x$, $x^2 = y$.

$$\left[\textbf{Ans.}: \frac{3}{28}\right]$$

11. $\iint xy(x + y)\,dx\,dy$, over the region bounded by the curve $x^2 = y$ and the line $x = y$.

$$\left[\textbf{Ans.}: \frac{3}{56}\right]$$

12. $\iint xy(x - 1)\,dx\,dy$, over the region bounded by the rectangular hyperbola $xy = 4$, the lines $y = 0$, $x = 1$, $x = 4$ and x-axis.

[Ans.: 8(3 – log 4)]

9.3 CHANGE OF ORDER OF INTEGRATION

Sometimes, evaluation of double integral becomes easier by changing the order of integration. To change the order of integration, first, we draw the region of integration with the help of the given limits. Then we draw a vertical or horizontal strip as per the required order of integration. This change of order also changes the limits of integration.

Type I Change of Order of Integration

Example 1

Change the order of integration of $\int_0^1\int_0^x f(x, y)\mathrm{d}x\,\mathrm{d}y$.

Solution

1. Since inner limits depends on x, the function is integrated first w.r.t. y and then w.r.t. x.

   The correct form of the integral

   $= \int_0^1\int_0^x f(x, y)\mathrm{d}y\,\mathrm{d}x.$

2. Limits of y : $y = 0$ to $y = x$, along vertical strip $A'B'$

   Limits of x : $x = 0$ to $x = 1$

3. The region is bounded by the lines $y = 0$, $y = x$, and $x = 1$.
4. The point of intersection of $y = x$ and $x = 1$ is $Q(1, 1)$.
5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from the line $y = x$ and terminates on the line $x = 1$.

   Limits of $x : x = y$ to $x = 1$

   Limits of $y : y = 0$ to $y = 1$

Hence, the given integral after change of order is

$$\int_0^1\int_0^x f(x, y)\mathrm{d}y\,\mathrm{d}x = \int_0^1\int_y^1 f(x, y)\mathrm{d}x\,\mathrm{d}y$$

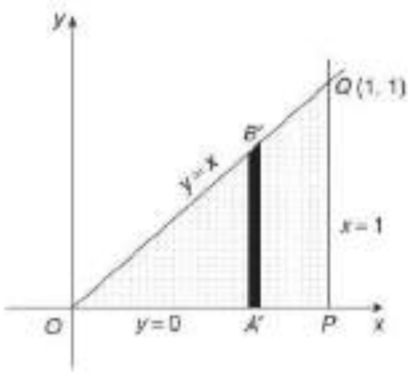

Fig. 9.23

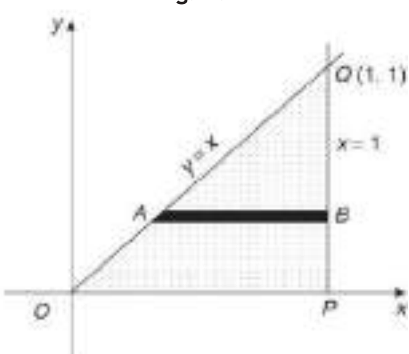

Fig. 9.24

Example 2

Change the order of integration of $\int_0^1\int_0^y f(x, y)\mathrm{d}y\,\mathrm{d}x$.

Solution

1. Since inner limits depend on y, the function is integrated first w.r.t. x and then w.r.t. y. The correct form of the integral $= \int_0^1\int_0^y f(x, y)\,dx\,dy.$
2. Limits of $x : x = 0$ to $x = y$, along horizontal strip $A'B'$
   Limits of $y : y = 0$ to $y = 1$
3. The region is bounded by the lines $x = 0$, $x = y$, and $y = 1$.
4. The point of intersection of $x = y$ and $y = 1$ is $Q\ (1, 1)$.
5. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from the line $x = y$ and terminates on the line $y = 1$.
   Limits of $y : y = x$ to $y = 1$
   Limits of $x : x = 0$ to $x = 1$

Hence, the given integral after change of order is

$$\int_0^1\int_0^y f(x, y)\,dx\,dy = \int_0^1\int_x^1 f(x, y)\,dy\,dx$$

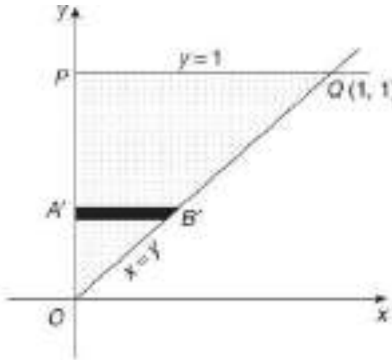

Fig. 9.25

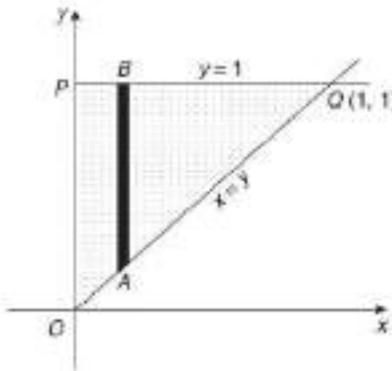

Fig. 9.26

Example 3

Change the order of integration of $\int_0^a\int_x^a f(x, y)\,dy\,dx.$

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of $y : y = x$ to $y = a$, along vertical strip $A'B'$.
   Limits of $x : x = 0$ to $x = a$
3. The region is bounded by the lines $y = x$, $y = a$ and $x = 0$.
4. The point of intersection of $y = x$ and $y = a$ is $Q\ (a, a)$.
5. To change the order of integration i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from

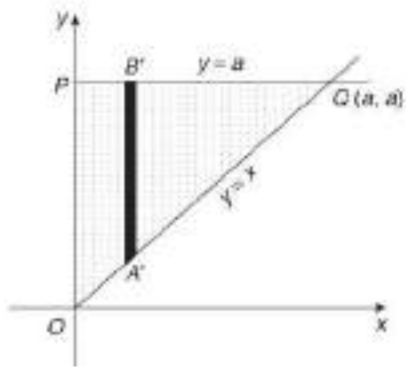

Fig. 9.27

the line $x = 0$ and terminates on the line $y = x$.

Limits of $x : x = 0$ to $x = y$

Limits of $y : y = 0$ to $y = a$

Hence, the given integral after change of order is

$$\int_0^a \int_x^a f(x, y)\,dy\,dx = \int_0^a \int_0^y f(x, y)\,dx\,dy$$

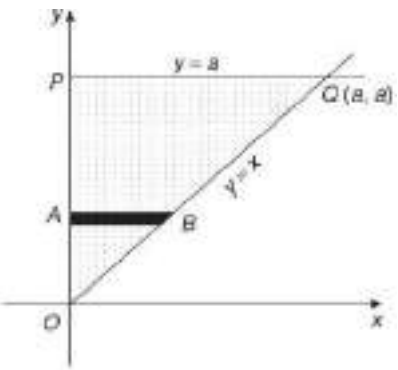

Fig. 9.28

Example 4

Change the order of integration of $\int_0^\infty \int_x^\infty f(x, y)\,dx\,dy$.

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y and then w.r.t. x.

   The correct form of the integral

   $= \int_0^\infty \int_x^\infty f(x, y)\,dy\,dx$

2. Limits of $y : y = x$ to $y \to \infty$, along vertical strip

   Limits of $x : x = 0$ to $x \to \infty$

3. The region is bounded by the lines $y = x$ and $x = 0$.

4. Here the only point of intersection is origin O.

5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from the line $x = 0$ and terminates on the line $y = x$.

   Limits of x : $x = 0$ to $x = y$

   Limits of y : $y = 0$ to $y \to \infty$

Hence, the given integral after change of order is

$$\int_0^\infty \int_x^\infty f(x, y)\,dy\,dx = \int_0^\infty \int_0^y f(x, y)\,dx\,dy$$

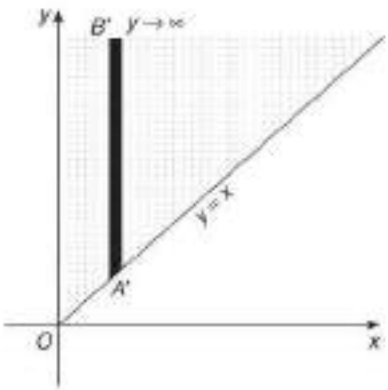

Fig. 9.29

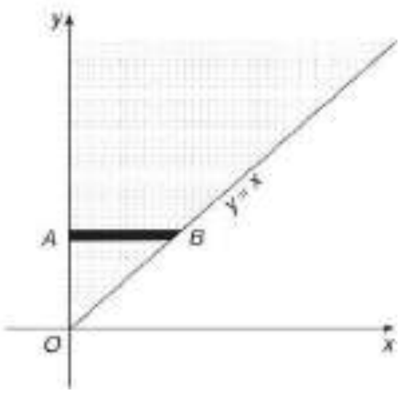

Fig. 9.30

Example 5

Change the order of integration of $\int_0^1 \int_x^{\sqrt{x}} f(x, y)\,dy\,dx$.

Solution

1. The function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of y : $y = x$ to $y = \sqrt{x}$

   Limits of x : $x = 0$ to $x = 1$
3. The region is bounded by the line $y = x$ and the parabola $y^2 = x$.
4. The points of intersection of $y^2 = x$ and $y = x$ are obtained as

   $x^2 = x$

   $x = 0, 1$

   $\therefore y = 0, 1.$

   The points of intersection are O (0, 0) and Q (1, 1).

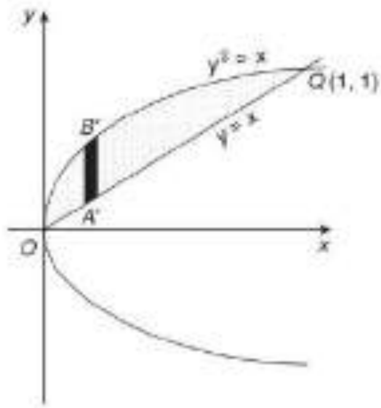

Fig. 9.31

5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from the parabola $y^2 = x$ and terminates on the line $y = x$.

   Limits of x : $x = y^2$ to $x = y$

   Limits of y : $y = 0$ to $y = 1$

Hence, the given integral after change of order is

$$\int_0^1 \int_x^{\sqrt{x}} f(x, y)\,dy\,dx = \int_0^1 \int_{y^2}^{y} f(x, y)\,dx\,dy$$

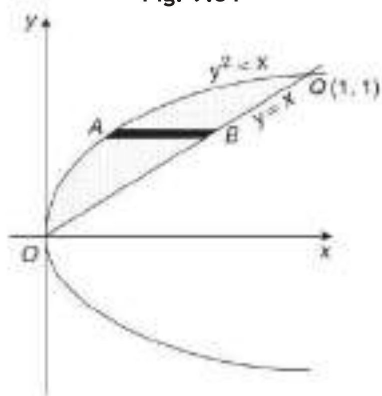

Fig. 9.32

Example 6

Change the order of integration of $\int_0^1 \int_{y^2}^{y^{\frac{1}{3}}} f(x, y)\,dx\,dy$.

Solution

1. The function is integrated first w.r.t. x and then w.r.t. y.

2. Limits of x : $x = y^2$ to $x = y^{\frac{1}{3}}$

   Limits of y : $y = 0$ to $y = 1$
3. The region is bounded by the parabola $y^2 = x$ and the cubical parabola $y = x^3$.
4. The points of intersection of $y^2 = x$ and $y = x^3$ are obtained as

   $x^6 = x$

   $x = 0, 1$

   $\therefore y = 0, 1.$

   The points of intersection are O (0, 0) and Q (1, 1).

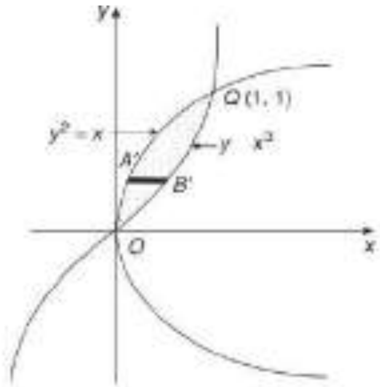

Fig. 9.33

5. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip parallel to y-axis which starts from the cubical parabola $y = x^3$ and terminates on the parabola $y^2 = x$.

   Limits of y : $y = x^3$ to $y = \sqrt{x}$

   Limits of x : $x = 0$ to $x = 1$

Hence, the given integral after change of order is

$$\int_0^1 \int_{y^2}^{y^{\frac{1}{3}}} f(x, y)\,dx\,dy = \int_0^1 \int_{x^3}^{\sqrt{x}} f(x, y)\,dy\,dx$$

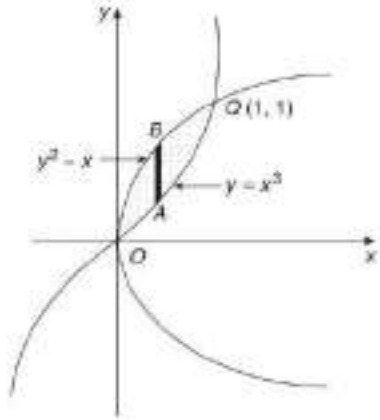

Fig. 9.34

Example 7

Change the order of integration of

$$\int_0^8 \int_{\frac{y-8}{4}}^{\frac{y}{4}} f(x, y)\,dx\,dy.$$

Solution

1. The function is integrated first w.r.t. x and then w.r.t. y.
2. Limits of x : $x = \frac{y-8}{4}$ to $x = \frac{y}{4}$

   Limits of y : $y = 0$ to $y = 8$
3. The region is bounded by the line $y = 4x + 8$, $y = 4x$, $y = 8$ and x-axis ($y = 0$).

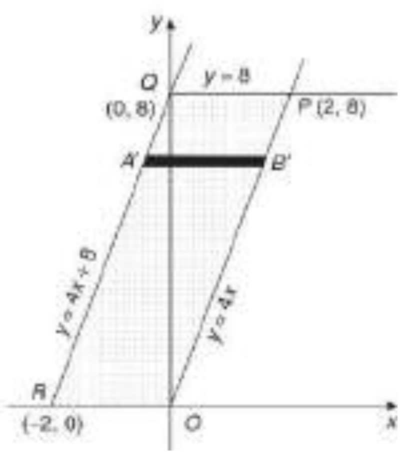

Fig. 9.35

4. The point of intersection of $y = 4x$ and $y = 8$ is obtained as

   $8 = 4x$

   $x = 2.$

   The point of intersection is P (2, 8).
5. To change the order of integration, i.e., to integrate first w.r.t. y, divide the region $OPQR$ into two subregions OQR and OPQ. Draw a vertical strip parallel to y-axis in each subregion.

   (i) In subregion OQR, strip AB starts from x-axis and terminates on the line $y = 4x + 8$.

   Limits of y : $y = 0$ to $y = 4x + 8$

   Limits of x : $x = -2$ to $x = 0$

   (ii) In subregion OPQ, strip CD starts from the line $y = 4x$ and terminates on the line $y = 8$.

   Limits of y : $y = 4x$ to $y = 8$

   Limits of x : $x = 0$ to $x = 2$

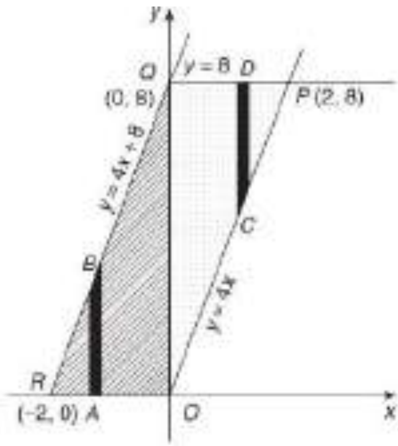

Fig. 9.36

Hence, the given integral after change of order is

$$\int_0^8 \int_{\frac{y-8}{4}}^{\frac{y}{4}} f(x, y)\,dx\,dy = \int_{-2}^0 \int_0^{4x+8} f(x, y)\,dy\,dx + \int_0^2 \int_{4x}^8 f(x, y)\,dy\,dx$$

Example 8

Change the order of integration of $\int_{-a}^{a} \int_0^{\frac{y^2}{a}} f(x, y)\,dx\,dy.$

Solution

1. The function is integrated first w.r.t. x and then w.r.t. y.
2. Limits of x : $x = 0$ to $x = \dfrac{y^2}{a}$

   Limits of y : $y = -a$ to $y = a$
3. The region is bounded by the y-axis, the parabola $y^2 = ax$, and the line $y = -a$, and $y = a$.
4. (i) The point of intersection of $y^2 = ax$ and $y = -a$ is obtained as

   $a^2 = ax$

   $x = a$

   The point of intersection is R (a, $-a$).

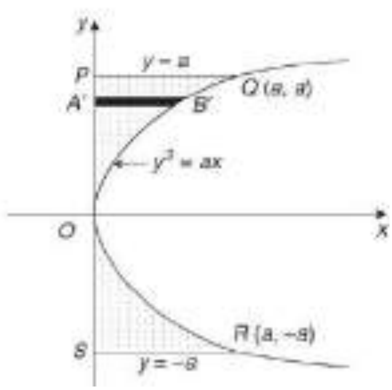

Fig. 9.37

(ii) The point of intersection of $y^2 = ax$ and $y = a$ is obtained as

$a^2 = ax$

$x = a$

The point of intersection is $Q\,(a, a)$.

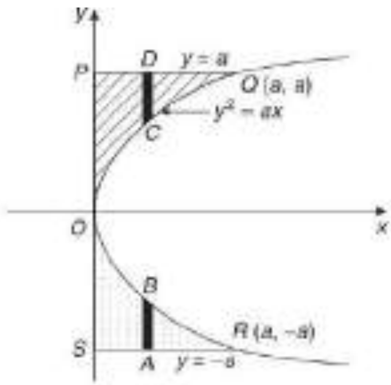

Fig. 9.38

5. To change the order of integration, i.e., to integrate first w.r.t. y, divide the region into two subregions ORS and OPQ. Draw a vertical strip parallel to y-axis in each subregion.

   (i) In subregion ORS, strip AB starts from the line $y = -a$ and terminates on the parabola $y^2 = ax$.

   Limits of y : $y = -a$ to $y = -\sqrt{ax}$ (part of the parabola below x-axis)

   Limits of x : $x = 0$ to $x = a$

   (ii) In subregion OPQ, strip CD starts from the parabola $y^2 = ax$ and terminates on the line $y = a$.

   Limits of y : $y = \sqrt{ax}$ to $y = a$

   Limits of x : $x = 0$ to $x = a$

Hence, the given integral after change of order is

$$\int_{-a}^{a}\int_{0}^{\frac{y^2}{a}} f(x, y)\,dx\,dy = \int_{0}^{a}\int_{-a}^{-\sqrt{ax}} f(x, y)\,dy\,dx + \int_{0}^{a}\int_{\sqrt{ax}}^{a} f(x, y)\,dy\,dx$$

Example 9

Change the order of integration of $\int_{0}^{2}\int_{y}^{2+\sqrt{4-2y}} f(x, y)\,dx\,dy$.

Solution

1. The function is integrated first w.r.t. x and then w.r.t. y.
2. Limits of x : $x = y$ to $x = 2 + \sqrt{4 - 2y}$

   Limits of y : $y = 0$ to $y = 2$
3. The region is bounded by the x-axis, the line $y = x$ and the parabola $(x - 2)^2 = 2(2 - y)$.
4. The points of intersection of $y = x$ and $(x - 2)^2 = 2(2 - y)$ are obtained as

   $(x - 2)^2 = 2(2 - x)$

   $x = 0, 2$

   $\therefore\ y = 0, 2.$

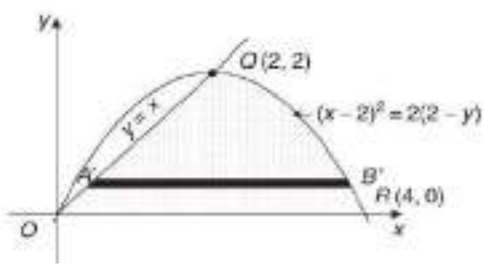

Fig. 9.39

The points of intersection are O (0, 0) and Q (2, 2).

5. To change the order of integration, i.e., to integrate first w.r.t. y, divide the region into two subregions OPQ and PQR. Draw a vertical strip parallel to y-axis in each subregion.

   (i) In subregion OPQ, strip AB starts from x-axis and terminates on the line $y = x$.
   Limits of y : $y = 0$ to $y = x$
   Limits of x : $x = 0$ to $x = 2$

   (ii) In subregion PQR, strip CD starts from x-axis and terminates on the parabola $(x - 2)^2 = 2(2 - y)$.
   Limits of y : $y = 0$ to $y = 2x - \dfrac{x^2}{2}$
   Limits of x : $x = 2$ to $x = 4$

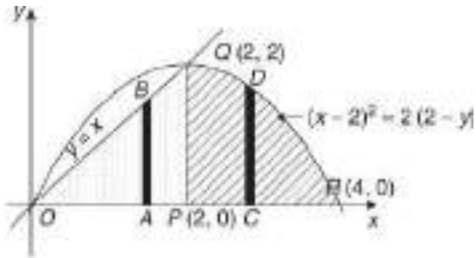

Fig. 9.40

Hence, the given integral after change of order is

$$\int_0^2 \int_y^{2+\sqrt{4-2y}} f(x, y)\,dx\,dy = \int_0^2 \int_0^x f(x, y)\,dy\,dx + \int_2^4 \int_0^{2x-\frac{x^2}{2}} f(x, y)\,dy\,dx$$

Example 10

Change the order of integration of $\displaystyle\int_0^{a\cos\alpha} \int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x, y)\,dy\,dx.$

Solution

1. The function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of y : $y = x\tan\alpha$ to $y = \sqrt{a^2 - x^2}$
   Limits of x : $x = 0$ to $x = a\cos\alpha$
3. The region is bounded by the line $y = x\tan\alpha$, the circle $x^2 + y^2 = a^2$ and y-axis. Since given limits of x and y are positive, the region lies in the first quadrant.
4. The points of intersection of $y = x\tan\alpha$ and $x^2 + y^2 = a^2$ are obtained as
   $x^2 + x^2\tan^2\alpha = a^2$
   $x = \pm a\cos\alpha$
   $\therefore\ y = \pm a\sin\alpha.$
   The points of intersection are P $(a\cos\alpha, a\sin\alpha)$ and $P'(-a\cos\alpha, -a\sin\alpha)$.

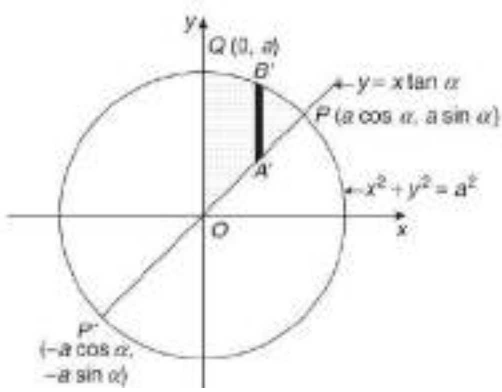

Fig. 9.41

5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into two subregions OPR and PQR. Draw a horizontal strip in each subregion.
 (i) In subregion OPR, strip AB starts from y-axis and terminates on the line $y = x \tan \alpha$.
 Limits of x : $x = 0$ to $x = y \cot \alpha$
 Limits of y : $y = 0$ to $y = a \sin \alpha$
 (ii) In subregion PQR, strip CD starts from y-axis and terminates on the circle $x^2 + y^2 = a^2$.
 Limits of x : $x = 0$ to $x = \sqrt{a^2 - y^2}$
 Limits of y : $y = a \sin \alpha$ to $y = a$

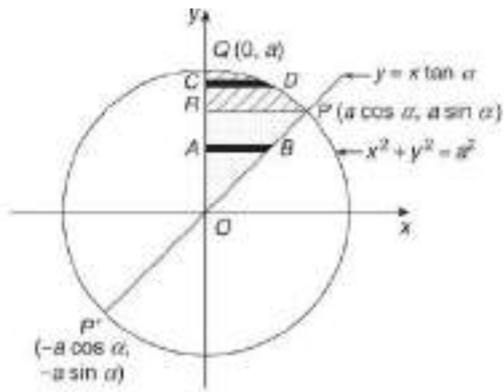

Fig. 9.42

Hence, given integral after change of order is

$$\int_0^{a\cos\alpha}\int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x, y)\,dy\,dx = \int_0^{a\sin\alpha}\int_0^{y\cot\alpha} f(x, y)\,dx\,dy + \int_{a\sin\alpha}^{a}\int_0^{\sqrt{a^2-y^2}} f(x, y)\,dx\,dy$$

Example 11

Change the order of integration of $\int_0^4\int_{\sqrt{4x-x^2}}^{\sqrt{4x}} f(x, y)\,dy\,dx.$

Solution

1. The function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of y : $y = \sqrt{4x - x^2}$ to $y = \sqrt{4x}$
 Limits of x : $x = 0$ to $x = 4$.
3. The region is bounded by the circle $x^2 + y^2 - 4x = 0$, the parabola $y^2 = 4x$ and the line $x = 4$.
4. (i) The point of intersection of $x^2 + y^2 - 4x = 0$ and $y^2 = 4x$ is obtained as
 $x^2 = 0$
 $x = 0$
 $\therefore\ y = 0.$
 The point of intersection is O (0, 0).
 (ii) The points of intersection of $y^2 = 4x$ and $x = 4$ are obtained as
 $y^2 = 16$
 $y = \pm 4$
 The points of intersection are Q (4, 4) and Q' (4, −4).

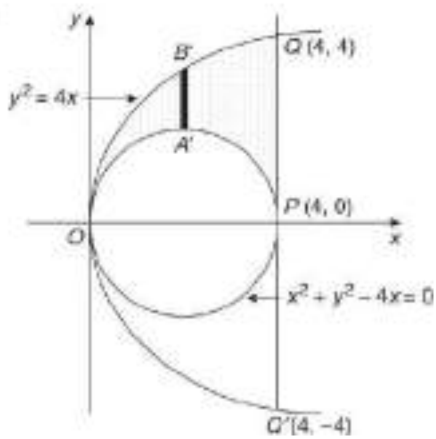

Fig. 9.43

5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into three subregions ORT, TPS and RSQ. Draw a horizontal strip parallel to x-axis in each subregion.

   (i) In subregion ORT, strip AB starts from the parabola $y^2 = 4x$ and terminates on the circle $x^2 + y^2 - 4x = 0$.

   Limits of x :

   $$x = \frac{y^2}{4} \text{ to } x = 2 - \sqrt{4 - y^2}$$

   (Part of the circle where $x < 2$)

   Limits of y : $y = 0$ to $y = 2$

   (ii) In subregion TPS, strip CD starts from the circle $x^2 + y^2 - 4x = 0$ and terminates on the line $x = 4$.

   Limits of x : $x = 2 + \sqrt{4 - y^2}$ (Part of circle where $x > 2$) to $x = 4$

   Limits of y : $y = 0$ to $y = 2$

   (iii) In subregion RSQ, strip EF starts from the parabola $y^2 = 4x$ and terminates on the line $x = 4$.

   Limits of x : $x = \frac{y^2}{4}$ to $x = 4$

   Limits of y : $y = 2$ to $y = 4$

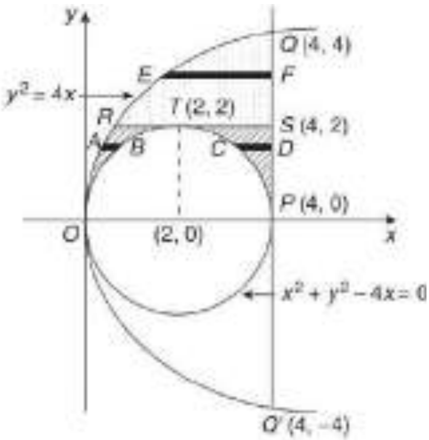

Fig. 9.44

Hence, given integral after change of order is

$$\int_0^4 \int_{\sqrt{4x-x^2}}^{\sqrt{4x}} f(x, y)\,dy\,dx = \int_0^2 \int_{\frac{y^2}{4}}^{2-\sqrt{4-y^2}} f(x, y)\,dx\,dy + \int_0^2 \int_{2+\sqrt{4-y^2}}^{4} f(x, y)\,dx\,dy + \int_2^4 \int_{\frac{y^2}{4}}^{4} f(x, y)\,dx\,dy$$

Example 12

Change the order of integration of $\int_0^2 \int_{\sqrt{4-x}}^{(4-x)^2} f(x, y)\,dy\,dx.$

Solution

1. The function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of y : $y = \sqrt{4-x}$ to $y = (4-x)^2$.

   Limits of x : $x = 0$ to $x = 2$
3. The region is enclosed by the parabolas $y^2 = 4 - x$, $y = (4 - x)^2$, the lines $x = 0$ and $x = 2$.
4. (i) The points of intersection of $x = 2$ and $y^2 = 4 - x$ are obtained as

$y^2 = (4 - 2)$

$y = \pm\sqrt{2}.$

The points of intersection are $Q(2, \sqrt{2})$ and $Q'(2, -\sqrt{2})$.

(ii) The point of intersection of $x = 2$ and $y = (4 - x)^2$ is obtained as

$y = (4 - 2)^2 = 4.$

The point of intersection is $S(2, 4)$.

(iii) The points of intersection of $x = 0$ and $y^2 = 4 - x$ are obtained as

$y^2 = 4$

$y = \pm 2.$

The points of intersection are $P(0, 2)$ and $P'(0, -2)$.

(iv) The point of intersection of $x = 0$ and $y = (4 - x)^2$ is obtained as

$y = 16.$

The point of intersection is $U(0, 16)$.

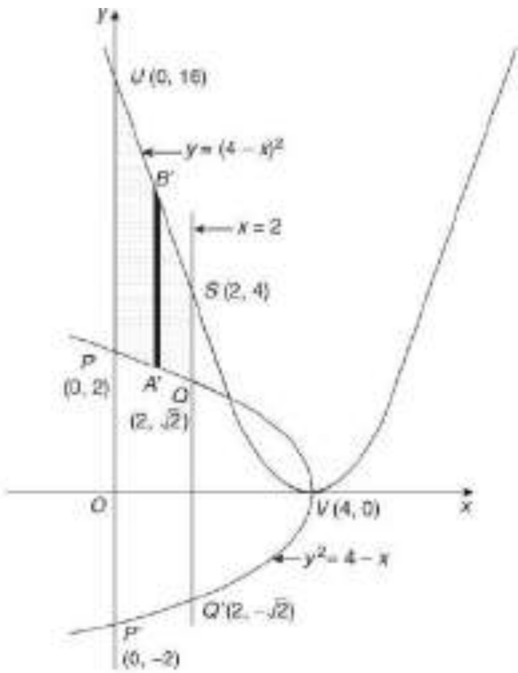

Fig. 9.45

5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into three subregions PQR, $PRST$ and STU. Draw a horizontal strip in each subregion.

(i) In subregion PQR, strip AB starts from the parabola $y^2 = 4 - x$ and terminates on the line $x = 2$.

Limits of x :

$x = 4 - y^2$ to $x = 2$

Limits of y :

$y = \sqrt{2}$ to $y = 2$

(ii) In subregion $PRST$, strip CD starts from y-axis and terminates on the line $x = 2$.

Limits of x : $x = 0$ to $x = 2$

Limits of y : $y = 2$ to $y = 4$

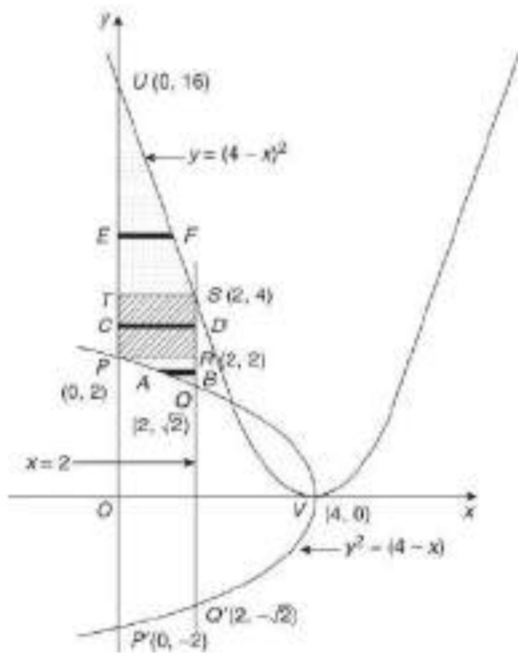

Fig. 9.46

(iii) In subregion *STU*, strip *EF* starts from y-axis and terminates on the parabola $y = (4 - x)^2$.

Limits of $x : x = 0$ to $x = 4 - \sqrt{y}$ (Part of the parabola where $x < 4$)

Limits of $y : y = 4$ to $y = 16$

Hence, the given integral after change of order is

$$\int_0^2 \int_{\sqrt{4-x}}^{(4-x)^2} f(x, y)\mathrm{d}y\,\mathrm{d}x = \int_{\sqrt{2}}^2 \int_{4-y^2}^2 f(x, y)\mathrm{d}x\,\mathrm{d}y + \int_2^4 \int_0^2 f(x, y)\mathrm{d}x\,\mathrm{d}y + \int_4^{16} \int_0^{4-\sqrt{y}} f(x, y)\mathrm{d}x\,\mathrm{d}y$$

Type II Evaluation of Double Integrals by Changing the Order of Integration

Example 1

Change the order of integration and evaluate $\int_0^a \int_x^a (x^2 + y^2)\mathrm{d}y\,\mathrm{d}x.$

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y.
2. Limits of $y : y = x$ to $y = a$, along vertical strip
   Limits of $x : x = 0$ to $x = a$
3. The region is bounded by the lines $y = x$, $y = a$ and $x = 0$
4. The point of intersection of $y = x$ and $y = a$ is $Q\,(a, a)$.
5. To change the order of integration, i.e. to integrate first w.r.t. x, draw a horizontal strip *AB* parallel to x-axis which starts from y-axis and terminates on the line $y = x$.
   Limits of $x : x = 0$ to $x = y$
   Limits of $y : y = 0$ to $y = a$

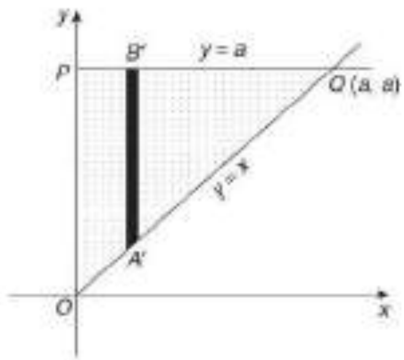

Fig. 9.47

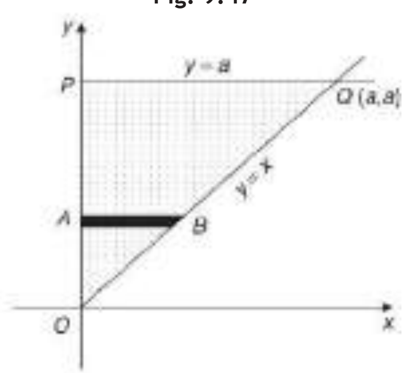

Fig. 9.48

Hence, the given integral after change of order is

$$\int_0^a \int_x^a (x^2 + y^2)\mathrm{d}y\,\mathrm{d}x = \int_0^a \int_0^y (x^2 + y^2)\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^a \left| \frac{x^3}{3} + y^2 x \right|_0^y \mathrm{d}y$$

$$= \int_0^a \left(\frac{y^3}{3} + y^3 \right) \mathrm{d}y$$

$$= \int_0^a \frac{4}{3} y^3 \, dy$$

$$= \frac{4}{3} \left| \frac{y^4}{4} \right|_0^a$$

$$= \frac{a^4}{3}$$

Example 2

Change the order of integration and evaluate $\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} \, dy \, dx.$

Solution

1. The function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.
2. Limits of $y : y = x$ to $y = \pi$
   Limits of $x : x = 0$ to $x = \pi$
3. The region is bounded by the line $y = x$, $y = \pi$ and $x = 0$.
4. The point of intersection of the line $y = x$ and the line $y = \pi$ is $P(\pi, \pi)$.
5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from y-axis and terminates on the line $y = x$.
   Limits of $x : x = 0$ to $x = y$
   Limits of $y : y = 0$ to $y = \pi$

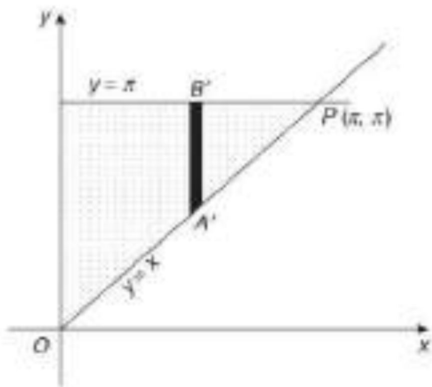

Fig. 9.49

Hence, the given integral after change of order is

$$\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} \, dy \, dx = \int_0^{\pi} \frac{\sin y}{y} \int_0^{y} dx \, dy$$

$$= \int_0^{\pi} \frac{\sin y}{y} |x|_0^y \, dy$$

$$= \int_0^{\pi} \frac{\sin y}{y} \cdot y \, dy$$

$$= \int_0^{\pi} \sin y \, dy$$

$$= |-\cos y|_0^{\pi}$$

$$= -\cos \pi + \cos 0$$

$$= 2$$

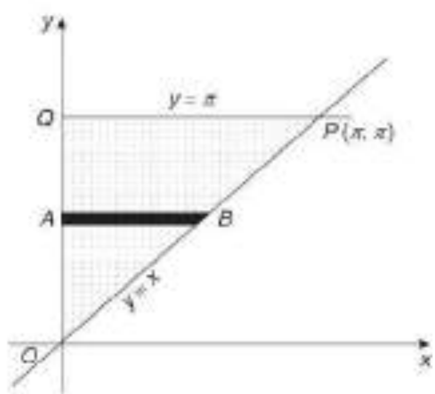

Fig. 9.50

Example 3

Evaluate the iterated integral $\int_0^1\int_x^1 \sin y^2 dydx$. **[Winter 2013]**

Solution

1. Since the inner limits depend on x, the function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.
2. Limits of y : $y = x$ to $y = 1$, along vertical strip
   Limits of x : $x = 0$ to $x = 1$
3. The region is bounded by the lines $y = x$, $y = 1$ and $x = 0$.
4. The point of intersection of the line $y = x$ and the line $y = 1$ is $P(1, 1)$.

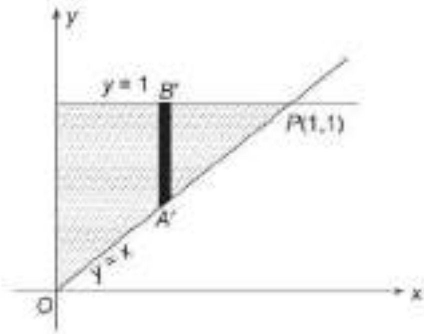

Fig. 9.51

5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from y-axis and terminates on the line $y = x$.
   Limits of x : $x = 0$ to $x = y$
   Limits of y : $y = 0$ to $y = 1$

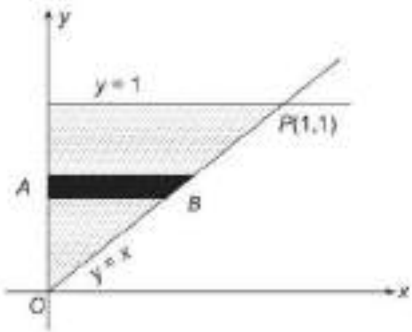

Fig. 9.52

Hence, the given integral after change of order is

$$\int_0^1\int_x^1 \sin y^2 dydx = \int_0^1\int_0^y \sin y^2 dxdy$$

$$= \int_0^1 \sin y^2 \left|x\right|_0^y dy$$

$$= \int_0^1 \sin y^2 \cdot y\, dy$$

$$= \frac{1}{2}\int_0^1 \sin y^2 \cdot 2y\, dy$$

$$= \frac{1}{2}\left|-\cos y^2\right|_0^1 dy \qquad \left[\because \int \sin f(y)\cdot f'(y)dy = -\cos f(y)\right]$$

$$= \frac{1}{2}\left[-\cos 1 + \cos 0\right]$$

$$= \frac{1}{2}[1 - \cos 1]$$

Example 4

Change the order of integration and evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx.$

[Winter 2015]

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y but evaluation becomes easier by changing the order of integration.
2. Limits of y : $y = x$ to $y \to \infty$, along vertical strip
   Limits of x : $x = 0$ to $x \to \infty$
3. The region is bounded by the lines $y = x$ and $x = 0$.
4. Here, the only point of intersection is origin.
5. To change the order of integration, i.e. to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from y-axis and terminates on the line $y = x$.
   Limits of x : $x = 0$ to $x = y$
   Limits of y : $y = 0$ to $y = \infty$

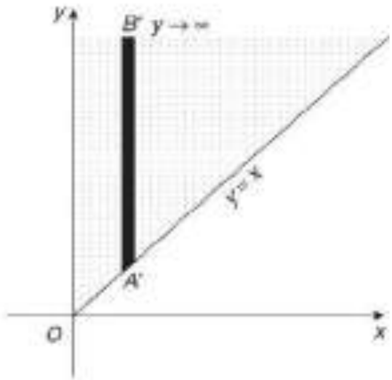

Fig. 9.53

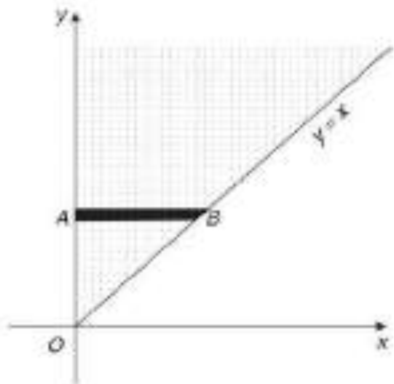

Fig. 9.54

Hence, the given integral after change of order is

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \left\{ \int_0^y dx \right\} \frac{e^{-y}}{y}\, dy$$

$$= \int_0^\infty |x|_0^y \frac{e^{-y}}{y}\, dy$$

$$= \int_0^\infty y \cdot \frac{e^{-y}}{y}\, dy$$

$$= \int_0^\infty e^{-y}\, dy$$

$$= \left| -e^{-y} \right|_0^\infty$$

$$= -(e^{-\infty} - e^0)$$

$$= 1$$

Example 5

Evaluate the integral $\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\, dx\, dy$ *by changing the order of integration.*

[Summer 2017, 2015]

Solution

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.
2. Limits of $x : x = \frac{y}{2}$ to $x = 1$
   along horizontal strip $A'B'$
   Limits of $y : y = 0$ to $y = 2$
3. The region is bounded by the lines $y = 2x$, $x = 1$, $y = 2$, and $y = 0$.
4. The point of intersection of $y = 2x$ and $x = 1$ is $x = 1, y = 2$.
   The point of intersection is $Q\,(1, 2)$.
5. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the line $y = 2x$.
   Limits of $y : y = 0$ to $y = 2x$
   Limits of $x : x = 0$ to $x = 1$

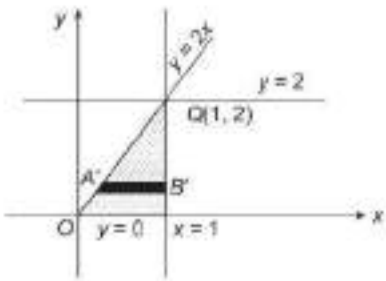

Fig. 9.55

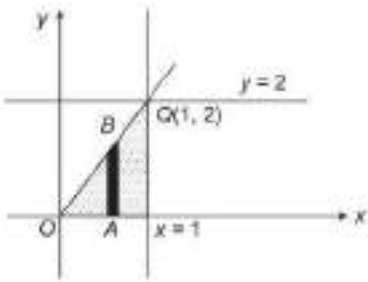

Fig. 9.56

Hence, the given integral after change of order is

$$\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\, dx\, dy = \int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx$$

$$= \int_0^1 \left\{ \int_0^{2x} dy \right\} e^{x^2}\, dx$$

$$= \int_0^1 2x \cdot e^{x^2}\, dx$$

$$= \left| e^{x^2} \right|_0^1 \qquad \left[\because \int e^{f(x)} f'(x)\, dx = e^{f(x)} \right]$$

$$= e^1 - e^0$$

$$= e - 1$$

Example 6

Change the order of integration and evaluate $\int_0^{\infty}\int_0^{x} xe^{-\frac{x^2}{y}}\,dy\,dx.$

Solution

1. The function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.
2. Limits of $y : y = 0$ to $y = x$.
   Limits of $x : x = 0$ to $x \to \infty$.
3. The region is the part of the first quadrant bounded between the lines $y = x$ and $y = 0$.
4. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip parallel to x-axis which starts from the line $y = x$ and extends up to infinity.
   Limits of $x : x = y$ to $x \to \infty$
   Limits of $y : y = 0$ to $y \to \infty$

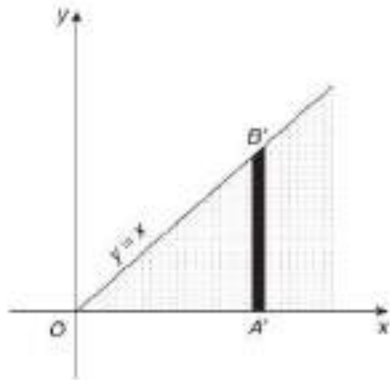

Fig. 9.57

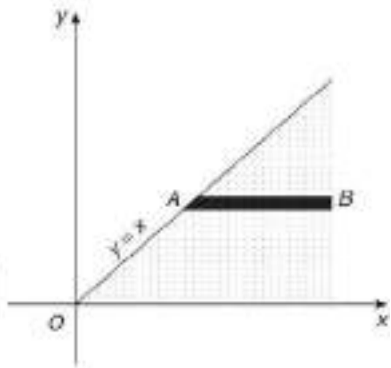

Fig. 9.58

Hence, the given integral after change of order is

$$\int_0^{\infty}\int_0^{x} xe^{-\frac{x^2}{y}}\,dy\,dx = \int_0^{\infty}\int_y^{\infty} xe^{-\frac{x^2}{y}}\,dx\,dy$$

$$= \int_0^{\infty}\left(-\frac{y}{2}\right)\int_y^{\infty} e^{-\frac{x^2}{y}}\left(-\frac{2x}{y}\right)dx\,dy$$

$$= -\frac{1}{2}\int_0^{\infty} y\left|e^{-\frac{x^2}{y}}\right|_y^{\infty} dy$$

$$\left[\because \int e^{f(x)} f'(x)\,dx = e^{f(x)}\right]$$

$$= -\frac{1}{2}\int_0^{\infty} y(0 - e^{-y})\,dy$$

$$= \frac{1}{2}\left|-ye^{-y} - e^{-y}\right|_0^{\infty}$$

$$= \frac{1}{2}$$

Example 7

Change the order of integration and evaluate $\int_0^{a}\int_y^{a} \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy.$

Solution

1. Since inner limits depend on y, the function is integrated first w.r.t. x but evaluation becomes easier by changing the order of integration.
2. Limits of $x: x = y$ to $x = a$, along horizontal strip $A'B'$
   Limits of $y: y = 0$ to $y = a$
3. The region is bounded by the lines $y = x$, $x = a$ and $y = 0$.
4. The point of intersection of $y = x$ and $x = a$ is $x = a$, $y = a$.
   The point of intersection is $Q\,(a, a)$.
5. To change the order of integration, i.e. to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the line $y = x$.
   Limits of $y: y = 0$ to $y = x$
   Limits of $x: x = 0$ to $x = a$

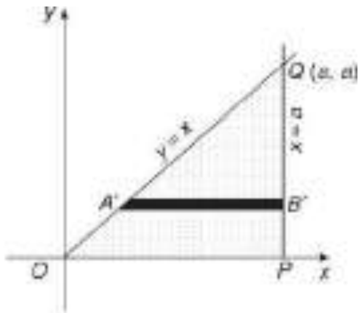

Fig. 9.59

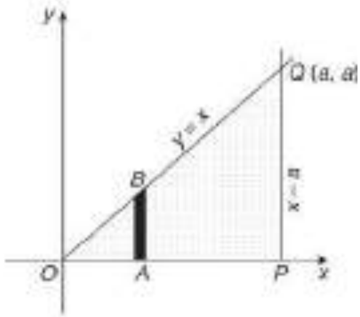

Fig. 9.60

Hence, the given integral after change of order is

$$\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^a \int_0^x \frac{x^2}{\sqrt{x^2+y^2}}\,dx$$

$$= \int_0^a \left\{ \int_0^x \frac{1}{\sqrt{x^2+y^2}}\,dy \right\} x^2\,dx$$

$$= \int_0^a \left| \log\left(y + \sqrt{y^2+x^2}\right) \right|_0^x x^2\,dx$$

$$= \int_0^a \left[\log\left(x + \sqrt{2x^2}\right) - \log x \right] x^2\,dx$$

$$= \int_0^a \left[\log \frac{x\left(1+\sqrt{2}\right)}{x} \right] x^2\,dx$$

$$= \log\left(1+\sqrt{2}\right) \int_0^a x^2\,dx$$

$$= \log\left(1+\sqrt{2}\right) \left| \frac{x^3}{3} \right|_0^a$$

$$= \log\left(1+\sqrt{2}\right) \frac{a^3}{3}$$

Example 8

Change the order of integration and evaluate **[Winter 2016]**

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}}\,dy\,dx.$$

Solution

1. The function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.
2. Limits of y : $y = 0$ to $y = \sqrt{1-x^2}$

   Limits of x : $x = 0$ to $x = 1$
3. Since given limits of x and y are positive, the region is the part of circle $x^2 + y^2 = 1$ in the first quadrant.

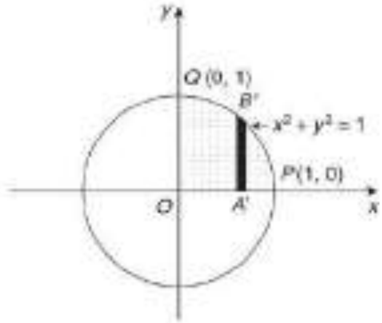

Fig. 9.61

4. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from y axis and terminates on the circle $x^2 + y^2 = 1$.

   Limits of x : $x = 0$ to $x = \sqrt{1-y^2}$

   Limits of y : $y = 0$ to $y = 1$

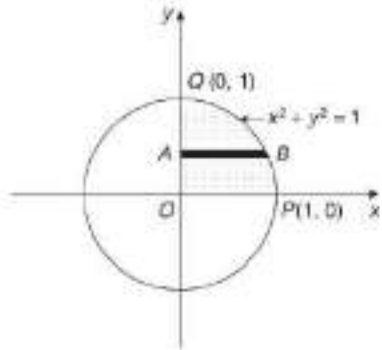

Fig. 9.62

Hence, the given integral after change of order is

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}}\,dy\,dx = \int_0^1 \frac{e^y}{e^y+1} \int_0^{\sqrt{1-y^2}} \frac{1}{\sqrt{(1-y^2)-x^2}}\,dx\,dy$$

$$= \int_0^1 \frac{e^y}{e^y+1} \left| \sin^{-1} \frac{x}{\sqrt{1-y^2}} \right|_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 \frac{e^y}{e^y+1} (\sin^{-1} 1 - \sin^{-1} 0)\,dy$$

$$= \int_0^1 \frac{e^y}{e^y+1} \cdot \frac{\pi}{2}\,dy$$

$$= \frac{\pi}{2} \left| \log(e^y+1) \right|_0^1 \qquad \left[\because \int \frac{f'(y)}{f(y)}\,dy = \log f(y) \right]$$

$$= \frac{\pi}{2} [\log(e+1) - \log 2]$$

$$= \frac{\pi}{2} \log\left(\frac{e+1}{2}\right)$$

Example 9

Change the order of integration and evaluate $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx\,dy$.

Solution

1. Since inner limits depend on y, the function is integrated first w.r.t. x.
2. Limits of $x : x = a - \sqrt{a^2 - y^2}$ to $x = a + \sqrt{a^2 - y^2}$, along horizontal strip $A'B'$

   Limits of $y : y = 0$ to $y = a$
3. The region is bounded by the circle $(x - a)^2 + y^2 = a^2$ and the line $y = 0$. Since limits of y are positive, the region is the part of the circle $(x - a)^2 + y^2 = a^2$ above x-axis.
4. The points of intersection of the circle with x-axis are O (0, 0) and Q (2a, 0).
5. To change the order of integration i.e. to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the circle

   $(x - a)^2 + y^2 = a^2$

   or $x^2 + y^2 - 2ax = 0$

   Limits of $y : y = 0$ to $y = \sqrt{2ax - x^2}$

   Limits of $x : x = 0$ to $x = 2a$

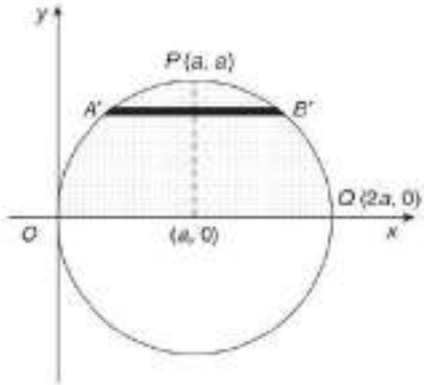

Fig. 9.63

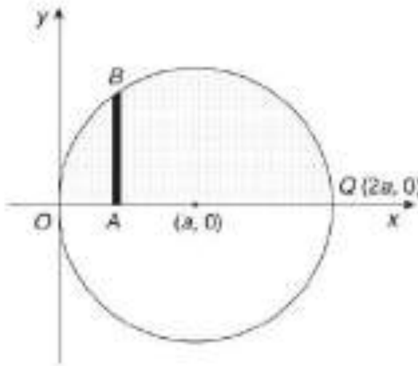

Fig. 9.64

Hence, the given integral after change of order is

$$\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx\,dy = \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} dy\,dx$$

$$= \int_0^{2a} |y|_0^{\sqrt{2ax-x^2}} dx$$

$$= \int_0^{2a} \sqrt{2ax - x^2}\,dx$$

$$= \int_0^{2a} \sqrt{a^2 - (x-a)^2}$$

$$= \left| \frac{(x-a)}{2} \sqrt{a^2 - (x-a)^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x-a}{a}\right) \right|_0^{2a}$$

$$= \frac{a}{2}\sqrt{0} + \frac{a^2}{2}\sin^{-1} 1 - \frac{(0-a)}{2}\sqrt{0} - \frac{a^2}{2}\sin^{-1}(-1)$$

$$= a^2 \sin^{-1} 1 \quad [\because \sin^{-1}(-1) = -\sin^{-1}(1)]$$

$$= a^2 \frac{\pi}{2}$$

$$= \frac{\pi a^2}{2}$$

Example 10

Evaluate $\displaystyle\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$ *by changing the order of integration.*

[Summer 2016]

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y.
2. Limits of $y : y = x$ to $y = \sqrt{2-x^2}$ along vertical setup $A'B'$
   Limits of $x : x = 0$ to $x = 1$
3. The region is bounded by y-axis, the line $y = x$ and the circle $x^2 + y^2 = 2$.
4. The point of intersection of the circle $y = \sqrt{2-x^2}$ and $y = x$ is obtained as

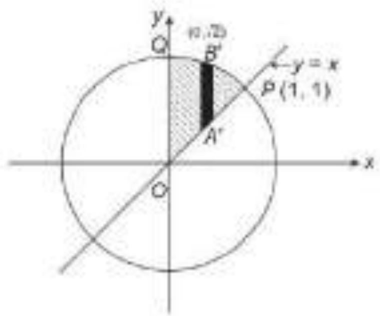

Fig. 9.65

$$x^2 = 2 - x^2$$
$$2x^2 = 2$$
$$x^2 = 1$$
$$x = \pm 1$$
$$\therefore \quad y = 1$$

Hence, $P(1, 1)$ is the point of intersection.

5. To change the order of integration, i.e. to integrate first w.r.t. x, divide the region into two subregions *OPR* and *PQR*. Draw a horizontal strip parallel to x-axis in each subregion.

   (i) In the subregion *OPR,* strip *AB* starts from y-axis and terminates on the line $y = x$.

   Limit of x: $x = 0$ to $x = y$

   Limit of y: $y = 0$ to $y = 1$

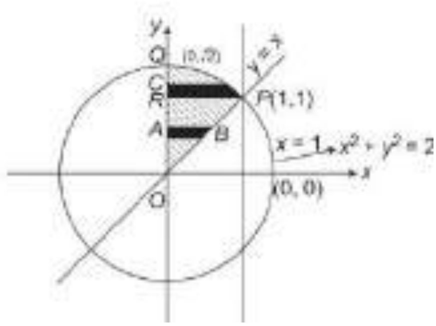

Fig. 9.66

(ii) In the subregion *PQR*, strip *CD* starts from y-axis and terminates on the circle $y=\sqrt{2-x^2}$.

Limit of x: $x=0$ to $x=\sqrt{2-y^2}$

Limit of y: $y=1$ to $y=\sqrt{2}$

$$\int_0^1\int_x^{\sqrt{2-x^2}}\frac{x}{\sqrt{x^2+y^2}}\,dy\,dx=\iint_{OPR}\frac{x}{\sqrt{x^2+y^2}}\,dy\,dx+\iint_{PQR}\frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$$

$$=\int_0^1\int_0^y\frac{x}{\sqrt{x^2+y^2}}\,dy\,dx+\int_1^{\sqrt{2}}\int_0^{\sqrt{2-y^2}}\frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$$

$$=\int_0^1\frac{1}{2}\left[\frac{(x^2+y^2)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^y dy+\int_1^{\sqrt{2}}\frac{1}{2}\left[\frac{(x^2+y^2)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^{\sqrt{2-y^2}} dy$$

$$=\int_0^1\left[\sqrt{2}y-y\right]dy+\int_1^{\sqrt{2}}\left[\sqrt{2}-y\right]dy$$

$$=(\sqrt{2}-1)\left|\frac{y^2}{2}\right|_0^1+\left[\sqrt{2}y-\frac{y^2}{2}\right]_1^{\sqrt{2}}$$

$$=(\sqrt{2}-1)\left(\frac{1}{2}-0\right)+\left(2-1-\sqrt{2}+\frac{1}{2}\right)$$

$$=\frac{\sqrt{2}}{2}-\frac{1}{2}+\frac{3}{2}-\sqrt{2}$$

$$=1-\sqrt{2}+\frac{1}{\sqrt{2}}$$

$$=1-\frac{1}{\sqrt{2}}$$

Example 11

Sketch the region of integration, reverse the order of integration and evaluate $\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dy\,dx.$ **[Summer 2014]**

Solution

1. Since inner limit depends on x, the function is integrated first w.r.t. y.
2. Limits of y : $y = 0$ to $y = 4 - x^2$ along vertical strip $A'B'$

   Limits of x : $x = 0$ to $x = 2$
3. The region is bounded by the parabola $x^2 = 4 - y$, x-axis and y-axis.
4. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip parallel to x-axis which starts from y-axis and terminates on the parabola $x^2 = 4 - y$.

   Limits of x : $x = 0$ to $x = \sqrt{4-y}$

   Limits of y : $y = 0$ to $y = 4$

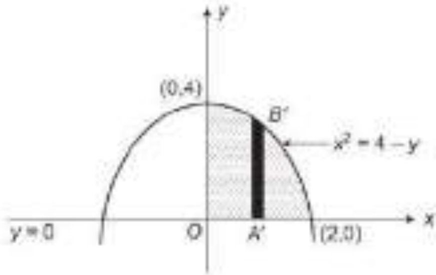

Fig. 9.67

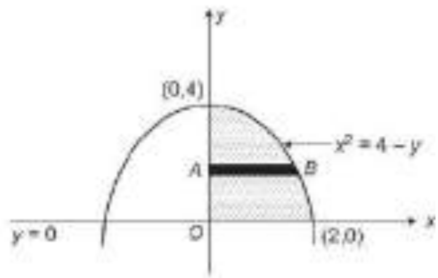

Fig. 9.68

Hence, the given integral after change of order is

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dy\,dx = \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\,dx\,dy$$

$$= \int_0^4 \frac{e^{2y}}{4-y}\left[\int_0^{\sqrt{4-y}} x\,dx\right]dy$$

$$= \int_0^4 \frac{e^{2y}}{4-y}\left|\frac{x^2}{2}\right|_0^{\sqrt{4-y}} dy$$

$$= \int_0^4 \frac{e^{2y}}{4-y}\left[\frac{1}{2}(4-y)\right]dy$$

$$= \int_0^4 \frac{e^{2y}}{2} dy$$

$$= \frac{1}{4} \left| e^{2y} \right|_0^4$$

$$= \frac{1}{4}(e^8 - 1)$$

Example 12

Change the order of integration and evaluate $\int_0^1 \int_{x^2}^{2-x} xy \, dx \, dy.$

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y.

   The correct form of integral

   $= \int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$

2. Limits of y : $y = x^2$ to $y = 2 - x$, along vertical strip $A'B'$

   Limits of x : $x = 0$ to $x = 1$

3. The region is bounded by y-axis, the line $x + y = 2$ and the parabola $x^2 = y$. Since given limits of x and y are positive, the region lies in the first quadrant.

4. The points of intersection of $x + y = 2$ and $x^2 = y$ are obtained as

$$x^2 = 2 - x$$
$$x^2 + x - 2 = 0$$
$$(x - 1)(x + 2) = 0$$
$$x = 1, -2$$
$$y = 1, 4$$

   The points of intersection are P (1, 1) and P' (–2, 4).

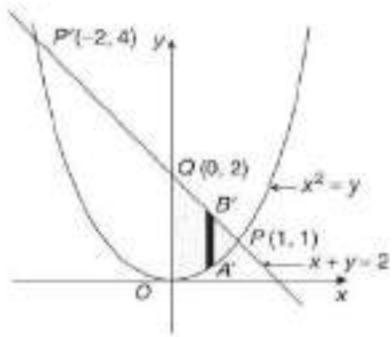

Fig. 9.69

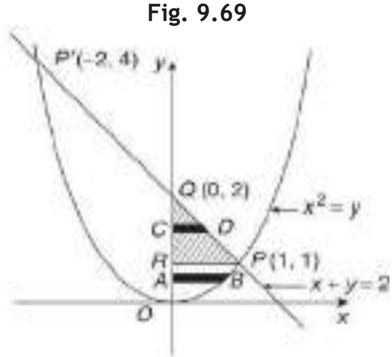

Fig. 9.70

5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into two subregions OPR and RPQ. Draw a horizontal strip parallel to x-axis in each subregion.

   (i) In subregion OPR, strip AB starts from y-axis and terminates on the parabola $x^2 = y$.

   Limits of x : $x = 0$ to $x = \sqrt{y}$

   Limits of y : $y = 0$ to $y = 1$

(ii) In subregion *RPQ*, strip *CD* starts from *y*-axis and terminates on the line $x + y = 2$.

Limits of $x : x = 0$ to $x = 2 - y$

Limits of $y : y = 1$ to $y = 2$

Hence, the given integral after change of order is

$$\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx = \int_0^1\int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2\int_0^{2-y} xy\,dx\,dy$$
$$= \int_0^1\left|\frac{x^2}{2}\right|_0^{\sqrt{y}} y\,dy + \int_1^2\left|\frac{x^2}{2}\right|_0^{2-y} y\,dy$$
$$= \frac{1}{2}\int_0^1 (y)y\,dy + \frac{1}{2}\int_1^2 (2-y)^2 y\,dy$$
$$= \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 (4y - 4y^2 + y^3)dy$$
$$= \frac{1}{2}\left|\frac{y^3}{3}\right|_0^1 + \frac{1}{2}\left|4\frac{y^2}{2} - 4\frac{y^3}{3} + \frac{y^4}{4}\right|_1^2$$
$$= \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left(8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4}\right)$$
$$= \frac{1}{6} + \frac{5}{24}$$
$$= \frac{9}{24}$$
$$= \frac{3}{8}$$

Example 13

Change the order of integration and evaluate $\int_0^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ax}} dy\,dx.$

[Winter 2014]

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y.
2. Limits of $y : y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$, along vertical strip $A'B'$
3. The region is bounded by the parabolas $x^2 = 4ay$ and $y^2 = 4ax$.
4. The points of intersection of $x^2 = 4ay$ and $y^2 = 4ax$ are obtained as

$$x^4 = 16a^2y^2$$
$$= 16a^2(4ax)$$

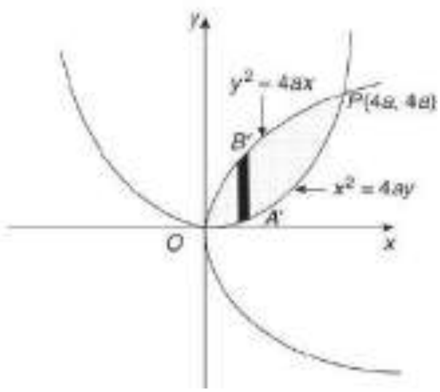

Fig. 9.71

$$x(x^3 - 64a^3) = 0$$
$$x = 0, x = 4a$$
$$\therefore y = 0, y = 4a$$

The points of intersection are O (0, 0) and P ($4a$, $4a$).

5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from the parabola $y^2 = 4ax$ and terminates on the parabola $x^2 = 4ay$.

Limits of $x: x = \frac{y^2}{4a}$ to $x = 2\sqrt{ay}$

Limits of $y: y = 0$ to $y = 4a$

Hence, the given integral after change of order is

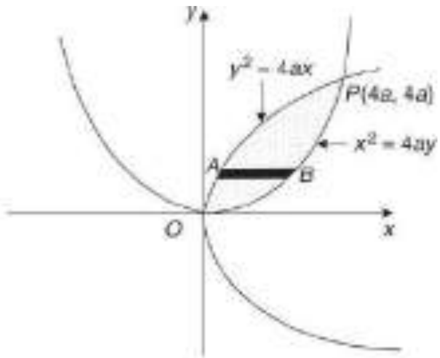

Fig. 9.72

$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dy\,dx = \int_0^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy$$
$$= \int_0^{4a} \left| x \right|_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$
$$= \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy$$
$$= 2\sqrt{a}\left|\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^{4a} - \frac{1}{4a}\left|\frac{y^3}{3}\right|_0^{4a}$$
$$= \frac{4}{3}\sqrt{a}(4)^{\frac{3}{2}}a^{\frac{3}{2}} - \frac{1}{12a}(64a^3)$$
$$= \frac{32}{3}a^2 - \frac{16}{3}a^2$$
$$= \frac{16}{3}a^2$$

Example 14

Change the order of integration and evaluate

$$\int_0^a\int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\,dx\,dy.$$

Solution

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.

2. Limits of x : $x = 0$ to $x = a - \sqrt{a^2 - y^2}$

   Limits of y : $y = 0$ to $y = a$
3. The region is bounded by the circle $(x - a)^2 + y^2 = a^2$, the lines $y = a$ and $x = 0$.
4. The point of intersection of $(x - a)^2 + y^2 = a^2$ and $y = a$ is obtained as

   $$(x - a)^2 + a^2 = a^2$$
   $$x = a$$

   The point of intersection is $P\,(a, a)$.

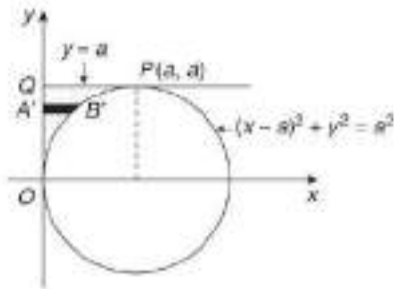

Fig. 9.73

5. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from the circle $(x - a)^2 + y^2 = a^2$ and terminates on the line $y = a$.

   Limits of y : $y = \sqrt{2ax - x^2}$ to $y = a$

   Limits of x : $x = 0$ to $x = a$

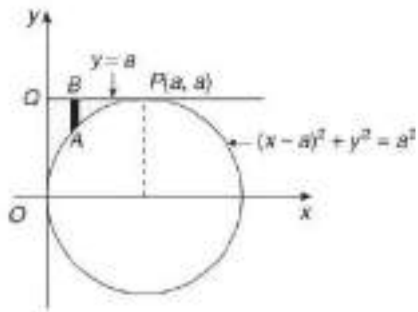

Fig. 9.74

Hence, the given integral after change of order is

$$\int_0^a \int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\,dx\,dy = \int_0^a \int_{\sqrt{2ax-x^2}}^a \frac{x\log(x+a)}{(x-a)^2}\,y\,dy\,dx$$

$$= \int_0^a \frac{x\log(x+a)}{(x-a)^2}\left|\frac{y^2}{2}\right|_{\sqrt{2ax-x^2}}^a dx$$

$$= \int_0^a \frac{x\log(x+a)}{(x-a)^2}\left(\frac{a^2-2ax+x^2}{2}\right)dx$$

$$= \frac{1}{2}\int_0^a x\log(x+a)\,dx$$

$$= \frac{1}{2}\left[\left|\frac{x^2}{2}\log(x+a)\right|_0^a - \int_0^a \frac{x^2}{2}\cdot\frac{1}{x+a}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{a^2}{2}\log 2a - \frac{1}{2}\int_0^a \left\{(x-a) + \frac{a^2}{x+a}\right\}dx\right]$$

$$= \frac{1}{2}\left[\frac{a^2}{2}\log 2a - \frac{1}{2}\left|\frac{x^2}{2} - ax + a^2\log(x+a)\right|_0^a\right]$$

$$= \frac{1}{4}\left(a^2\log 2a - \frac{a^2}{2} + a^2 - a^2\log 2a + a^2\log a\right)$$

$$= \frac{1}{4}\left(\frac{a^2}{2} + a^2 \log a\right)$$

$$= \frac{a^2}{8}(1 + 2\log a)$$

Example 15

Change the order of integration and evaluate

$$\int_0^{\frac{1}{2}} \int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\,dx\,dy.$$

Solution

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.
2. Limits of $x : x = 0$ to $x = \sqrt{1-4y^2}$

   Limits of $y : y = 0$ to $y = \frac{1}{2}$
3. Since the limits of x and y are positive, the region is the part of the ellipse in the first quadrant.
4. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the ellipse $x^2 + 4y^2 = 1$.

   Limits of $y : y = 0$ to $y = \frac{1}{2}\sqrt{1-x^2}$

   Limits of $x : x = 0$ to $x = 1$

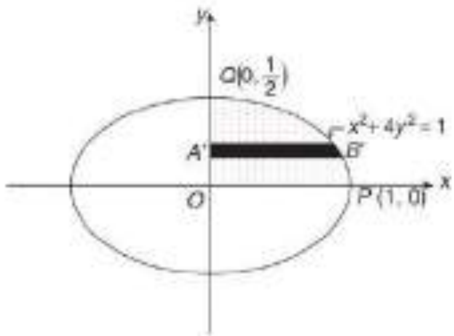

Fig. 9.75

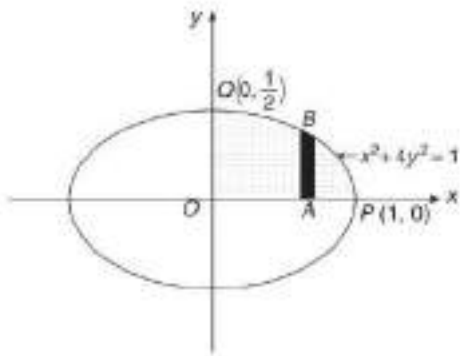

Fig. 9.76

Hence, the given integral after change of order is

$$\int_0^{\frac{1}{2}} \int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\,dx dy = \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}} \int_0^{\frac{1}{2}\sqrt{1-x^2}} \frac{1}{\sqrt{(1-x^2)-y^2}}\,dy dx$$

$$= \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}} \left| \sin^{-1} \frac{y}{\sqrt{1-x^2}} \right|_0^{\frac{1}{2}\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}\left(\sin^{-1}\frac{1}{2}-\sin^{-1} 0\right)\mathrm{d}x$$

$$= \int_0^1 \frac{2-(1-x^2)}{\sqrt{1-x^2}}\cdot\frac{\pi}{6}\,\mathrm{d}x$$

$$= \frac{\pi}{6}\int_0^1\left(\frac{2}{\sqrt{1-x^2}}-\sqrt{1-x^2}\right)\mathrm{d}x$$

$$= \frac{\pi}{6}\left|2\sin^{-1}x-\frac{x}{2}\sqrt{1-x^2}-\frac{1}{2}\sin^{-1}x\right|_0^1$$

$$\left[\because \int\sqrt{a^2-x^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{\pi}{6}\left(\frac{3}{2}\sin^{-1}1\right)$$

$$= \frac{\pi}{4}\cdot\frac{\pi}{2}$$

$$= \frac{\pi^2}{8}$$

Example 16

Change the order of integration and evaluate

$$\int_0^a\int_0^y \frac{x\,\mathrm{d}y\,\mathrm{d}x}{\sqrt{(a^2-x^2)(a-y)(y-x)}}.$$

Solution

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.
2. Limits of $x : x = 0$ to $x = y$
   Limits of $y : y = 0$ to $y = a$
3. The region is bounded by the line $y = x$, $y = a$ and $x = 0$.
4. The point of intersection of $y = a$ and $y = x$ is $P\,(a, a)$.
5. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from the line $y = x$ and terminates on the line $y = a$.

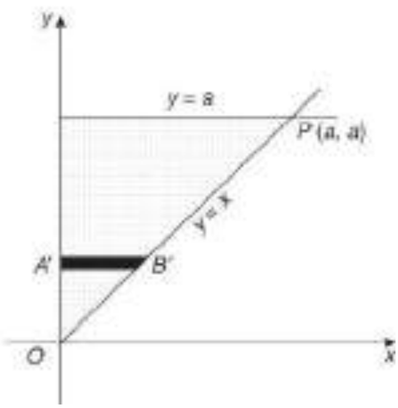

Fig. 9.77

Limits of y : $y = x$ to $y = a$

Limits of x : $x = 0$ to $x = a$

Hence, the given integral after change of order is

$$\int_0^a \int_0^y \frac{x\,dy\,dx}{\sqrt{(a^2-x^2)(a-y)(y-x)}}$$

$$= \int_0^a \int_x^a \frac{x\,dy\,dx}{\sqrt{(a^2-x^2)(a-y)(y-x)}}$$

$$= \int_0^a \frac{x}{\sqrt{a^2-x^2}}\left[\int_x^a \frac{dy}{\sqrt{(a-y)(y-x)}}\right]dx$$

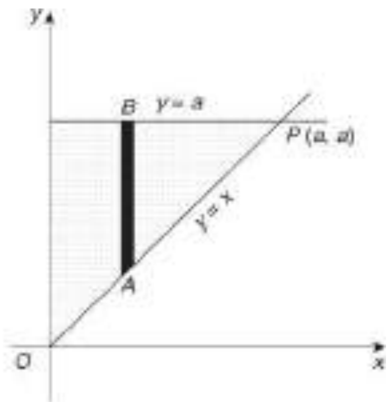

Fig. 9.78

Putting $y - x = t^2$, $dy = 2t\,dt$

When $y = x$, $t = 0$

When $y = a$, $t = \sqrt{a-x}$

$$\int_0^a \int_0^y \frac{x\,dy\,dx}{\sqrt{(a^2-x^2)(a-y)(y-x)}} = \int_0^a \frac{x}{\sqrt{a^2-x^2}} \int_0^{\sqrt{a-x}} \frac{2t\,dt}{\sqrt{(a-x-t^2)t^2}}\,dx$$

$$= 2\int_0^a \frac{x}{\sqrt{a^2-x^2}} \int_0^{\sqrt{a-x}} \frac{dt}{\sqrt{(a-x)-t^2}}\,dx$$

$$= 2\int_0^a \frac{x}{\sqrt{a^2-x^2}} \left|\sin^{-1}\frac{t}{\sqrt{a-x}}\right|_0^{\sqrt{a-x}} dx$$

$$= 2\int_0^a \frac{x}{\sqrt{a^2-x^2}} (\sin^{-1}1 - \sin^{-1}0)\,dx$$

$$= 2\cdot\frac{\pi}{2}\int_0^a \left[-\frac{1}{2}(a^2-x^2)^{-\frac{1}{2}}(-2x)\right]dx$$

$$= -\frac{\pi}{2}\left|2(a^2-x^2)^{\frac{1}{2}}\right|_0^a \qquad \left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= \pi a$$

Example 17

Change the order of integration and evaluate

$$\int_0^1 \int_x^{\frac{1}{x}} \frac{y}{(1+xy)^2(1+y^2)}\,dy\,dx.$$

Solution

1. The function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.

2. Limits of y : $y = x$ to $y = \frac{1}{x}$

   Limits of x : $x = 0$ to $x = 1$

3. The region is bounded by the rectangular hyperbola $xy = 1$, the line $y = x$ and y-axis in the first quadrant.
4. The point of intersection of $xy = 1$ and $y = x$ in the first quadrant is obtained as

   $x^2 = 1$

   $x = 1$

   $\therefore \; y = 1$

   The point of intersection is P (1, 1).

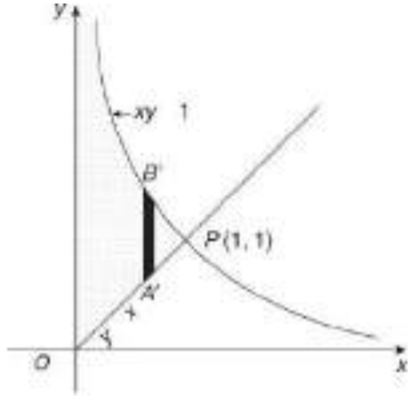

Fig. 9.79

5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into two subregions OPQ and QPR. Draw a horizontal strip parallel to x-axis in each subregion.
   (i) In subregion OPQ, strip AB starts from y-axis and terminates on the line $y = x$.

   Limits of x : $x = 0$ to $x = y$
   Limits of y : $y = 0$ to $y = 1$

   (ii) In subregion QPR, strip CD starts from y-axis and terminates on the rectangular hyperbola $xy = 1$.

   Limits of x : $x = 0$ to $x = \frac{1}{y}$

   Limits of y : $y = 1$ to $y \to \infty$.

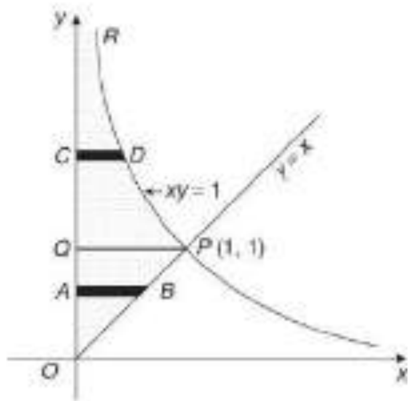

Fig. 9.80

Hence, the given integral after change of order is

$$\int_0^1 \int_x^{\frac{1}{x}} \frac{y}{(1+xy)^2(1+y^2)}\,dy\,dx = \int_0^1 \int_0^y \frac{y}{1+y^2}\frac{1}{(1+xy)^2}\,dx\,dy + \int_1^\infty \int_0^{\frac{1}{y}} \frac{y}{1+y^2}\frac{1}{(1+xy)^2}\,dx\,dy$$

$$= \int_0^1 \frac{y}{1+y^2}\left|-\frac{1}{y(1+xy)}\right|_0^y dy + \int_1^\infty \frac{y}{1+y^2}\left|-\frac{1}{y(1+xy)}\right|_0^{\frac{1}{y}} dy$$

$$= -\int_0^1 \frac{1}{1+y^2}\left(\frac{1}{1+y^2}-1\right)dy - \int_1^\infty \frac{1}{1+y^2}\left(\frac{1}{2}-1\right)dy$$

$$= -\int_0^1 \left[\frac{1}{(1+y^2)^2} - \frac{1}{1+y^2}\right]dy + \frac{1}{2}\int_1^\infty \frac{1}{1+y^2}\,dy$$

Putting $y = \tan\theta$ in the first term of first integral, $dy = \sec^2\theta\, d\theta$,

When $y = 0$, $\theta = 0$

When $y = 1$, $\theta = \frac{\pi}{4}$

$$\int_0^1\int_x^{\frac{1}{x}} \frac{y}{(1+xy)^2(1+y^2)}\,dy\,dx = -\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\, d\theta}{\sec^4\theta} + \left|\tan^{-1} y\right|_0^1 + \frac{1}{2}\left|\tan^{-1} y\right|_1^{\infty}$$

$$= -\int_0^{\frac{\pi}{4}} \frac{(1+\cos 2\theta)}{2}\,d\theta + (\tan^{-1}1 - \tan^{-1}0) + \frac{1}{2}(\tan^{-1}\infty - \tan^{-1}1)$$

$$= -\frac{1}{2}\left|\theta + \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{4}} + \frac{\pi}{4} + \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= -\frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2} + \frac{3\pi}{8}$$

$$= \frac{\pi - 1}{4}$$

Example 18

Change the order of integration and evaluate

$$\int_0^1\int_0^{\sqrt{1-y^2}} \frac{\cos^{-1} x}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\,dx\,dy.$$

Solution

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.
2. Limits of x : $x = 0$ to $x = \sqrt{1-y^2}$
   Limits of y : $y = 0$ to $y = 1$
3. Since given limits of x and y are positive, the region is the part of the circle $x^2 + y^2 = 1$ in the first quadrant.
4. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis in the region. AB starts from x-axis and terminates on the circle $x^2 + y^2 = 1$.
   Limits of y : $y = 0$ to $y = \sqrt{1-x^2}$
   Limits of x : $x = 0$ to $x = 1$.

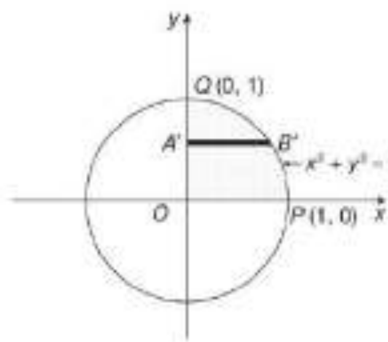

Fig. 9.81

Hence, the given integral after change of order is

$$\int_0^1\int_0^{\sqrt{1-x^2}} \frac{\cos^{-1}x}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\,dx\,dy$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} \frac{\cos^{-1}x}{\sqrt{1-x^2}}\frac{1}{\sqrt{(1-x^2)-y^2}}\,dy\,dx$$

$$= \int_0^1 \frac{\cos^{-1}x}{\sqrt{1-x^2}}\left|\sin^{-1}\frac{y}{\sqrt{1-x^2}}\right|_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{\cos^{-1}x}{\sqrt{1-x^2}}(\sin^{-1}1-\sin^{-1}0)dx$$

$$= -\frac{\pi}{2}\int_0^1 \cos^{-1}x\left(-\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$= -\frac{\pi}{2}\left|\frac{(\cos^{-1}x)^2}{2}\right|_0^1 \quad \left[\because \int f(x)f'(x)dx = \frac{[f(x)]^2}{2}\right]$$

$$= -\frac{\pi}{4}\left[(\cos^{-1}1)^2-(\cos^{-1}0)^2\right]$$

$$= -\frac{\pi}{4}\left[0-\left(\frac{\pi}{2}\right)^2\right]$$

$$= \frac{\pi^3}{16}$$

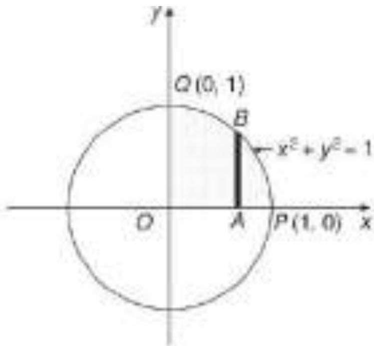

Fig. 9.82

EXERCISE 9.3

Change the order of integration of the following integrals:

1. $\int_0^6\int_{2-x}^{2+x} f(x, y)dy\,dx$

$$\left[\textbf{Ans.:} \int_{-4}^{2}\int_{2-y}^{6} f(x, y)dy\,dx + \int_2^8\int_{y-2}^{6} f(x, y)dy\,dx\right]$$

2. $\int_0^1\int_x^{2x} f(x, y)dy\,dx$

$$\left[\textbf{Ans.:} \int_0^1\int_{\frac{y}{2}}^{y} f(x, y)dx\,dy + \int_1^2\int_{\frac{y}{2}}^{1} f(x, y)dx\,dy\right]$$

3. $\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)dx\,dy$

$$\left[\textbf{Ans.:} \int_{-1}^{1}\int_{x^2}^{1} f(x, y)dx\,dy\right]$$

4. $\int_{-a}^{a}\int_{0}^{\frac{y^2}{a}} f(x, y)\,dx\,dy$

$$\left[\textbf{Ans.: } \int_0^a\int_{-a}^{-\sqrt{ax}} f(x, y)\,dy\,dx + \int_0^a\int_{\sqrt{ax}}^{a} f(x, y)\,dy\,dx\right]$$

5. $\int_0^a\int_{\frac{x^2}{a}}^{2a-x} f(x, y)\,dy\,dx$

$$\left[\textbf{Ans.: } \int_0^a\int_0^{\sqrt{ay}} f(x, y)\,dx\,dy + \int_a^{2a}\int_0^{2a-y} f(x, y)\,dx\,dy\right]$$

6. $\int_{-2}^{3}\int_{y^2-6}^{y} f(x, y)\,dx\,dy$

$$\left[\textbf{Ans.: } \int_{-6}^{-2}\int_{-\sqrt{x+6}}^{\sqrt{x+6}} f(x, y)\,dy\,dx + \int_{-2}^{3}\int_{x}^{\sqrt{x+6}} f(x, y)\,dy\,dx\right]$$

7. $\int_0^1\int_{2y}^{2(1+\sqrt{1-y})} f(x, y)\,dx\,dy$

$$\left[\textbf{Ans.: } \int_0^2\int_0^{\frac{x}{2}} f(x, y)\,dy\,dx + \int_2^4\int_0^{\frac{4x-x^2}{4}} f(x, y)\,dy\,dx\right]$$

8. $\int_0^2\int_{\frac{x^2+4}{4}}^{\frac{6-x}{2}} f(x, y)\,dy\,dx$

$$\left[\textbf{Ans.: } \int_1^2\int_0^{2\sqrt{y-1}} f(x, y)\,dx\,dy + \int_2^3\int_0^{6-2y} f(x, y)\,dx\,dy\right]$$

9. $\int_0^2\int_{\sqrt{4-x^2}}^{x+6a} f(x, y)\,dy\,dx$

$$\left[\textbf{Ans.: } \int_0^2\int_{\sqrt{4-y^2}}^{2} f(x, y)\,dx\,dy + \int_2^{6a}\int_0^2 f(x, y)\,dx\,dy + \int_{6a}^{6a+2}\int_{y-6a}^{2} f(x, y)\,dx\,dy\right]$$

10. $\int_0^a\int_{\sqrt{a^2-y^2}}^{y+a} f(x, y)\,dx\,dy$

$$\left[\textbf{Ans.: } \int_0^a\int_{\sqrt{a^2-x^2}}^{a} f(x, y)\,dy\,dx + \int_a^{2a}\int_{x-a}^{a} f(x, y)\,dy\,dx\right]$$

11. $\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x, y)\,dx\,dy$

$$\left[\begin{array}{l}\textbf{Ans.: }\\ \int_0^a\int_{a+\sqrt{a^2-y^2}}^{2a} f(x, y)\,dx\,dy + \int_0^a\int_{\frac{y^2}{2a}}^{a-\sqrt{a^2-y^2}} f(x, y)\,dx\,dy + \int_a^{2a}\int_{\frac{y^2}{2a}}^{2a} f(x, y)\,dx\,dy\end{array}\right]$$

12. $\int_0^a \int_{\sqrt{\frac{a^2-x^2}{4}}}^{\sqrt{a^2-x^2}} f(x, y)\,dy\,dx$

$$\left[\textbf{Ans.:} \int_0^{\frac{a}{2}} \int_{\sqrt{a^2-4y^2}}^{\sqrt{a^2-y^2}} f(x, y)\,dx\,dy + \int_{\frac{a}{2}}^{a} \int_0^{\sqrt{a^2-y^2}} f(x, y)\,dx\,dy\right]$$

13. $\int_0^a \int_x^{\frac{a^2}{x}} f(x, y)\,dy\,dx$

$$\left[\textbf{Ans.:} \int_0^a \int_0^y f(x, y)\,dx\,dy + \int_a^{\infty} \int_0^{\frac{a^2}{y}} f(x, y)\,dx\,dy\right]$$

14. $\int_a^b \int_{\frac{k}{x}}^{mx} f(x, y)\,dy\,dx$

$$\left[\textbf{Ans.:} \int_{\frac{k}{b}}^{\frac{k}{a}} \int_{\frac{k}{y}}^{b} f(x, y)\,dx\,dy + \int_{\frac{k}{a}}^{ma} \int_a^b f(x, y)\,dx\,dy + \int_{ma}^{mb} \int_{\frac{y}{m}}^{b} f(x, y)\,dx\,dy\right]$$

15. $\int_0^1 \int_1^{e^x} f(x, y)\,dy\,dx$

$$\left[\textbf{Ans.:} \int_1^e \int_{\log y}^1 f(x, y)\,dx\,dy\right]$$

16. $\int_0^2 \int_0^{x^3} f(x, y)\,dy\,dx$

$$\left[\textbf{Ans.:} \int_0^8 \int_{y^{\frac{1}{3}}}^2 f(x, y)\,dx\,dy\right]$$

17. $\int_0^{\frac{1}{2}} \int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2-y^2}}\,dx\,dy$

$$\left[\textbf{Ans.:} \int_0^1 \int_0^{\frac{\sqrt{1-x^2}}{2}} \frac{1+x^2}{\sqrt{1-x^2-y^2}}\,dy\,dx = \frac{2\pi}{3}\right]$$

18. $\int_0^2 \int_0^{\frac{x^2}{2}} \frac{x}{\sqrt{x^2+y^2+1}}\,dy\,dx$

$$\left[\textbf{Ans.:} \int_0^2 \int_{\sqrt{2y}}^2 \frac{x}{\sqrt{x^2+y^2+1}}\,dx\,dy = \frac{1}{4}(5\log 5 - 4)\right]$$

19. $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dy\,dx$

$$\left[\textbf{Ans.:} \frac{\pi a}{4}\right]$$

9.4 DOUBLE INTEGRALS IN POLAR COORDINATES

The integral $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\mathrm{d}r\mathrm{d}\theta$ represents the polar form of the double integration. This integral is first integrated w.r.t. r keeping θ constant and then the resulting expression is integrated w.r.t. θ.

Limits of Integration

If the limits of integration are not given then these limits are obtained from the equations of the given curves. Let the region be bounded by the curves $r = r_1(\theta)$, $r = r_2(\theta)$ and the lines $\theta = \theta_1$, $\theta = \theta_2$.

The region of integration is *PQRS*. Draw an elementary radius vector *AB* from origin which enters in the region from the curve $r = r_1(\theta)$ and leaves at the curve $r = r_2(\theta)$. Therefore, limits for r are $r_1(\theta)$ to $r_2(\theta)$.

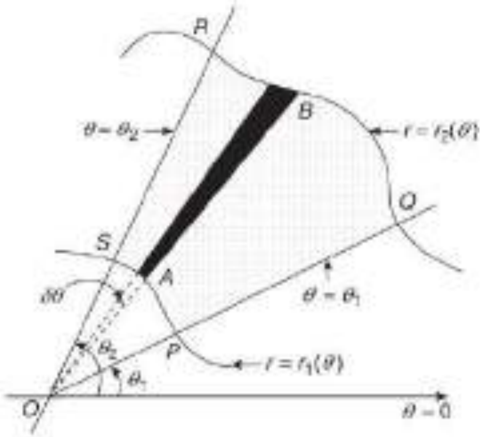

Fig. 9.83

To cover the entire region *PQRS*, rotate elementary radius vector *AB* from *PQ* to *RS*. Therefore, θ varies from θ_1 to θ_2.

$$\iint f(r,\theta)\,\mathrm{d}r\,\mathrm{d}\theta = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\,\mathrm{d}r\,\mathrm{d}\theta$$

Type I Evaluation of Double Integrals in Polar Coordinates

Example 1

Evaluate $\int_0^{\frac{\pi}{4}}\int_0^1 r\,\mathrm{d}r\,\mathrm{d}\theta$.

Solution

$$\int_0^{\frac{\pi}{4}}\int_0^1 r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\frac{\pi}{4}}\left[\int_0^1 r\,\mathrm{d}r\right]\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{4}}\left|\frac{r^2}{2}\right|_0^1 \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{4}}\frac{1}{2}\mathrm{d}\theta$$

$$= \frac{1}{2}\left|\theta\right|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2} \cdot \frac{\pi}{4}$$

$$= \frac{\pi}{8}$$

Another method: Since both the limits are constant and integrand (function) is explicit in r and θ, the integral can be written as

$$\int_0^{\frac{\pi}{4}} \int_0^1 r \, dr \, d\theta = \int_0^{\frac{\pi}{4}} d\theta \cdot \int_0^1 r \, dr$$

$$= |\theta|_0^{\frac{\pi}{4}} \cdot \left| \frac{r^2}{2} \right|_0^1$$

$$= \frac{\pi}{4} \cdot \frac{1}{2}$$

$$= \frac{\pi}{8}$$

Example 2

Evaluate $\int_0^{\pi} \int_0^{\sin\theta} r \, dr \, d\theta.$

Solution

$$\int_0^{\pi} \int_0^{\sin\theta} r \, dr \, d\theta = \int_0^{\pi} \left[\int_0^{\sin\theta} r \, dr \right] d\theta$$

$$= \int_0^{\pi} \left| \frac{r^2}{2} \right|_0^{\sin\theta} d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \sin^2 \theta \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \frac{1 - \cos 2\theta}{2} d\theta$$

$$= \frac{1}{4} \left| \theta - \frac{\sin 2\theta}{2} \right|_0^{\pi}$$

$$= \frac{1}{4} \left(\pi - \frac{\sin 2\pi}{2} \right)$$

$$= \frac{\pi}{4}$$

Example 3

Evaluate the integral $\int_0^{\frac{\pi}{2}}\int_0^{1-\sin\theta} r^2 \cos\theta \, dr \, d\theta$. **[Summer 2014]**

Solution

$$\int_0^{\frac{\pi}{2}}\int_0^{1-\sin\theta} r^2 \cos\theta \, dr \, d\theta = \int_0^{\frac{\pi}{2}}\left[\int_0^{1-\sin\theta} r^2 \, dr\right]\cos\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left|\frac{r^3}{3}\right|_0^{1-\sin\theta} \cos\theta \, d\theta$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{2}}(1-\sin\theta)^3 \cos\theta \, d\theta$$

$$= \frac{1}{3}\left|\frac{(1-\sin\theta)^4}{4}\right|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{3}\left[0-\frac{1}{4}\right]$$

$$= -\frac{1}{12}$$

Example 4

Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^2 \sin\theta \, d\theta \, dr.$

Solution

Since inner limits depend on θ, the function is integrated first w.r.t. r.

The correct form of the integral $= \int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^2 \sin\theta \, dr \, d\theta$

$$\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^2 \sin\theta \, dr \, d\theta = \int_0^{\frac{\pi}{2}}\left[\int_0^{2a\cos\theta} r^2 dr\right]\sin\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left|\frac{r^3}{3}\right|_0^{2a\cos\theta} \sin\theta \, d\theta$$

$$= \frac{8a^3}{3}\int_0^{\frac{\pi}{2}}\cos^3\theta \sin\theta \, d\theta$$

$$= -\frac{8a^3}{3}\int_0^{\frac{\pi}{2}}\cos^3\theta(-\sin\theta) \, d\theta$$

$$= -\frac{8a^3}{3}\left|\frac{\cos^4\theta}{4}\right|_0^{\frac{\pi}{2}} \qquad \left[\because \int [f(\theta)]^n f'(\theta)\mathrm{d}\theta = \frac{[f(\theta)]^{n+1}}{n+1}, n \neq -1\right]$$

$$= -\frac{8a^3}{12}\left(\cos^4\frac{\pi}{2} - \cos^4 0\right)$$

$$= \frac{2}{3}a^3$$

Example 5

Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{a(1+\sin\theta)} r^2\cos\theta\,\mathrm{d}\theta\,\mathrm{d}r.$

Solution

Since inner limits depend on θ, the function is integrated first w.r.t. r.

The correct form of the integral $= \int_0^{\frac{\pi}{2}}\int_0^{a(1+\sin\theta)} r^2\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta$

$$\int_0^{\frac{\pi}{2}}\int_0^{a(1+\sin\theta)} r^2\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}}\left[\int_0^{a(1+\sin\theta)} r^2\mathrm{d}r\right]\cos\theta\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}}\left|\frac{r^3}{3}\right|_0^{a(1+\sin\theta)}\cos\theta\,\mathrm{d}\theta$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{2}} a^3(1+\sin\theta)^3\cos\theta\,\mathrm{d}\theta$$

$$= \frac{a^3}{3}\left|\frac{(1+\sin\theta)^4}{4}\right|_0^{\frac{\pi}{2}} \qquad \left[\because \int [f(\theta)]^n f'(\theta)\mathrm{d}\theta = \frac{[f(\theta)]^{n+1}}{n+1}, n \neq -1\right]$$

$$= \frac{a^3}{12}\left[\left(1+\sin\frac{\pi}{2}\right)^4 - (1+\sin 0)^4\right]$$

$$= \frac{a^3}{12}[2^4 - 1]$$

$$= \frac{5}{4}a^3$$

Example 6

Evaluate $\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\mathrm{d}r\,\mathrm{d}\theta.$

Solution

$$\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{\cos 2\theta}} \frac{1}{2}\cdot\frac{2r}{(1+r^2)^2}\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}}\left[\int_0^{\sqrt{\cos 2\theta}} (1+r^2)^{-2}\cdot 2r\,dr\right]d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}}\left|-(1+r^2)^{-1}\right|_0^{\sqrt{\cos 2\theta}} d\theta \qquad \left[\because \int [f(r)]^n f'(r)\,dr = \frac{[f(r)]^{n+1}}{n+1},\ n \neq -1\right]$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{4}}\left(\frac{1}{1+\cos 2\theta}-1\right)d\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{4}}\left(\frac{1}{2\cos^2\theta}-1\right)d\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{4}}\left(\frac{1}{2}\sec^2\theta-1\right)d\theta$$

$$= -\frac{1}{2}\left|\frac{1}{2}\tan\theta-\theta\right|_0^{\frac{\pi}{4}}$$

$$= -\frac{1}{2}\left(\frac{1}{2}\tan\frac{\pi}{4}-\frac{\pi}{4}\right)$$

$$= \frac{1}{8}(\pi-2)$$

Example 7

Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin(\theta+\phi)\,d\theta\,d\phi.$

Solution

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin(\theta+\phi)\,d\theta\,d\phi = \int_0^{\frac{\pi}{2}}\left[\int_0^{\frac{\pi}{2}} \sin(\theta+\phi)\,d\theta\right]d\phi$$

$$= \int_0^{\frac{\pi}{2}}\left|-\cos(\theta+\phi)\right|_0^{\frac{\pi}{2}} d\phi$$

$$= -\int_0^{\frac{\pi}{2}}\left[\cos\left(\frac{\pi}{2}+\phi\right)-\cos\phi\right]d\phi$$

$$= -\int_0^{\frac{\pi}{2}}(-\sin\phi-\cos\phi)\,d\phi$$

$$= -\left|\cos\phi - \sin\phi\right|_0^{\frac{\pi}{2}}$$

$$= -\left(\cos\frac{\pi}{2} - \sin\frac{\pi}{2} - \cos 0 + \sin 0\right)$$

$$= 2$$

Type II Evaluation of Double Integrals Over a Given Region in Polar Coordinates

Example 1

Evaluate $\iint\limits_R r^3 \sin 2\theta \, \mathrm{d}r \, \mathrm{d}\theta$ *over the area bounded in the first quadrant between the circle* $r = 2$ *and* $r = 4$. **[Summer 2016]**

Solution

1. The region of integration is the interior of the circle between $r = 2$ to $r = 4$.
2. Draw an elementary radius vector OAB from the origin which enters in the region from the circle $r = 2$ and leaves at the circle $r = 4$.
3. Limits of r: $r = 2$ to $r = 4$

   Limits of θ: $\theta = 0$ to $\theta = \frac{\pi}{2}$

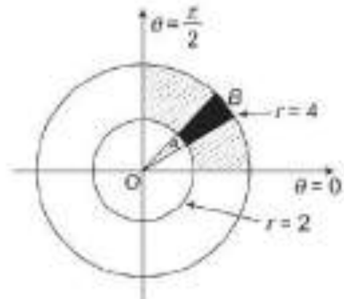

Fig. 9.84

$$I = \iint r^3 \sin 2\theta \, \mathrm{d}r \, \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_2^4 r^3 \sin 2\theta \, \mathrm{d}r \, \mathrm{d}\theta$$

$$= \left|\frac{r^4}{4}\right|_2^4 \left| -\frac{\cos 2\theta}{2}\right|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[(4)^4 - (2)^4\right]\left(\frac{1}{2}\right)(\cos\pi - \cos 0)$$

$$= \frac{1}{2}[256 - 16]\left(-\frac{1}{2}\right)(-2)$$

$$= \frac{1}{2}[240]$$

$$= 120$$

Example 2

Evaluate $\iint r\sqrt{a^2-r^2}\,\mathrm{d}r\,\mathrm{d}\theta$ *over the upper half of the circle* $r = a\cos\theta$. **[Summer 2017]**

Solution

1. The region of integration is the upper half of the circle $r = a\cos\theta$.
2. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = a\cos\theta$.
3. Limits of $r : r = 0$ to $r = a\cos\theta$

   Limits of $\theta : \theta = 0$ to $\theta = \frac{\pi}{2}$

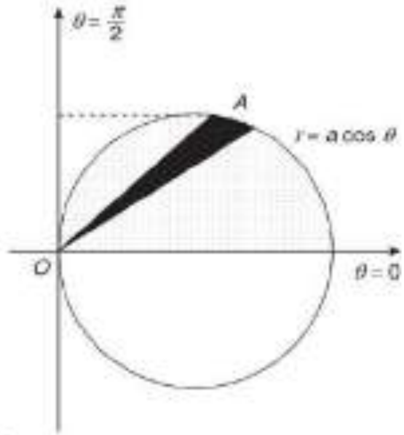

Fig. 9.85

$$I = \iint r\sqrt{a^2-r^2}\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta}\left(-\frac{1}{2}\right)(a^2-r^2)^{\frac{1}{2}}(-2r)\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}}\left|\frac{2(a^2-r^2)^{\frac{3}{2}}}{3}\right|_0^{a\cos\theta}\mathrm{d}\theta \qquad \left[\because \int [f(r)]^n f'(r)\,\mathrm{d}r = \frac{[f(r)]^{n+1}}{n+1}\right]$$

$$= -\frac{1}{3}\int_0^{\frac{\pi}{2}}(a^3\sin^3\theta - a^3)\,\mathrm{d}\theta$$

$$= -\frac{a^3}{3}\int_0^{\frac{\pi}{2}}\left(\frac{3\sin\theta - \sin 3\theta}{4} - 1\right)\mathrm{d}\theta$$

$$= -\frac{a^3}{3}\left|\frac{1}{4}\left(-3\cos\theta + \frac{\cos 3\theta}{3}\right) - \theta\right|_0^{\frac{\pi}{2}}$$

$$= -\frac{a^3}{3}\left(-\frac{3}{4}\cos\frac{\pi}{2} + \frac{1}{12}\cos\frac{3\pi}{2} - \frac{\pi}{2} + \frac{3}{4}\cos 0 - \frac{1}{12}\cos 0\right)$$

$$= -\frac{a^3}{3}\left(-\frac{\pi}{2} + \frac{3}{4} - \frac{1}{12}\right)$$

$$= -\frac{a^3}{3}\left(\frac{2}{3} - \frac{\pi}{2}\right)$$

Example 3

Evaluate $\iint r^4 \cos^3\theta \, dr \, d\theta$ *over the interior of the circle* $r = 2a\cos\theta$.

Solution

1. The region of integration is the interior of the circle $r = 2a\cos\theta$.
2. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = 2a\cos\theta$.
3. Limits of $r : r = 0$ to $r = 2a\cos\theta$

   Limits of $\theta : \theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$

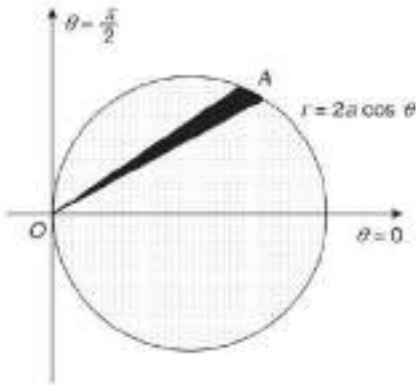

Fig. 9.86

$$I = \iint r^4 \cos^3\theta \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta \int_0^{2a\cos\theta} r^4 \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta \left| \frac{r^5}{5} \right|_0^{2a\cos\theta} d\theta$$

$$= \frac{1}{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta \, (2a\cos\theta)^5 \, d\theta$$

$$= \frac{32a^5}{5} \cdot 2\int_0^{\frac{\pi}{2}} \cos^8\theta \, d\theta$$

$$= \frac{32a^5}{5} \cdot B\left(\frac{9}{2}, \frac{1}{2}\right) \qquad \left[\because B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \, d\theta \right]$$

$$= \frac{32a^5}{5} \cdot \frac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(5)}$$

$$= \frac{32a^5}{5} \cdot \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{24}$$

$$= \frac{7\pi}{4} a^5$$

Example 4

Evaluate $\iint r^2 \sin\theta \, dr \, d\theta$ *over the cardioid* $r = a\,(1 + \cos\theta)$ *above the initial line.*

Solution

1. The region of integration is the part of the cardioid $r = a(1 + \cos\theta)$ above the initial line ($\theta = 0$).
2. Draw an elementary radius vector OA which starts from the origin and terminates on the cardioid $r = a(1 + \cos\theta)$.
3. Limits of r: $r = 0$ to $r = a(1+\cos\theta)$

   Limits of θ : $\theta = 0$ to $\theta = \pi$

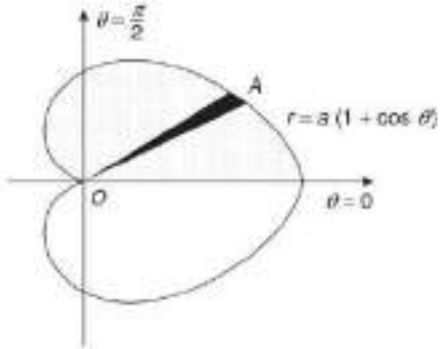

Fig. 9.87

$$I = \iint r^2 \sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2 \sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi} \left| \frac{r^3}{3} \right|_0^{a(1+\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{1}{3}\int_0^{\pi} a^3 (1+\cos\theta)^3 \sin\theta \cdot d\theta$$

$$= -\frac{a^3}{3}\int_0^{\pi} (1+\cos\theta)^3 (-\sin\theta) d\theta$$

$$= -\frac{a^3}{3}\left| \frac{(1+\cos\theta)^4}{4} \right|_0^{\pi} \qquad \left[\because \int [f(\theta)]^n f'(\theta) d\theta = \frac{[f(\theta)]^{n+1}}{n+1} \right]$$

$$= -\frac{a^3}{12}\left[(1+\cos\pi)^4 - (1+\cos 0)^4 \right]$$

$$= -\frac{a^3}{12}(0-16)$$

$$= \frac{4}{3}a^3$$

Example 5

Evaluate $\iint \frac{r \, dr \, d\theta}{\sqrt{r^2 + a^2}}$ *over one loop of the lemniscate* $r^2 = a^2 \cos 2\theta$.

Solution

1. The region of integration is one loop of the lemniscate $r^2 = a^2 \cos 2\theta$ bounded between the lines $\theta = -\frac{\pi}{4}$ and $\theta = \frac{\pi}{4}$.

2. Draw an elementary radius vector OA which starts from the origin and terminates on the lemniscate $r^2 = a^2 \cos 2\theta$.
3. Limits of r: $r = 0$ to $r = a\sqrt{\cos 2\theta}$

   Limits of $\theta: \theta = -\frac{\pi}{4}$ to $\theta = \frac{\pi}{4}$

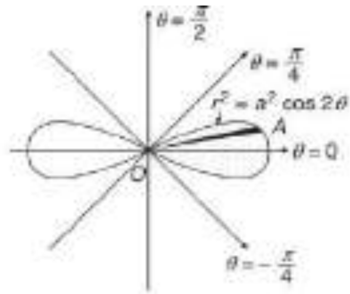

Fig. 9.88

$$I = \iint \frac{r\,\mathrm{d}r\,\mathrm{d}\theta}{\sqrt{r^2 + a^2}}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{a\sqrt{\cos 2\theta}} \frac{r\,\mathrm{d}r\,\mathrm{d}\theta}{\sqrt{r^2 + a^2}}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{a\sqrt{\cos 2\theta}} \frac{1}{2}(r^2 + a^2)^{-\frac{1}{2}}(2r)\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left| 2(r^2 + a^2)^{\frac{1}{2}} \right|_0^{a\sqrt{\cos 2\theta}} \mathrm{d}\theta \qquad \left[\because \int [f(r)]^n f'(r)\,\mathrm{d}r = \frac{[f(r)]^{n+1}}{n+1}, n \neq -1 \right]$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2a \left[(\cos 2\theta + 1)^{\frac{1}{2}} - 1 \right] \mathrm{d}\theta$$

$$= a\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\sqrt{2}\cos\theta - 1 \right) \mathrm{d}\theta$$

$$= a\left| \sqrt{2}\sin\theta - \theta \right|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= a\left[\sqrt{2}\sin\frac{\pi}{4} - \frac{\pi}{4} - \sqrt{2}\sin\left(-\frac{\pi}{4}\right) + \left(-\frac{\pi}{4}\right) \right]$$

$$= a\left(2 - \frac{\pi}{2} \right)$$

$$= \frac{a}{2}(4 - \pi)$$

Example 6

Evaluate $\iint r^2\,\mathrm{d}r\,\mathrm{d}\theta$ *over the area between the circles* $r = a \sin\theta$ *and* $r = 2a \sin\theta$.

Solution

1. The region of integration is the area bounded between the circle $r = a \sin \theta$ and $r = 2a \sin \theta$.

2. Draw an elementary radius vector OAB from the origin which enters in the region from the circle $r = a \sin\theta$ and leaves at the circle $r = 2a \sin\theta$.
3. Limits of r : $r = a\sin\theta$ to $r = 2a\sin\theta$
   Limits of θ : $\theta = 0$ to $\theta = \pi$

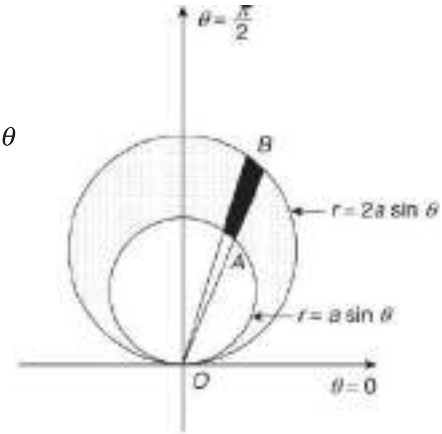

Fig. 9.89

$$
\begin{aligned}
I &= \iint r^2\, dr\, d\theta \\
&= \int_0^{\pi}\int_{a\sin\theta}^{2a\sin\theta} r^2\, dr\, d\theta \\
&= \int_0^{\pi} \left| \frac{r^3}{3} \right|_{a\sin\theta}^{2a\sin\theta} d\theta \\
&= \frac{1}{3}\int_0^{\pi} (8a^3\sin^3\theta - a^3\sin^3\theta)\, d\theta \\
&= \frac{7a^3}{3}\int_0^{\pi} \sin^3\theta\, d\theta \\
&= \frac{7a^3}{3}\int_0^{\pi} \frac{3\sin\theta - \sin 3\theta}{4}\, d\theta \\
&= \frac{7a^3}{12}\left| -3\cos\theta + \frac{\cos 3\theta}{3} \right|_0^{\pi} \\
&= \frac{7a^3}{12}\left[-3(\cos\pi - \cos 0) + \frac{1}{3}(\cos 3\pi - \cos 0) \right] \\
&= \frac{7a^3}{12}\left(\frac{16}{3} \right) \\
&= \frac{28}{9}a^3
\end{aligned}
$$

EXERCISE 9.4

Evaluate the following integrals:

1. $\iint r e^{-\frac{r^2}{a^2}} \cos\theta \sin\theta\, dr\, d\theta$ over the upper half of the circle $r = 2a\cos\theta$.

$$\left[\textbf{Ans.:} \frac{a^2}{16}\left(3 + \frac{1}{e^4} \right) \right]$$

2. $\iint r^3\, dr\, d\theta$ over the region between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

$$\left[\textbf{Ans.:} \frac{45\pi}{2} \right]$$

3. $\iint r\sin\theta\, dA$ over the cardioid $r = a\,(1+\cos\theta)$ above the initial line.

$$\left[\text{Ans.}: \frac{4}{3}a^3\right]$$

4. $\iint \frac{r}{\sqrt{r^2+4}}\,dr\,d\theta$ over one loop of the lemniscate $r^2 = 4\cos 2\theta$.

$$\left[\text{Ans.}: (4-\pi)\right]$$

9.5 MULTIPLE INTEGRALS BY SUBSTITUTION

9.5.1 Change of Variables from Cartesian to Polar Coordinates

The double integral can be changed from Cartesian coordinates (x, y) to polar coordinates (r, θ) by putting $x = r\cos\theta$, $y = r\sin\theta$. Then $\iint f(x, y)\,dy\,dx = \iint f(r\cos\theta, r\sin\theta)|J|\,dr\,d\theta$ where J is the Jacobian (functional determinant) defined as

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

Hence, $\iint f(x, y)\,dy\,dx = \iint f(r\cos\theta, r\sin\theta)|r|\,dr\,d\theta$

$$= \iint f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Example 1

Evaluate $\iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$ *over the first quadrant of the circle* $x^2 + y^2 = 1$.

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the circle $x^2 + y^2 = 1$ is obtained as $r = 1$.
2. The region of integration is the part of the circle $r = 1$ in the first quadrant.

3. Draw an elementary radius vector *OA* which starts from the origin and terminates on the circle $r = 1$.
4. Limits of r : $r = 0$ to $r = 1$

   Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{2}$

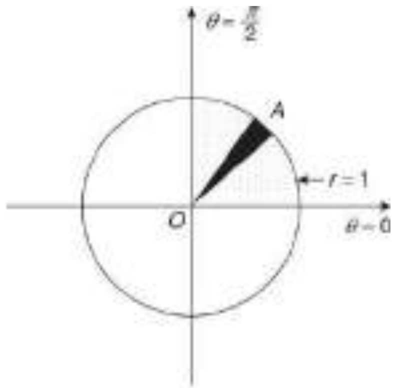

Fig. 9.90

Hence, the polar form of the given integral is

$$I = \iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr\,d\theta$$

Putting $r^2 = \cos 2t$, $2r\,dr = -2\sin 2t\,dt$

When $r = 0, t = \frac{\pi}{4}$

When $r = 1, t = 0$

$$I = \int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^0 \sqrt{\frac{1-\cos 2t}{1+\cos 2t}}(-\sin 2t\,dt)\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}} \sqrt{\frac{2\sin^2 t}{2\cos^2 t}}\sin 2t\,dt\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}} \frac{\sin t}{\cos t}\,2\sin t\cos t\,dt\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{4}} (1-\cos 2t)\,dt$$

$$= \left|\theta\right|_0^{\frac{\pi}{2}} \left| t - \frac{\sin 2t}{2} \right|_0^{\frac{\pi}{4}} d\theta$$

$$= \frac{\pi}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

$$= \frac{\pi}{8}(\pi - 2)$$

Example 2

Evaluate $\iint \frac{4xy}{x^2+y^2} e^{-x^2-y^2}\,dx\,dy$ *over the region bounded by the circle* $x^2 + y^2 - x = 0$ *in the first quadrant.*

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the circle $x^2 + y^2 - x = 0$ is $r^2 - r\cos\theta = 0$, $r = \cos\theta$.

2. The region of integration is the part of the circle $r = \cos\theta$ in the first quadrant.
3. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = \cos\theta$.
4. Limits of $r : r = 0$ to $r = \cos\theta$

   Limits of $\theta : \theta = 0$ to $\theta = \frac{\pi}{2}$

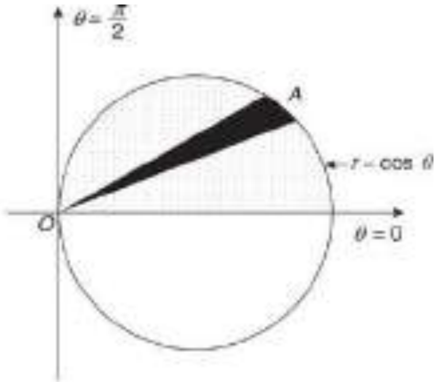

Fig. 9.91

Hence, the polar form of the given integral is

$$
\begin{aligned}
I &= \iint \frac{4xy}{x^2+y^2} e^{-x^2-y^2} \, dx\, dy \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\cos\theta} \frac{4r^2 \cos\theta \sin\theta}{r^2} e^{-r^2} r \, dr \, d\theta \\
&= -2\int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \left[\int_0^{\cos\theta} e^{-r^2} (-2r) \, dr \right] d\theta \\
&= -2\int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \left| e^{-r^2} \right|_0^{\cos\theta} d\theta \qquad \left[\because \int e^{f(r)} f'(r) \, dr = e^{f(r)} \right] \\
&= -2\int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \left(e^{-\cos^2\theta} - 1 \right) d\theta \\
&= -\int_0^{\frac{\pi}{2}} \left[e^{-\cos^2\theta} (2\cos\theta \sin\theta) - \sin 2\theta \right] d\theta \\
&= -\left| e^{-\cos^2\theta} + \frac{\cos 2\theta}{2} \right|_0^{\frac{\pi}{2}} \\
&= -\left(e^0 + \frac{\cos\pi}{2} - e^{-1} - \frac{\cos 0}{2} \right) \\
&= \frac{1}{e}
\end{aligned}
$$

Example 3

Evaluate $\iint \frac{x^2 y^2}{(x^2+y^2)} \, dx\, dy$ *over the region bounded by the circles* $x^2 + y^2 = a^2$ *and* $x^2 + y^2 = b^2$ $(a > b)$.

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of
   (i) the circle $x^2 + y^2 = a^2$ is $r^2 = a^2$, $r = a$.
   (ii) the circle $x^2 + y^2 = b^2$ is $r^2 = b^2$, $r = b$.

2. The region of integration is the part bounded between the circles $r = a$ and $r = b$.
3. Draw an elementary radius vector OAB from the origin which enters in the region from the circle $r = b$ and terminates on the circle $r = a$.
4. Limits of $r : r = b$ to $r = a$
   Limits of $\theta : \theta = 0$ to $\theta = 2\pi$

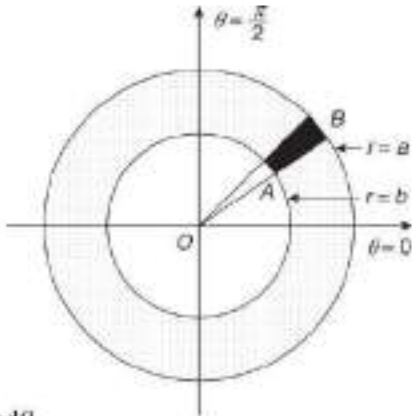

Fig. 9.92

Hence, the polar form of the given integral is

$$I = \iint \frac{x^2 y^2}{(x^2 + y^2)} \mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^{2\pi} \int_b^a \frac{r^4 \cos^2\theta \sin^2\theta}{r^2} \cdot r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{2\pi} \cos^2\theta \sin^2\theta \left| \frac{r^4}{4} \right|_b^a \mathrm{d}\theta$$

$$= \int_0^{2\pi} \frac{\sin^2 2\theta}{4} \cdot \frac{(a^4 - b^4)}{4} \mathrm{d}\theta$$

$$= \frac{a^4 - b^4}{16} \int_0^{2\pi} \frac{(1 - \cos 4\theta)}{2} \mathrm{d}\theta$$

$$= \left(\frac{a^4 - b^4}{32} \right) \left| \theta - \frac{\sin 4\theta}{4} \right|_0^{2\pi}$$

$$= \left(\frac{a^4 - b^4}{32} \right) (2\pi)$$

$$= \frac{\pi}{16} (a^4 - b^4)$$

Example 4

Evaluate $\iint \frac{(x^2 + y^2)^2}{x^2 y^2} \mathrm{d}x\,\mathrm{d}y$ *over the region common to the circles* $x^2 + y^2 = ax$ *and* $x^2 + y^2 = by$ $(a, b > 0)$.

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of
   (i) the circle $x^2 + y^2 = ax$ is $r^2 = ar\cos\theta$, $r = a\cos\theta$.
   (ii) the circle $x^2 + y^2 = by$ is $r^2 = br\sin\theta$, $r = b\sin\theta$.
2. The region of integration is the common part of the circles $r = a\cos\theta$ and $r = b\sin\theta$.

3. The point of intersection of the circle $r = a\cos\theta$ and $r = b\sin\theta$, is obtained as

$$b\sin\theta = a\cos\theta$$
$$\tan\theta = \frac{a}{b}$$
$$\theta = \tan^{-1}\frac{a}{b}$$

Hence, $\theta = \tan^{-1}\frac{a}{b}$ at P.

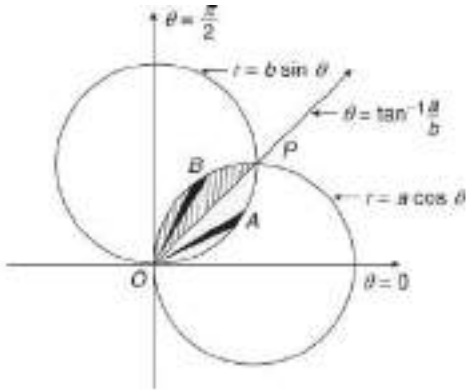

Fig. 9.93

4. Divide the region into two subregions *OAP* and *OBP*. Draw an elementary radius vector *OA* and *OB* in each subregion.
   (i) In subregion *OAP*, elementary radius vector *OA* starts from the origin and terminates on the circle $r = b\sin\theta$.
   Limits of r: $r = 0$ to $r = b\sin\theta$
   Limits of θ: $\theta = 0$ to $\theta = \tan^{-1}\frac{a}{b}$
   (ii) In subregion *OBP*, elementary radius vector *OB* starts from the origin and terminates on the circle $r = a\cos\theta$.
   Limits of r: $r = 0$ to $r = a\cos\theta$
   Limits of θ: $\theta = \tan^{-1}\frac{a}{b}$ to $\theta = \frac{\pi}{2}$

Hence, the polar form of the given integral is

$$I = \iint \frac{(x^2+y^2)^2}{x^2y^2}\,dx\,dy$$

$$= \int_0^{\tan^{-1}\frac{a}{b}}\int_0^{b\sin\theta} \frac{r^4}{r^4\sin^2\theta\cos^2\theta}\cdot r\,dr\,d\theta + \int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}}\int_0^{a\cos\theta} \frac{r^4}{r^4\sin^2\theta\cos^2\theta}\cdot r\,dr\,d\theta$$

$$= \int_0^{\tan^{-1}\frac{a}{b}} \frac{1}{\sin^2\theta\cos^2\theta}\left|\frac{r^2}{2}\right|_0^{b\sin\theta} d\theta + \int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}} \frac{1}{\sin^2\theta\cos^2\theta}\left|\frac{r^2}{2}\right|_0^{a\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\tan^{-1}\frac{a}{b}} \frac{1}{\sin^2\theta\cos^2\theta}\cdot b^2\sin^2\theta\,d\theta + \frac{1}{2}\int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}} \frac{1}{\sin^2\theta\cos^2\theta}\cdot a^2\cos^2\theta\,d\theta$$

$$= \frac{b^2}{2}\int_0^{\tan^{-1}\frac{a}{b}} \sec^2\theta\,d\theta + \frac{a^2}{2}\int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}} \operatorname{cosec}^2\theta\,d\theta$$

$$= \frac{b^2}{2}\left|\tan\theta\right|_0^{\tan^{-1}\frac{a}{b}} + \frac{a^2}{2}\left|-\cot\theta\right|_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}}$$

$$= \frac{b^2}{2}\left[\tan\tan^{-1}\left(\frac{a}{b}\right) - \tan 0\right] - \frac{a^2}{2}\left[\cot\frac{\pi}{2} - \cot\left(\tan^{-1}\frac{a}{b}\right)\right]$$

$$= \frac{b^2}{2}\left[\frac{a}{b} - 0\right] - \frac{a^2}{2}\left[0 - \frac{b}{a}\right]$$

$$= \frac{ab}{2} + \frac{ab}{2}$$

$$= ab$$

Example 5

Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y.$

Solution

1. Limits of $x : x = 0$ to $x \to \infty$
   Limits of $y : y = 0$ to $y \to \infty$
2. The region of integration is the first quadrant.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, the integral changes to polar form.
4. Draw an elementary radius vector which starts from the origin and extends up to infinity.
   Limits of $r :\ r = 0$ to $r \to \infty$

   Limits of $\theta : \theta = 0$ to $\theta = \frac{\pi}{2}$

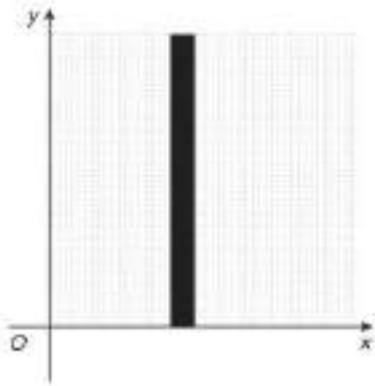

Fig. 9.94

Hence, the polar form of the given integral is

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\, r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}(-2r)\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \left|e^{-r^2}\right|_0^\infty \mathrm{d}\theta \quad \left[\because \int e^{f(r)} f'(r)\,\mathrm{d}r = e^{f(r)}\right]$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} (0 - e^0)\,\mathrm{d}\theta$$

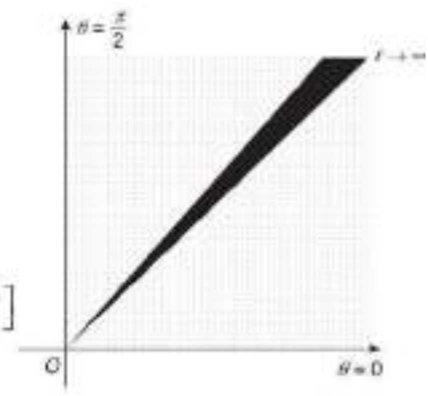

Fig. 9.95

$$= -\frac{1}{2}\left|-\theta\right|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}$$

Example 6

Evaluate $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\,dy}{(1+x^2+y^2)^{\frac{3}{2}}}$.

Solution

1. Limits of $x: x \to -\infty$ to $x \to \infty$
   Limits of $y: y \to -\infty$ to $y \to \infty$
2. The region of integration is the entire coordinate plane.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, integral changes to polar form.
4. Draw an elementary radius vector which starts from origin and extends up to ∞.
   Limits of r: $r = 0$ to $r \to \infty$
   Limits of θ : $\theta = 0$ to $\theta = 2\pi$

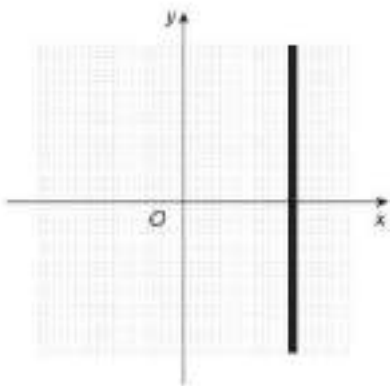

Fig. 9.96

Hence, the polar form of the given integral is

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\,dy}{(1+x^2+y^2)^{\frac{3}{2}}}$$

$$= \int_0^{2\pi}\int_0^{\infty} \frac{r\,dr\,d\theta}{(1+r^2)^{\frac{3}{2}}}$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\infty} (1+r^2)^{-\frac{3}{2}}(2r)\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left|-2(1+r^2)^{-\frac{1}{2}}\right|_0^{\infty} d\theta$$

$$\left[\because \int [f(r)]^n f'(r)\,dr = \frac{[f(r)]^{n+1}}{n+1}\right]$$

$$= -\int_0^{2\pi}\left|\frac{1}{\sqrt{1+r^2}}\right|_0^{\infty} d\theta$$

$$= -\int_0^{2\pi}(0-1)\,d\theta$$

$$= \left|\theta\right|_0^{2\pi}$$

$$= 2\pi$$

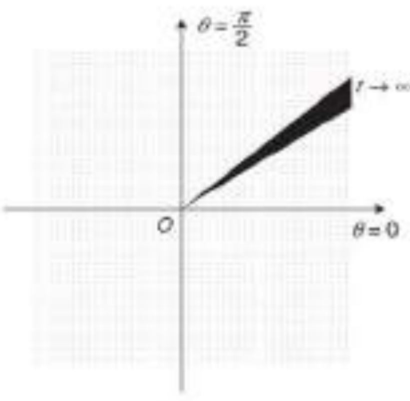

Fig. 9.97

Example 7

Evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$ *by transforming into polar coordinates.*

[Winter 2013]

Solution

1. Limits of $x : x = y$ to $x = a$
   Limits of $y : y = 0$ to $y = a$
2. The region of integration is bounded by the lines $y = x$, $x = a$ and $y = 0$.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of
   (i) the line $y = x$ is $r\sin\theta = r\cos\theta$, $\tan\theta = 1$, $\theta = \frac{\pi}{4}$.
   (ii) the line $x = a$ is $r\cos\theta = a$, $r = a\sec\theta$.
4. Draw an elementary radius vector OA which starts from the origin and terminates on the line $r = a\sec\theta$.
   Limits of $r : r = 0$ to $r = a\sec\theta$
   Limits of $\theta : \theta = 0$ to $\theta = \frac{\pi}{4}$

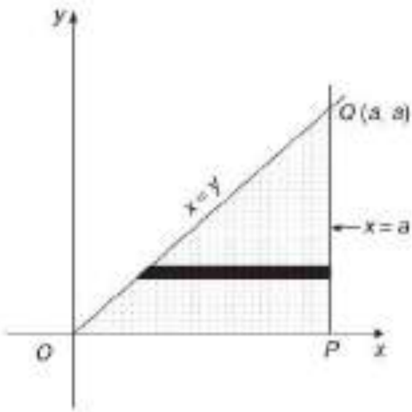

Fig. 9.98

Hence, the polar form of the given integral is

$$
\begin{aligned}
I &= \int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy \\
&= \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} \frac{r\cos\theta}{r^2}\cdot r\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} \cos\theta\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{4}} \left| r \right|_0^{a\sec\theta} \cos\theta\,d\theta \\
&= \int_0^{\frac{\pi}{4}} a\sec\theta\cos\theta\,d\theta \\
&= a\int_0^{\frac{\pi}{4}} d\theta \\
&= a\left|\theta\right|_0^{\frac{\pi}{4}} \\
&= a\cdot\frac{\pi}{4} \\
&= \frac{\pi a}{4}
\end{aligned}
$$

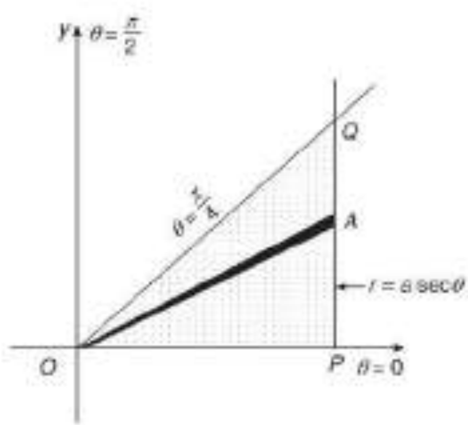

Fig. 9.99

Example 8

Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\, dy\, dx$ *by transforming into polar coordinates.*

Solution

1. Limits of y : $y = 0$ to $y = \sqrt{2x - x^2}$
   Limits of x : $x = 0$ to $x = 2$
2. The region of integration is bounded by the circle $x^2 + y^2 - 2x = 0$ and the lines $y = 0$, $x = 0$. Since the limits of x and y are positive, the region of integration is the part of the circle in the first quadrant.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the circle $x^2 + y^2 - 2x = 0$ is

$$r^2 - 2r\cos\theta = 0$$
$$r = 2\cos\theta.$$

4. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = 2\cos\theta$.

   Limits of r : $r = 0$ to $r = 2\cos\theta$

   Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{2}$

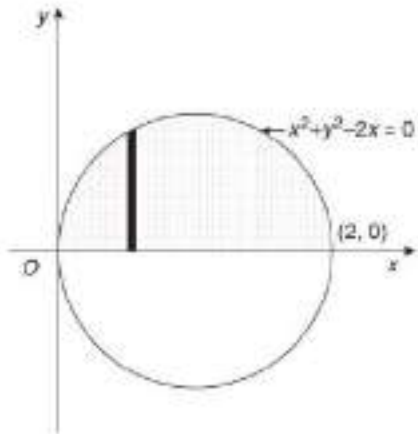

Fig. 9.100

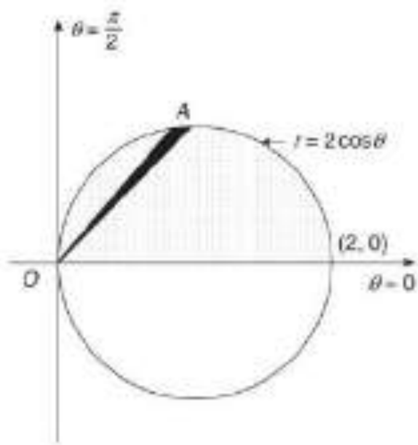

Fig. 9.101

Hence, the polar form of the given integral is

$$I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\, dy\, dx$$
$$= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \frac{r\cos\theta}{r^2} \cdot r\, dr\, d\theta$$
$$= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \cos\theta\, dr\, d\theta$$
$$= \int_0^{\frac{\pi}{2}} \left| r \right|_0^{2\cos\theta} \cos\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{2}} 2\cos^2\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)\, d\theta$$
$$= \left| \theta + \frac{\sin 2\theta}{2} \right|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + \frac{1}{2}\sin\pi - \frac{1}{2}\sin 0$$
$$= \frac{\pi}{2}$$

Example 9

Evaluate $\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2 + y^2)\,dx\,dy$.

Solution

1. Limits of y : $y = x$ to $y = \sqrt{2x - x^2}$
   Limits of x : $x = 0$ to $x = 1$
2. The region of integration is bounded by the line $y = x$ and the circle $x^2 + y^2 - 2x = 0$.

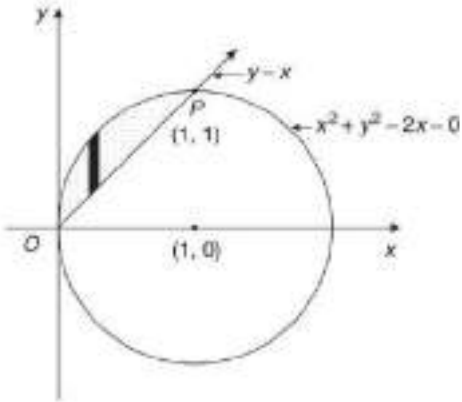

Fig. 9.102

3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of
   (i) the line $y = x$ is
   $$r\sin\theta = r\cos\theta,\ \tan\theta = 1,\ \theta = \frac{\pi}{4}.$$
   (ii) the circle $x^2 + y^2 - 2x = 0$ is
   $$r^2 - 2r\cos\theta = 0$$
   $$r = 2\cos\theta.$$
4. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = 2\cos\theta$.
   Limits of r : $r = 0$ to $r = 2\cos\theta$
   Limits of θ : $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$

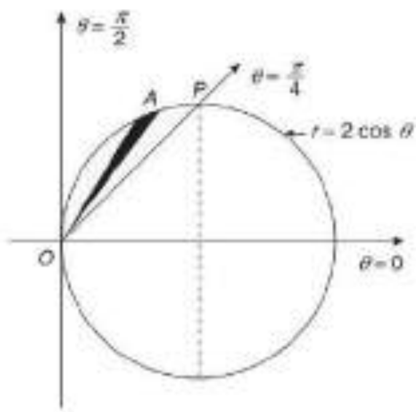

Fig. 9.103

Hence, the polar form of the given integral is

$$I = \int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2 + y^2)\,dx\,dy$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 \cdot r\,dr\,d\theta$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left| \frac{r^4}{4} \right|_0^{2\cos\theta} d\theta$$
$$= 4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^4\theta\,d\theta$$

$$= 4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 + 2\cos 2\theta + \cos^2 2\theta)\, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta$$

$$= \left|\frac{3}{2}\theta + \frac{2\sin 2\theta}{2} + \frac{\sin 4\theta}{8}\right|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{3}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) - \left(\sin \pi - \sin\frac{\pi}{2}\right) + \frac{1}{8}(\sin 2\pi - \sin \pi)$$

$$= \frac{3\pi}{8} + 1$$

Example 10

Evaluate the integral $\int_0^a \int_0^{\sqrt{a^2-y^2}} y^2 \sqrt{x^2+y^2}\, dy\, dx$ *by changing into polar coordinates.* **[Summer 2014]**

Solution

1. Limits of x : $x = 0$ to $x = \sqrt{a^2 - y^2}$
   Limits of y : $y = 0$ to $y = a$
2. The region of integration is bounded by $x = 0$, $x = \sqrt{a^2 - y^2}$, $y = 0$ and $y = a$.

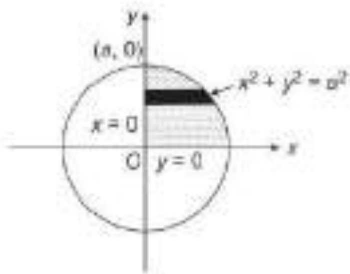

Fig. 9.104

3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the circle $x^2 + y^2 = a^2$ is
   (i) $r^2 = a^2$
   $\therefore\ r = a$
4. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = a$.
   Limits of r: $r = 0$ to $r = a$

   Limit of θ: $\theta = 0$ to $\theta = \dfrac{\pi}{2}$

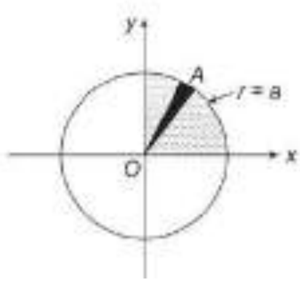

Fig. 9.105

Hence, the polar form of the given integral is

$$I = \int_0^{\frac{\pi}{2}} \int_0^a r^2 \sin^2 \theta \cdot r \cdot r \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 \theta \left[\int_0^a r^4 \, dr \right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 \theta \left| \frac{r^5}{5} \right|_0^a d\theta$$

$$= \frac{a^5}{5} \int_0^{\frac{\pi}{2}} \sin^2 \theta \, d\theta$$

$$= \frac{a^5}{5} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{a^5 \pi}{20}$$

Example 11

Evaluate $\int_0^1 \int_{\sqrt{x-x^2}}^{\sqrt{1-x^2}} \frac{xy e^{-(x^2+y^2)}}{x^2+y^2} \, dx \, dy.$

Solution

1. Limits of $y: y = \sqrt{x - x^2}$ to $y = \sqrt{1 - x^2}$

   Limits of $x: x = 0$ to $x = 1$
2. The region of integration is the part of the first quadrant bounded by the circles $x^2 + y^2 - x = 0$ and $x^2 + y^2 = 1$.
3. Putting $x = r \cos \theta$, $y = r \sin \theta$, polar form of
   (i) the circle $x^2 + y^2 - x = 0$ is $r^2 - r \cos \theta = 0$, $r = \cos \theta$.
   (ii) the circle $x^2 + y^2 = 1$ is $r^2 = 1$, $r = 1$.
4. Draw an elementary radius vector *OAB* from the origin which enters in the region from the circle $r = \cos \theta$ and terminates on the circle $r = 1$.

   Limits of $r: r = \cos \theta$ to $r = 1$

   Limits of $\theta: \theta = 0$ to $\theta = \frac{\pi}{2}$

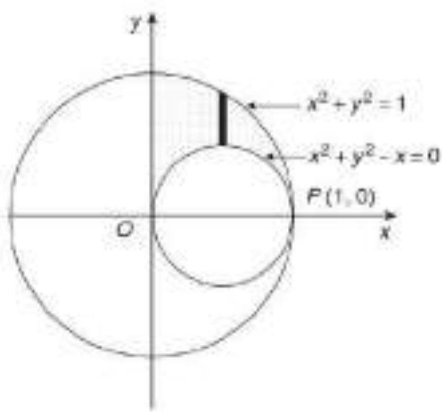

Fig. 9.106

Hence, the polar form of the given integral is

$$I=\int_0^1\int_{\sqrt{x-x^2}}^{\sqrt{1-x^2}}\frac{xye^{-(x^2+y^2)}}{x^2+y^2}\,dx\,dy$$

$$=\int_0^{\frac{\pi}{2}}\int_{\cos\theta}^{1}\frac{r^2\sin\theta\cos\theta e^{-r^2}}{r^2}\cdot r\,dr\,d\theta$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\int_{\cos\theta}^{1}e^{-r^2}(-2r)\,dr\,d\theta$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\left|e^{-r^2}\right|_{\cos\theta}^{1}d\theta$$

$$\left[\because\int e^{f(r)}f'(r)dr=e^{f(r)}\right]$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta(e^{-1}-e^{-\cos^2\theta})d\theta$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1}{2}\left(\frac{1}{e}\sin 2\theta-e^{-\cos^2\theta}\cdot 2\sin\theta\cos\theta\right)d\theta$$

$$=-\frac{1}{4}\left|\frac{1}{e}\left(-\frac{\cos 2\theta}{2}\right)-e^{-\cos^2\theta}\right|_0^{\frac{\pi}{2}}\quad\left[\because\int e^{f(\theta)}f'(\theta)d\theta=e^{f(\theta)}\right]$$

$$=-\frac{1}{4}\left[-\frac{1}{2e}(\cos\pi-\cos 0)-e^{-\left(\cos^2\frac{\pi}{2}\right)}+e^{-\cos^2 0}\right]$$

$$=-\frac{1}{4}\left[-\frac{1}{2e}(-2)-e^{0}+e^{-1}\right]$$

$$=-\frac{1}{4}\left[\frac{1}{e}-1+\frac{1}{e}\right]$$

$$=\frac{1}{4}\left[1-\frac{2}{e}\right]$$

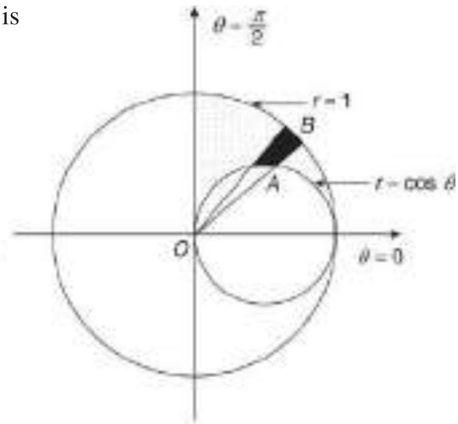

Fig. 9.107

Example 12

Evaluate $\int_0^{4a}\int_{\frac{y^2}{4a}}^{y}\frac{x^2-y^2}{x^2+y^2}dx\,dy.$ **[Winter 2013]**

Solution

1. Limits of $x: x=\frac{y^2}{4a}$ to $x=y$

   Limits of $y: y=0$ to $y=4a$.

2. The region of integration is bounded by the line $y = x$ and the parabola $y^2 = 4ax$.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of
   (i) the line $y = x$ is $r\sin\theta = r\cos\theta$,
   $\tan\theta = 1$, $\theta = \dfrac{\pi}{4}$.
   (ii) the parabola $y^2 = 4ax$ is
   $r^2\sin^2\theta = 4ar\cos\theta$, $r = 4a\cot\theta\operatorname{cosec}\theta$.
4. Draw an elementary radius vector OA which starts from the origin and terminates on the parabola $r = 4a\cot\theta\operatorname{cosec}\theta$.

   Limits of r: $r = 0$ to $r = 4a\cot\theta\operatorname{cosec}\theta$

   Limits of θ: $\theta = \dfrac{\pi}{4}$ to $\theta = \dfrac{\pi}{2}$

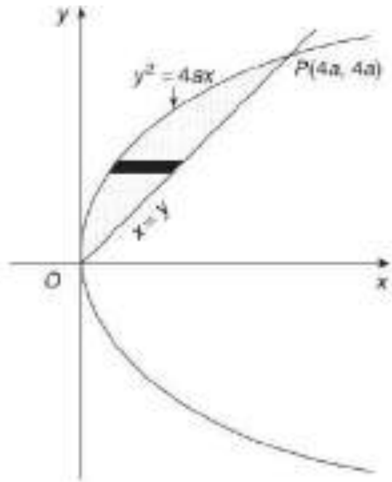

Fig. 9.108

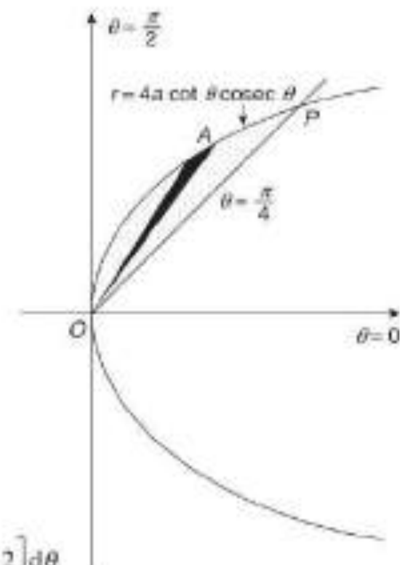

Fig. 9.109

Hence, the polar form of the given integral is

$$I = \int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2 - y^2}{x^2 + y^2}\,dx\,dy$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{4a\cot\theta\operatorname{cosec}\theta} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2}\cdot r\,dr\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 - 2\sin^2\theta)\left|\frac{r^2}{2}\right|_0^{4a\cot\theta\operatorname{cosec}\theta} d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 - 2\sin^2\theta)(4a)^2\cot^2\theta\operatorname{cosec}^2\theta\,d\theta$$

$$= 8a^2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cot^2\theta\operatorname{cosec}^2\theta - 2\cot^2\theta)\,d\theta$$

$$= 8a^2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\left\{-(\cot^2\theta)(-\operatorname{cosec}^2\theta)\right\} - 2\operatorname{cosec}^2\theta + 2\right] d\theta$$

$$= 8a^2\left|-\frac{\cot^3\theta}{3} + 2\cot\theta + 2\theta\right|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \qquad \left[\because \int [f(\theta)]^n f'(\theta)\,d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= 8a^2\left[-\frac{1}{3}\left(\cot^3\frac{\pi}{2} - \cot^3\frac{\pi}{4}\right) + 2\left(\cot\frac{\pi}{2} - \cot\frac{\pi}{4}\right) + 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right]$$

$$= 8a^2\left[-\frac{1}{3}(-1)+2(-1)+2\cdot\frac{\pi}{4}\right]$$
$$= 8a^2\left[-\frac{5}{3}+\frac{\pi}{2}\right]$$
$$= 8a^2\left[\frac{\pi}{2}-\frac{5}{3}\right]$$

Example 13

Evaluate $\int_0^a \int_{2\sqrt{ax}}^{\sqrt{5ax-x^2}} \frac{\sqrt{x^2+y^2}}{y^2}\,dx\,dy.$

Solution

1. Limits of $y : y = 2\sqrt{ax}$ to $y = \sqrt{5ax - x^2}$
   Limits of x : $x = 0$ to $x = a$
2. Since the limits of x and y are positive, the region of integration is the part of the first quadrant bounded by the parabola $y^2 = 4ax$ and the circle $x^2 + y^2 - 5ax = 0$
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of

   (i) the parabola $y^2 = 4ax$ is
   $$r^2\sin^2\theta = 4ar\cos\theta,$$
   $$r = 4a\cot\theta\ \text{cosec}\,\theta.$$
   (ii) the circle $x^2 + y^2 - 5ax = 0$ is
   $$r^2 - 5ar\cos\theta = 0,$$
   $$r = 5a\cos\theta.$$

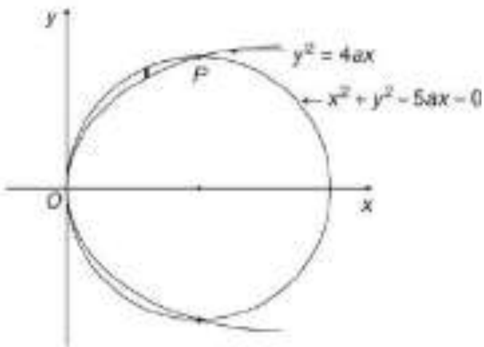

Fig. 9.110

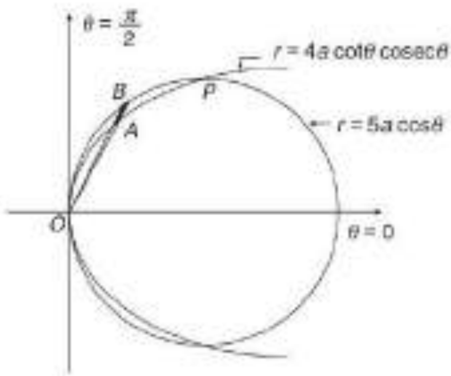

Fig. 9.111

4. The points of intersection of $r = 4a\cot\theta\ \text{cosec}\,\theta$ and $r = 5a\cos\theta$ are obtained as
   $$4a\cot\theta\ \text{cosec}\,\theta = 5a\cos\theta$$
   $$\sin^2\theta = \frac{4}{5}$$
   $$\theta = \pm\sin^{-1}\frac{2}{\sqrt{5}}$$

   Hence, $\theta = \sin^{-1}\frac{2}{\sqrt{5}}$ at P.

5. Draw an elementary radius vector *OAB* from the origin which enters in the region from the parabola $r = 4a\cot\theta\,\text{cosec}\,\theta$ and terminates on the circle $r = 5a\cos\theta$.
   Limits of r : $r = 4a\cot\theta\,\text{cosec}\,\theta$ to $r = 5a\cos\theta$

   Limits of $\theta : \theta = \sin^{-1}\dfrac{2}{\sqrt{5}}$ to $\theta = \dfrac{\pi}{2}$

Hence, the polar form of the given integral is

$$I = \int_0^a \int_{2\sqrt{ax}}^{\sqrt{5ax - x^2}} \frac{\sqrt{x^2+y^2}}{y^2}\,dx\,dy$$

$$= \int_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \int_{4a\cot\theta\,\text{cosec}\,\theta}^{5a\cos\theta} \frac{r}{r^2\sin^2\theta}\, r\,dr\,d\theta$$

$$= \int_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \text{cosec}^2\theta \left| r \right|_{4a\cot\theta\,\text{cosec}\,\theta}^{5a\cos\theta} d\theta$$

$$= \int_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \text{cosec}^2\theta (5a\cos\theta - 4a\cot\theta\,\text{cosec}\,\theta)\,d\theta$$

$$= \int_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \left[5a\cot\theta\,\text{cosec}\,\theta + 4a\,\text{cosec}^2\theta(-\text{cosec}\,\theta\cot\theta)\right] d\theta$$

$$= \left| -5a\,\text{cosec}\,\theta + 4a\frac{\text{cosec}^3\theta}{3} \right|_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \qquad \left[\because \int [f(\theta)]^n f'(\theta)\,d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= \left[-5a\,\text{cosec}\,\frac{\pi}{2} + 5a\,\text{cosec}\left(\sin^{-1}\frac{2}{\sqrt{5}}\right) + \frac{4a}{3}\text{cosec}^3\frac{\pi}{2} - \frac{4a}{3}\text{cosec}^3\left(\sin^{-1}\frac{2}{\sqrt{5}}\right)\right]$$

$$= \left[-5a + 5a\frac{\sqrt{5}}{2} + \frac{4a}{3} - \frac{4a}{3}\left(\frac{\sqrt{5}}{2}\right)^3\right] \qquad \left[\because \text{cosec}\left(\sin^{-1}\frac{2}{\sqrt{5}}\right) = \text{cosec}\left(\text{cosec}^{-1}\frac{\sqrt{5}}{2}\right) = \frac{\sqrt{5}}{2}\right]$$

$$= \frac{a}{3}(5\sqrt{5} - 11)$$

Example 14

Evaluate $\iint xy\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right)^{\frac{n}{2}} dx\,dy$ *over the first quadrant of the ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

Solution

Let $\quad x = ar\cos\theta,\ y = br\cos\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr$$

$$dx\,dy = |J|\,dr\,d\theta = abr\,dr\,d\theta$$

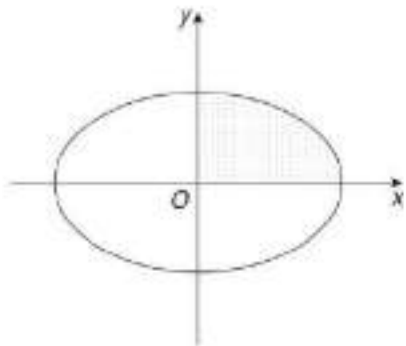

Fig. 9.112

Under the transformation $x = ar\cos\theta$, $y = br\sin\theta$, the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the xy-plane gets transformed to $r^2 = 1$ or $r = 1$, circle with centre (0, 0) and radius 1 in the $r\theta$-plane.

The region of integration is the part of the circle $r = 1$ in first quadrant in the $r\theta$-plane. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = 1$.

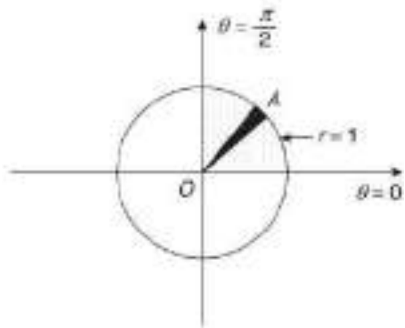

Fig. 9.113

Limits of r : $\ r = 0 \quad$ to $\quad r = 1$

Limits of θ : $\theta = 0 \quad$ to $\quad \theta = \frac{\pi}{2}$

Hence, the polar form of the given integral is

$$I = \iint xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}} dx\,dy$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 abr^2\cos\theta\sin\theta(r^2)^{\frac{n}{2}}\,abr\,dr\,d\theta$$

$$= a^2b^2\int_0^{\frac{\pi}{2}}\frac{\sin 2\theta}{2}\int_0^1 (r)^{n+3}\,dr$$

$$= \frac{a^2b^2}{2}\left|-\frac{\cos 2\theta}{2}\right|_0^{\frac{\pi}{2}}\left|\frac{r^{n+4}}{n+4}\right|_0^1$$

$$= \frac{a^2b^2}{4}(-\cos\pi+\cos 0)\cdot\frac{1}{n+4}$$

$$= \frac{a^2b^2}{2(n+4)}$$

EXERCISE 9.5

Change to polar coordinates and evaluate the following integrals:

1. $\iint \frac{1}{\sqrt{xy}}\,dx\,dy$ over the region bounded by the semicircle $x^2 + y^2 - x = 0$, $y \geq 0$.

$$\left[\textbf{Ans.}: \frac{\pi}{\sqrt{2}}\right]$$

2. $\iint y^2\,dx\,dy$ over the area outside the circle $x^2 + y^2 - ax = 0$ and inside the circle $x^2 + y^2 - 2ax = 0$.

$$\left[\textbf{Ans.}: \frac{15\pi a^4}{64}\right]$$

3. $\iint \sin(x^2 + y^2)\,dx\,dy$ over the circle $x^2 + y^2 = a^2$.

$$\left[\textbf{Ans.}: \pi(1 - \cos a^2)\right]$$

4. $\iint xy(x^2 + y^2)^{\frac{3}{2}}\,dx\,dy$ over the first quadrant of the circle $x^2 + y^2 = a^2$.

$$\left[\textbf{Ans.}: \frac{a^7}{14}\right]$$

5. $\int_0^3 \int_0^{\sqrt{3x}} \frac{dy\,dx}{\sqrt{x^2 + y^2}}$

$$\left[\textbf{Ans.}: \frac{3}{2}\log 3\right]$$

6. $\int_0^a \int_0^x \frac{x^3\,dx\,dy}{\sqrt{x^2 + y^2}}$

$$\left[\textbf{Ans.}: \frac{a^4}{4}\log\left(1 + \sqrt{2}\right)\right]$$

7. $\int_0^a \int_0^{\sqrt{a^2 - x^2}} \sin\left[\frac{\pi}{a^2}(a^2 - x^2 - y^2)\right] dx\,dy$

$$\left[\textbf{Ans.}: \frac{a^2}{2}\right]$$

8. $\int_0^a \int_0^{\sqrt{a^2 - x^2}} e^{-x^2 - y^2}\,dx\,dy$

$$\left[\textbf{Ans.}: \frac{\pi}{4}\left(1 - e^{-a^2}\right)\right]$$

9. $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2+y^2)\,dx\,dy$

$\left[\textbf{Ans.:}\ \frac{3\pi a^4}{4}\right]$

10. $\int_0^1\int_0^{\sqrt{x-x^2}}\frac{4xy}{x^2+y^2}e^{-(x^2+y^2)}\,dx\,dy$

$\left[\textbf{Ans.:}\ \frac{1}{e}\right]$

11. $\int_0^{\frac{a}{\sqrt{2}}}\int_y^{\sqrt{a^2-y^2}}\log e(x^2+y^2)\,dx\,dy$

$\left[\textbf{Ans.:}\ \frac{\pi}{4}a^2\left(\log a-\frac{1}{2}\right)\right]$

12. $\int_0^a\int_y^{a+\sqrt{a^2-y^2}}\frac{dx\,dy}{(4a^2+x^2+y^2)^3}$

$\left[\textbf{Ans.:}\ \frac{1}{8a^2}\left(\frac{\pi}{4}-\frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\right)\right]$

13. $\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}}\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$

$[\textbf{Ans.:}\ a]$

14. $\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}}\frac{xy}{x^2+y^2}e^{-(x^2+y^2)}\,dx\,dy$

$\left[\textbf{Ans.:}\ \frac{1}{4a^2}\left[1-(1+a^2)e^{-a^2}\right]\right]$

15. $\int_0^1\int_{x^2}^{x}\frac{dx\,dy}{\sqrt{x^2+y^2}}$

$\left[\textbf{Ans.:}\ \sqrt{2}-1\right]$

9.5.2 Change of Variables from Cartesian to Other Coordinates

In some cases, evaluation of double integral becomes easier by changing the variables. Let the variables x, y be replaced by new variables u, v by the transformation $x = f_1(u,v), y = f_2(u,v)$, then

$$\iint f(x,y)\mathrm{d}x\,\mathrm{d}y = \iint f(f_1,f_2)\,|J|\,\mathrm{d}u\mathrm{d}v \qquad \text{... (1)}$$

where

$$\text{Jacobian } J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$$

Using Eq. (1), the double integral can be transformed to new variables.

Example 1

Using the transformation $x - y = u, x + y = v$, *evaluate* $\iint \cos\left(\frac{x-y}{x+y}\right) dxdy$ *over the region bounded by the lines* $x = 0, y = 0, x + y = 1.$

Solution

$$x - y = u, x + y = v$$

$$x = \frac{u+v}{2}, y = \frac{v-u}{2}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$dxdy = |J| dudv = \frac{1}{2} dudv$$

The region bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$ in xy-plane is a triangle *OPQ.*

Under the transformation $x = \frac{u+v}{2}$ and $y = \frac{v-u}{2}$,

(i) the line $x = 0$ gets transformed to the line $u = -v$

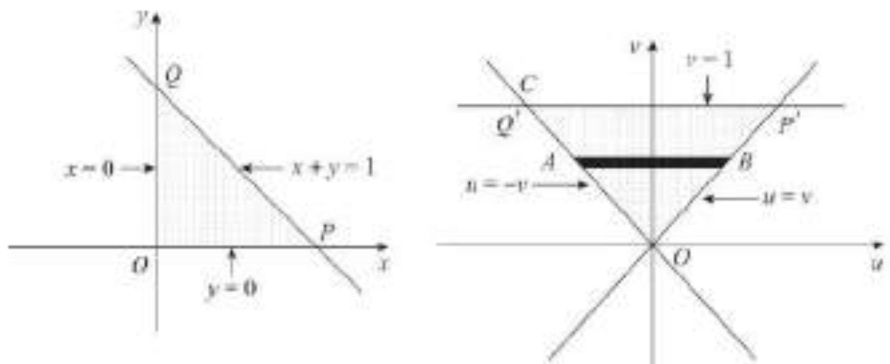

Fig. 9.114

(ii) the line $y = 0$ gets transformed to the line $u = v$

(iii) the line $x + y = 1$ gets transformed to the line $v = 1$

Thus, triangle *OPQ* in xy-plane gets transformed to triangle *OP'Q'* in uv-plane bounded by the lines $u = v$, $u = -v$ and $v = 1$.

In the region, draw a horizontal strip *AB* parallel to u-axis which starts from the line $u = -v$ and terminates on the line $u = v$.

Limits of u : $u = -v$ to $u = v$

Limits of v : $v = 0$ to $v = 1$

$$I = \iint \cos\left(\frac{x-y}{x+y}\right) dx\,dy$$

$$= \int_0^1 \int_{-v}^{v} \cos\left(\frac{u}{v}\right) \frac{1}{2} du\,dv$$

$$= \frac{1}{2} \int_0^1 \left| v \sin\left(\frac{u}{v}\right) \right|_{-v}^{v} dv$$

$$= \frac{1}{2} \int_0^1 v\left[\sin 1 - \sin(-1)\right] dv$$

$$= \frac{1}{2} \cdot 2\sin 1 \left| \frac{v^2}{2} \right|_0^1$$

$$= \frac{1}{2} \sin 1$$

Example 2

Using the transformation $x^2 - y^2 = u$, $2xy = v$, *find* $\iint (x^2 + y^2)\, dx\, dy$ *over the region in the first quadrant bounded by* $x^2 - y^2 = 1$, $x^2 - y^2 = 2$, $xy = 4$, $xy = 2$.

Solution

$$x^2 - y^2 = u,\ 2xy = v$$

It is difficult to express x and y in terms of u and v, therefore we write Jacobian of u, v in terms of x and y.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2)$$

$$du\,dv = |J|\,dx\,dy = 4\left(x^2 + y^2\right) dx\,dy$$

$$dx\,dy = \frac{1}{4\left(x^2 + y^2\right)} du\,dv$$

The region in xy-plane bounded by the curves $x^2 - y^2 = 1$, $x^2 - y^2 = 2$, $xy = 4$, $xy = 2$ is transformed to a square in uv-plane bounded by the lines $u = 1$, $u = 2$, $v = 4$, $v = 8$.

In the region, draw a vertical strip AB parallel to the v-axis which starts from the line $v = 4$ and terminates on the line $v = 8$.

$$I = \iint (x^2 + y^2)\, dx\, dy$$

$$= \int_1^2 \int_4^8 (x^2 + y^2) \frac{1}{4(x^2 + y^2)} \, du \, dv$$

$$= \frac{1}{4} \left| u \right|_1^2 \left| v \right|_4^8$$

$$= 1$$

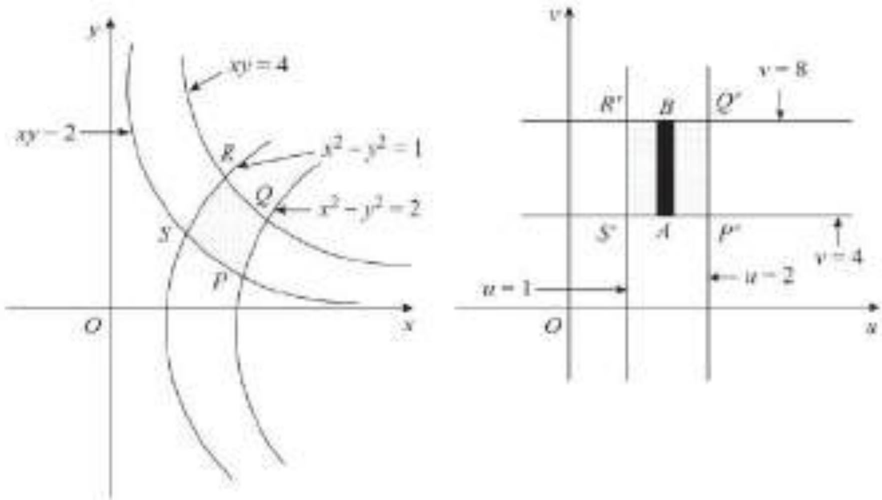

Fig. 9.115

Example 3

Evaluate $\iint_R (x^2 + y^2)\, dA$ *by changing the variables, where R is the region lying in the first quadrant and bounded by the hyperbola* $x^2 - y^2 = 1$, $x^2 - y^2 = 9$, $xy = 2$ *and* $xy = 4$. **[Summer 2014]**

Solution

Let $\quad u = x^2 - y^2 \qquad v = xy$

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix}$$

$$= 2x^2 + 2y^2 = 2(x^2 + y^2)$$

$$du\, dv = |J|\, dx\, dy = 2(x^2 + y^2)\, dx\, dy$$

$$dx\, dy = \frac{1}{2(x^2 + y^2)} \cdot du\, dv$$

The region in the xy-plane bounded by the curves $x^2 - y^2 - 1$, $x^2 - y^2 = 9$, $xy = 2$ and $xy = 4$ is transformed to a square in the uv-plane bounded by the lines $u = 1$, $u = 9$, $v = 2$, $v = 4$.

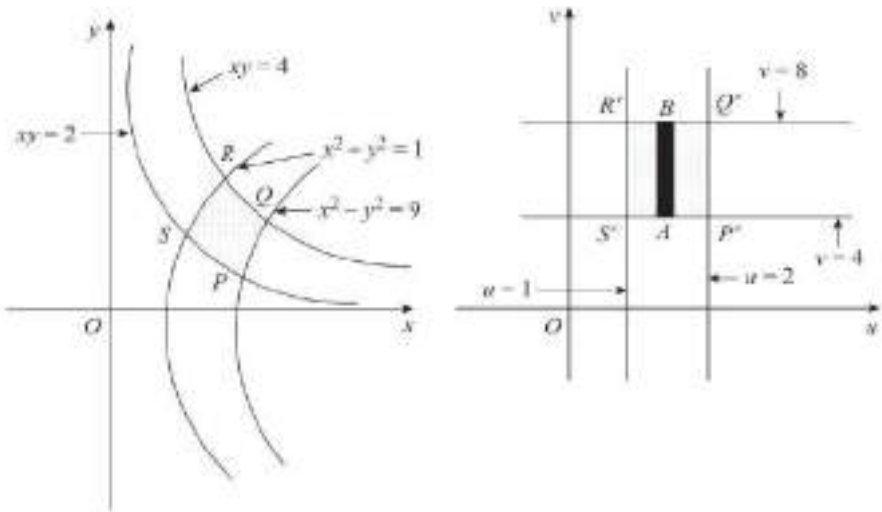

Fig. 9.116

$$
\begin{aligned}
I &= \iint_R (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_R (x^2 + y^2)\cdot \frac{1}{2(x^2 + y^2)}\,\mathrm{d}u\,\mathrm{d}v \\
&= \frac{1}{2}\iint \mathrm{d}u\,\mathrm{d}v \\
&= \frac{1}{2}\int_2^4\int_1^9 \mathrm{d}u\,\mathrm{d}v \\
&= \frac{1}{2}\left| v \right|_2^4 \quad \left| u \right|_1^9 \\
&= \frac{1}{2}(4-2)(9-1) \\
&= \frac{1}{2}(2)(8) \\
&= 8
\end{aligned}
$$

Example 4

Using the transformation $x + y = u$, $y = uv$, show that

$$\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{2}\,(e-1).$$

[Winter 2014]

Solution

$$x + y = u, y = uv$$

$$x = u(1-v), y = uv$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = (1-v)u + uv = u$$

$$\mathrm{d}x\mathrm{d}y = |J|\,\mathrm{d}u\mathrm{d}v = u\mathrm{d}u\,\mathrm{d}v$$

Limits of y : $y = 0$ to $y = 1 - x$

Limits of x : $x = 0$ to $x = 1$.

The region in xy-plane is the triangle OPQ bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$.
Under the transformation $x = u\,(1 - v)$ and $y = uv$,

(i) the line $x = 0$ gets transformed to the line $u = 0$ or $v = 1$
(ii) the line $y = 0$ gets transformed to the line $u = 0$ or $v = 0$
(iii) the line $x + y = 1$ gets transformed to the line $u = 1$

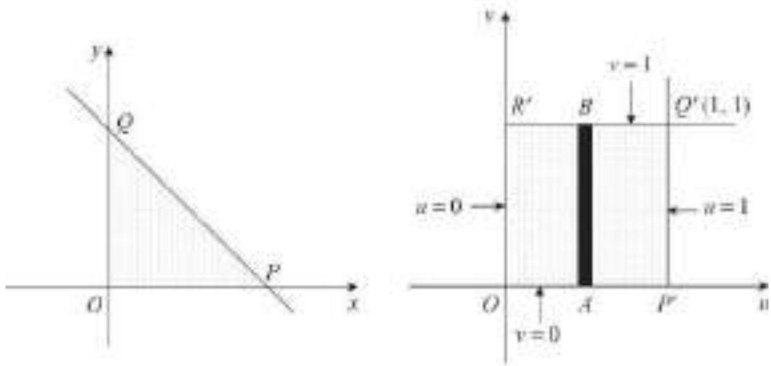

Fig. 9.117

Thus, the triangle OPQ in the xy-plane gets transformed to the square $OP'Q'R'$ in uv-plane bounded by the lines $u = 0$, $v = 0$, $u = 1$ and $v = 1$.
In the region, draw a vertical strip AB parallel to the v-axis which starts from the u-axis and terminates on the line $v = 1$.

Limits of v : $v = 0$ to $v = 1$

Limits of u : $u = 0$ to $u = 1$

$$I = \int_0^1 \int_0^{1-x} e^{\frac{y}{x+y}} \mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 \int_0^1 e^v u\mathrm{d}u\,\mathrm{d}v$$

$$= \left| e^v \right|_0^1 \left| \frac{u^2}{2} \right|_0^1$$

$$= (e^1 - e^0) \cdot \frac{1}{2}$$

$$= \frac{1}{2}(e - 1)$$

Example 5

Evaluate $\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy$ *by applying the transformations* $u = \frac{2x-y}{2},\ v = \frac{y}{2}$. *Draw both regions.* **[Winter 2015]**

Solution

1. The function is integrated first w.r.t. x.
2. Limits of x : $x = \frac{y}{2}$ to $x = \frac{y}{2}+1$

   Limits of y : $y = 0$ to $y = 4$.
3. The region is the parallelogram bounded by the lines $x = \frac{y}{2}, x = \frac{y}{2}+1,\ y = 0$ and $y = 4$ in xy-plane.

   Applying the transformations

$$u = \frac{2x-y}{2}, \qquad v = \frac{y}{2}$$

$$= x - \frac{y}{2}$$

(i) the line $x - \frac{y}{2} = 0$ mapped to the line $u = 0$

(ii) the line $x - \frac{y}{2} = 1$ mapped to the line $u = 1$

(iii) the line $y = 0$ mapped to the line $v = 0$

(iv) the line $y = 4$ mapped to the line $v = 2$

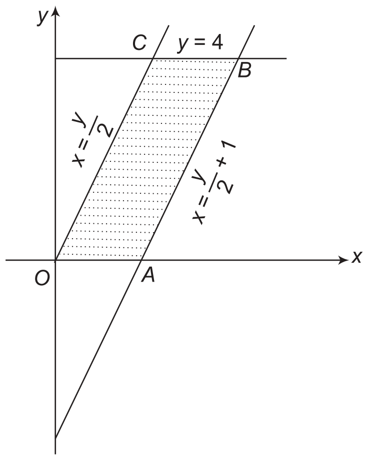

Fig. 9.118

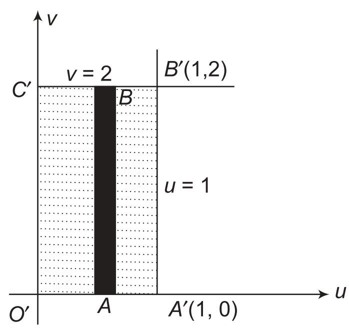

Fig. 9.119

Hence, the parallelogram $OABC$ in the xy-plane mapped to the rectangle $O'A'B'C'$ in uv-plane, bounded by the lines $u = 0$, $u = 1$, $v = 0$ and $v = 2$. In the region, draw a vertical strip AB parallel to the v-axis which starts from the u-axis and terminates on the line $v = 2$.

Limits of u : $u = 0$ to $u = 1$

Limits of v : $v = 0$ to $v = 2$

$$dx\,dy = |J|\,du\,dv$$

where Jacobian, $J = \frac{\partial(x, y)}{\partial(u, v)}$

$$J^* = \frac{1}{J} = \frac{\partial(u, v)}{\partial(x, y)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{2}$$

$$\therefore \ \mathrm{d}x\mathrm{d}y = \frac{1}{2}\mathrm{d}u\mathrm{d}v$$

Hence, the new form of the integral is

$$I = \int_0^4 \int_{\frac{y}{2}}^{y+1} \frac{2x-y}{2}\mathrm{d}x\mathrm{d}y$$

$$= \int_{v=0}^{2} \int_{u=0}^{1} u \cdot \frac{1}{2}\mathrm{d}u\,\mathrm{d}v$$

$$= \frac{1}{2}\int_0^2 \left|\frac{u^2}{2}\right|_0^1 \mathrm{d}v$$

$$= \frac{1}{4}\left|v\right|_0^2$$

$$= \frac{1}{4}(2)$$

$$= 2$$

Example 6

Using the transformation $x = u(1+v), y = v(1+u), u \geq 0, v \geq 0,$

evaluate $\int_0^2 \int_0^y \left[(x-y)^2 + 2(x+y) + 1\right]^{-\frac{1}{2}}\mathrm{d}y\mathrm{d}x.$

Solution

$$x = u(1+v), y = v(1+u)$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = 1+u+v$$

$$\mathrm{d}x\mathrm{d}v = |J|\,\mathrm{d}u\mathrm{d}v = (1+u+v)\,\mathrm{d}u\mathrm{d}v$$

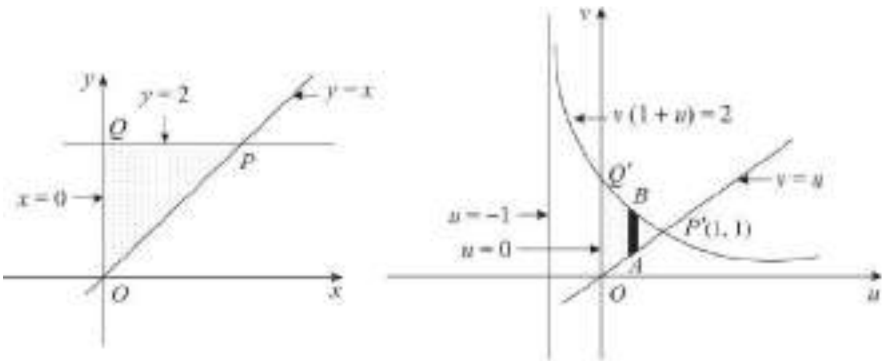

Fig. 9.120

Limits of x : $x = 0$ to $x = y$

Limits of y : $y = 0$ to $y = 2$.

The region in the xy-plane is the ΔOPQ bounded by the lines $x = 0$, $y = 2$ and $y = x$.

Under the transformation $x = u(1+v), y = v(1+u), u \ge 0, v \ge 0$

(i) the line $x = 0$ gets transformed to the line $u = 0$

(ii) the line $y = 2$ gets transformed to the curve $v\,(1 + u) = 2$

(iii) the line $y = x$ gets transformed to the line $u = v$

Thus, the triangle OPQ in the xy-plane gets transformed to the region $OP'Q'$ in uv plane bounded by the lines $u = 0$, $u = v$ and the curve $v\,(1 + u) = 2$.

The point of intersection of $u = v$ and $v\,(1 + u) = 2$ is obtained as $u^2 + u - 2 = 0$, $u = 1, -2$ and $v = 1, -2$.

The point of intersection is $P'\,(1,1)$.

In the region, draw a vertical strip AB parallel to the v-axis which starts from the line $u = v$ and terminates on the curve $v\,(1 + u) = 2$.

Limits of v : $v = u$ to $v = \dfrac{2}{1+u}$

Limits of u : $u = 0$ to $u = 1$

$$I = \int_0^2 \int_0^y \left[(x-y)^2 + 2(x+y) + 1\right]^{-\frac{1}{2}} \mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^1 \int_u^{\frac{2}{1+u}} \left[(u-v)^2 + 2(u+v+2uv) + 1\right]^{-\frac{1}{2}} (1+u+v)\,\mathrm{d}u\,\mathrm{d}v$$

$$= \int_0^1 \int_u^{\frac{2}{1+u}} (1+u+v)^{-1}(1+u+v)\,\mathrm{d}v\,\mathrm{d}u$$

$$= \int_0^1 \int_u^{\frac{2}{1+u}} \mathrm{d}v\,\mathrm{d}u$$

$$= \int_0^1 \left| v \right|_u^{\frac{2}{1+u}} \mathrm{d}u$$

$$= \int_0^1 \left(\frac{2}{1+u} - u \right) \mathrm{d}u$$

$$= \left| 2\log(1+u) - \frac{u^2}{2} \right|_0^1$$

$$= 2\log 2 - \frac{1}{2}$$

Example 7

Evaluate $\iint xy\,dxdy$ *by changing the variables over the region in the first quadrant bounded by the hyperbolas* $x^2 - y^2 = a^2$, $x^2 - y^2 = b^2$ *and the circles* $x^2 + y^2 = c^2$, $x^2 + y^2 = d^2$ *with* $0 < a < b < c < d$.

Solution

Let $\quad x^2 - y^2 = u, x^2 + y^2 = v$

$$x^2 = \frac{u+v}{2}, y^2 = \frac{v-u}{2}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{4x} & \frac{1}{4x} \\ -\frac{1}{4y} & \frac{1}{4y} \end{vmatrix} = \frac{1}{8xy}$$

$$dxdy = |J|\,dudv = \frac{1}{8xy}dudv$$

$$xydxdy = \frac{dudv}{8}$$

The region bounded by the hyperbolas $x^2 - y^2 = a^2$, $x^2 - y^2 = b^2$ and the circles $x^2 + y^2 = c^2$, $x^2 + y^2 = d^2$ in xy-plane is the curvilinear rectangle $PQRS$.

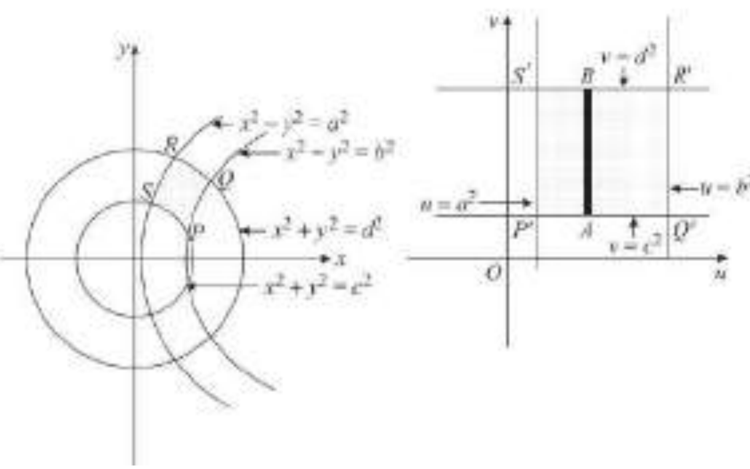

Fig. 9.121

Under the transformation $x^2 - y^2 = u$ and $x^2 + y^2 = v$,

(i) the hyperbolas $x^2 - y^2 = a^2$, $x^2 - y^2 = b^2$ get transformed to the lines $u = a^2$, $u = b^2$ respectively.

(ii) the circles $x^2 + y^2 = c^2$, $x^2 + y^2 = d^2$ get transformed to the lines $v = c^2$, $v = d^2$ respectively.

Thus, the curvilinear rectangle *PQRS* in the *xy*-plane gets transformed to the rectangle *P'Q'R'S'* in *uv*-plane bounded by the lines $u = a^2$, $u = b^2$, $v = c^2$ and $v = d^2$.
In the region, draw a vertical strip *AB* parallel to *v*-axis which starts from the line $v = c^2$ and terminates on the line $v = d^2$.

Limits of v : $v = c^2$ to $v = d^2$

Limits of u : $u = a^2$ to $u = b^2$

$$I = \iint xy\,dx\,dy$$
$$= \int_{u=a^2}^{b^2}\int_{v=c^2}^{d^2} \frac{1}{8}\,du\,dv$$
$$= \frac{1}{8}\left|u\right|_{a^2}^{b^2}\left|v\right|_{c^2}^{d^2}$$
$$= \frac{1}{8}(b^2 - a^2)(d^2 - c^2)$$

Example 8

Evaluate $\iint (x+y)^2\,dx\,dy$, *by changing the variables over the parallelogram with vertices* (1, 0), (3, 1), (2, 2), (0, 1).

Solution

The region of integration in *xy*-plane is the parallelogram PQRS.
Equations of the sides of the parallelogram are obtained as

(i) $PQ : y - 0 = \frac{1-0}{3-1}(x-1)$

$$2y = x - 1$$
$$x - 2y = 1$$

(ii) $RS : y - 1 = \frac{2-1}{2-0}(x-0)$

$$2y - 2 = x$$
$$x - 2y = -2$$

(iii) $PS : y - 0 = \frac{1-0}{0-1}(x-1)$

$$x + y = 1$$

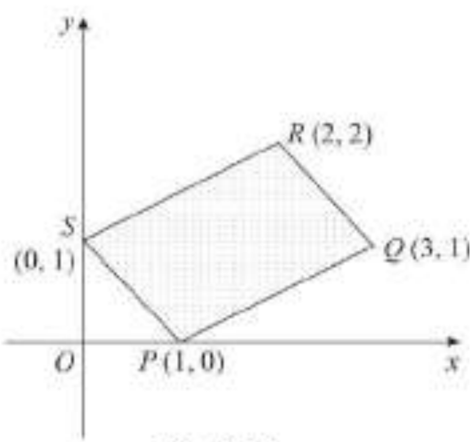

Fig. 9.122

(iv) $QR: y-1=\frac{2-1}{2-3}(x-3)$

$$y-1=-x+3$$
$$x+y=4$$

Let $x-2y=u, x+y=v$

$$x=\frac{u+2v}{3}, y=\frac{v-u}{3}$$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{3} & \frac{2}{3}\\ -\frac{1}{3} & \frac{1}{3}\end{vmatrix}=\frac{1}{3}$$

$$\mathrm{d}x\mathrm{d}y=|J|\,\mathrm{d}u\mathrm{d}v=\frac{1}{3}\mathrm{d}u\mathrm{d}v$$

Under the transformation $x-2y=u$, and $x+y=v$

(i) the lines $x-2y=1$, $x-2y=-2$ get transformed to the lines $u=1$, $u=-2$ respectively.

(ii) the lines $x+y=1$, $x+y=4$ get transformed to the lines $v=1$, $v=4$ respectively

Thus, the parallelogram $PQRS$ in the xy-plane gets transformed to a square $P'Q'R'S'$ in uv-plane bounded by the lines $u=1$, $u=-2$, $v=1$ and $v=4$.

In the region, draw a vertical strip AB parallel to v-axis which starts from the line $v=1$ and terminates on the line $v=4$.

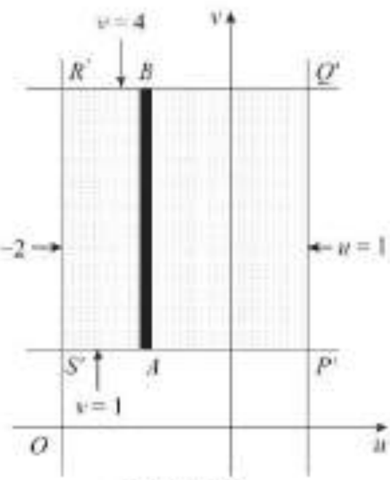

Fig. 9.123

Limits of v : $v=1$ to $v=4$

Limits of u : $u=-2$ to $u=1$

$$I=\iint (x+y)^2\,\mathrm{d}x\mathrm{d}y$$
$$=\int_{u=2}^{1}\int_{v=1}^{4} v^2\,\frac{1}{3}\mathrm{d}u\mathrm{d}v$$
$$=\frac{1}{3}\left|u\right|_{-2}^{1}\left|\frac{v^3}{3}\right|_{1}^{4}$$
$$=21$$

Example 9

Evaluate $\iint xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}}$ *over the first quadrant of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Solution

Let $x = ar\cos\theta$, $y = br\sin\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr$$

$$\mathrm{d}x\,\mathrm{d}y = |J|\,\mathrm{d}r\,\mathrm{d}\theta = abr\,\mathrm{d}r\,\mathrm{d}\theta$$

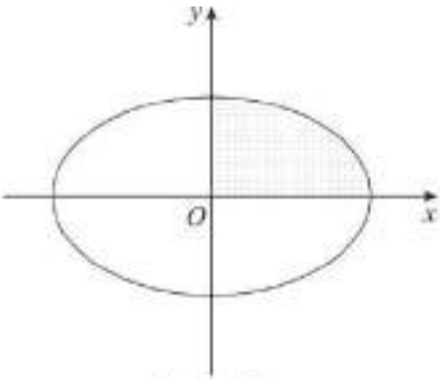

Fig. 9.124

Under the transformation $x = ar\cos\theta$, $y = br\sin\theta$, the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ in the xy-plane gets transformed to $r^2 = 1$ or $r = 1$, circle with centre (0, 0) and radius 1 in the $r\theta$-plane.

The region of integration is the part of the circle $r = 1$ in first quadrant in the $r\theta$-plane. In the region, draw an elementary radius vector OA from the pole which terminates on the circle $r = 1$.

Limits of r : $r = 0$ to $r = 1$

Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{2}$

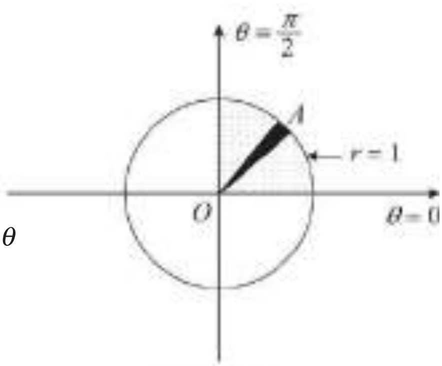

Fig. 9.125

$$I = \iint xy\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{\frac{n}{2}} \mathrm{d}x\mathrm{d}y$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 abr^2\cos\theta\sin\theta(r^2)^{\frac{n}{2}}\,abr\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= a^2b^2\int_0^{\frac{\pi}{2}}\frac{\sin 2\theta}{2}\int_0^1 (r)^{n+3}\mathrm{d}r$$

$$= \frac{a^2b^2}{2}\left|-\frac{\cos 2\theta}{2}\right|_0^{\frac{\pi}{2}}\left|\frac{r^{n+4}}{n+4}\right|_0^1$$

$$= \frac{a^2b^2}{4}(-\cos\pi + \cos 0)\cdot\frac{1}{n+4}$$

$$= \frac{a^2b^2}{2(n+4)}$$

EXERCISE 9.6

1. Using the transformation $x + y = u$, $x - y = v$, evaluate $\iint e^{\frac{x-y}{x+y}} dxdy$ over the region bounded by $x = 0$, $y = 0$ and $x + y = 1$.

$$\left[\textbf{Ans.: } \frac{1}{4}\left(e - \frac{1}{e}\right)\right]$$

2. Using the transformation $x^2 - y^2 = u$, $2xy = v$, evaluate $\iint (x^2 - y^2) dxdy$ over the region bounded by the hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = 9$, $xy = 2$ and $xy = 4$.

[Ans.: 4]

3. Using the transformation $x + y = u$, $y = uv$, evaluate
$\int_0^\infty \int_0^\infty e^{-(x+y)} x^{p-1} y^{q-1} dx\, dy$.

$$\left[\textbf{Ans.: } \Gamma p \, \Gamma q\right]$$

4. Using the transformation $x = u$, $y = uv$, evaluate $\int_0^1 \int_0^x \sqrt{x^2 + y^2}\, dx\, dy$.

$$\left[\textbf{Ans.: } \frac{1}{3}\left[\frac{\sqrt{2}}{2} + \frac{1}{2}\log\left(1 + \sqrt{2}\right)\right]\right]$$

5. Evaluate $\iint (x + y)^2 dx\, dy$ by changing the variables over the region bounded by the parallelogram with sides $x + y = 0$, $x + y = 2$, $3x - 2y = 0$ and $3x - 2y = 3$.

$$\left[\textbf{Ans. : } \frac{8}{5}\right]$$

6. Evaluate $\iint (x - y)^4 e^{x+y} dx\, dy$, by changing the variables over the region bounded by the square with vertices at (1, 0), (2, 1), (1, 2), (0, 1).

$$\left[\textbf{Ans. : } \frac{e^3 - e}{5}\right]$$

7. Evaluate $\iint [xy(1 - x - y)]^{\frac{1}{2}} dx\, dy$, by changing the variables over the region bounded by the triangle with sides $x = 0$, $y = 0$, $x + y = 1$.

$$\left[\textbf{Ans.: } \frac{2\pi}{105}\right]$$

9.6 TRIPLE INTEGRALS

Let $f(x, y, z)$ be a continuous function defined in a closed and bounded region V in 3-dimensional space. Divide the region V into small elementary parallelopipeds by drawing planes parallel to the coordinate planes. Let the total number of complete parallelopipeds which lie inside the region V be n. Let δV_r be the volume of the r^{th} parallelopiped and (x_r, y_r, z_r) be any point in this parallelopiped. Consider the sum

$$S = \sum_{r=1}^{n} f(x_r, y_r, z_r)\delta V_r \qquad \ldots(1)$$

where, $\delta V_r = \delta x_r \cdot \delta y_r \cdot \delta z_r$

If we increase the number of elementary parallelopipeds, n, then the volume of each parallelopiped decreases. Hence as $n \to \infty, \delta V_r \to 0$.

The limit of the sum given by Eq. (1), if it exists is called the triple integral of $f(x, y, z)$ over the region V and is denoted by $\iiint_V f(x, y, z)\,\mathrm{d}V$

Hence, $$\iiint_V f(x, y, z)\,\mathrm{d}V = \lim_{\substack{n \to \infty \\ \delta V_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r, z_r)\delta V_r$$

where $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

9.6.1 Triple Integrals in Cartesian Coordinates

Triple integral of a continuous function $f(x, y, z)$ over a region V can be evaluated by three successive integrations.

Let the region V be bounded below by a surface $z = z_1(x, y)$ and above by a surface $z = z_2(x, y)$. Let the projection of region V in xy-plane be R which be bounded by the curves $y = y_1(x)$, $y = y_2(x)$ and $x = a$, $x = b$. Then the triple integral is defined as

$$I = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} \left\{ \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\,\mathrm{d}z \right\} \mathrm{d}y \right] \mathrm{d}x$$

Note: The order of variables in $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ indicates the order of integration. In some cases this order is not maintained. Therefore, it is advisable to identify the order of integration with the help of the limits.

9.6.2 Triple Integrals in Cylindrical Coordinates

Cylindrical coordinates r, θ, z are used to evaluate the integral in the regions which are bounded by cylinders along z-axis, planes through z-axis, planes perpendicular to the z-axis.

Relations between Cartesian (rectangular) coordinates (x, y, z) and cylindrical coordinates (r, θ, ϕ) are given as $x = r\cos\theta$

$$y = r\sin\theta$$

$$z = z$$

Then $\iiint f(x, y, z)\,dx\,dy\,dz$

$$= \iiint f(r\cos\theta, r\sin\theta, z)\,|J|\,dz\,dr\,d\theta$$

where, $J = \dfrac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos\theta(r\cos\theta) + r\sin\theta(\sin\theta)$$

$$= r$$

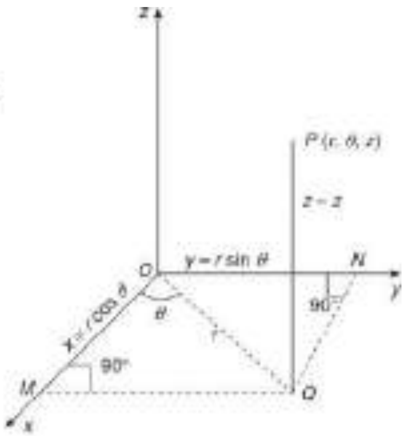

Fig. 9.126

Hence, $\iiint f(x, y, z)\,dx\,dy\,dz = \iiint f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$

9.6.3 Triple Integrals in Spherical Coordinates

Spherical coordinates (r, θ, ϕ) are used to evaluate the integral in the regions which are bounded by the sphere with centre at the origin.

Relations between cartesian (rectangular) coordinates (x, y, z) and spherical coordinates (r, θ, ϕ) are given as

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

Then

$$\iiint f(x, y, z)\,dx\,dy\,dz$$

$$= \iiint f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\,|J|\,dr\,d\theta\,d\phi$$

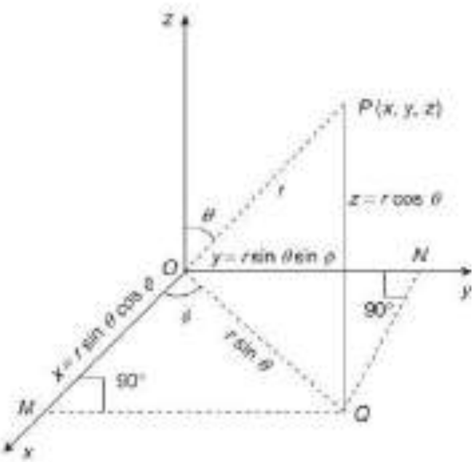

Fig. 9.127

where
$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$
$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$
$$= \sin\theta\cos\phi(r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi(-r\sin\theta\cos\phi\cos\theta)$$
$$- r\sin\theta\sin\phi(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi)$$
$$= r^2\sin\theta\cos^2\phi(\sin^2\theta + \cos^2\theta) + r^2\sin\theta\sin^2\phi$$
$$= r^2\sin\theta$$

Hence, $\iiint f(x, y, z)\,dx\,dy\,dz = \iiint f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)r^2\sin\theta\,dr\,d\theta\,d\phi$

***Note*:** If the region of integration is a sphere $x^2 + y^2 + z^2 = a^2$ with centre at (0, 0, 0) and radius a, then limits of r, θ, ϕ are

(i) For positive octant of the sphere,
$$r: r = 0 \quad \text{to} \quad r = a$$
$$\theta: \theta = 0 \quad \text{to} \quad \theta = \frac{\pi}{2}$$
$$\phi: \phi = 0 \quad \text{to} \quad \phi = \frac{\pi}{2}$$

(ii) For hemisphere,
$$r: r = 0 \quad \text{to} \quad r = a$$
$$\theta: \theta = 0 \quad \text{to} \quad \theta = \frac{\pi}{2}$$
$$\phi: \phi = 0 \quad \text{to} \quad \phi = 2\pi$$

(iii) For complete sphere,
$$r: r = 0 \quad \text{to} \quad r = a$$
$$\theta: \theta = 0 \quad \text{to} \quad \theta = \pi$$
$$\phi: \phi = 0 \quad \text{to} \quad \phi = 2\pi$$

9.6.4 Change of Variables

In some cases, evaluation of a triple integral becomes easier by changing the variables. Let the variables x, y, z be replaced by new variables u, v, w by the transformation $x = f_1(u, v, w)$, $y = f_2(u, v, w)$, $z = f_3(u, v, w)$.

Then $$\iiint f(x, y, z)\,dx\,dy\,dz = \iiint f(f_1, f_2, f_3)\,|J|\,du\,dv\,dw$$

where, $$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

9.6.5 Working Rule for Evaluation of Triple Integrals

1. Draw all the planes and surfaces and identify the region of integration.
2. Draw an elementary volume parallel to z (y or x) axis.
3. Find the variation of z (y or x) along the elementary volume.
4. Lower and upper limits of z (y or x) are obtained from the equation of the surface (or plane) where elementary volume starts and terminates respectively.
5. Find the projection of the region on xy (zx or yz) plane.
6. Draw the region of projection in xy (zx or yz) plane.
7. Follow the steps of double integration to find the limits of x and y (z and x or y and z).

Note: (1) If the region is bounded by the cylinders along the z-axis, planes through z-axis, the planes perpendicular to the z-axis, then the cylindrical coordinates are used.
(2) If the region is bounded by the sphere, then the spherical coordinates are used.

Type I Evaluation of Triple Integrals when Limits are Given

Example 1

Evaluate $\int_0^1\int_0^2\int_0^e dy\,dx\,dz$.

Solution

$$\int_0^1\int_0^2\int_0^e dy\,dx\,dz = \int_0^1\int_0^2\left[\int_0^e dy\right]dx\,dz$$

$$= \int_0^1\int_0^2 |y|_0^e\,dx\,dz$$

$$= \int_0^1\left[\int_0^2 e\,dx\right]dz$$

$$= e\int_0^1 |x|_0^2\,dz$$

$$= e\int_0^1 2\,dz$$

$$= 2e|z|_0^1$$
$$= 2e$$

Another method

Since all the limits are constant and integrand (function) is explicit in x, y and z, the integral can be written as

$$\int_0^1\int_0^2\int_0^e dy\,dx\,dz = \int_0^1 dz \cdot \int_0^2 dx \cdot \int_0^e dy$$
$$= |z|_0^1 \cdot |x|_0^2 \cdot |y|_0^e$$
$$= 1 \cdot 2 \cdot e$$
$$= 2e.$$

Example 2

Evaluate $\int_0^2\int_1^3\int_1^2 xy^2z\,dz\,dy\,dx.$

Solution

Since all the limits are constant and integrand (function) is explicit in x, y and z, the integral can be written as

$$\int_0^2\int_1^3\int_1^2 xy^2z\,dz\,dy\,dx = \int_0^2 x\,dx \cdot \int_1^3 y^2dy \cdot \int_1^2 z\,dz$$
$$= \left|\frac{x^2}{2}\right|_0^2 \cdot \left|\frac{y^3}{3}\right|_1^3 \cdot \left|\frac{z^2}{2}\right|_1^2$$
$$= 2 \cdot \frac{26}{3} \cdot \frac{3}{2}$$
$$= 26$$

Example 3

Evaluate $\int_0^1\int_0^\pi\int_0^\pi y\sin z\,dx\,dy\,dz.$ **[Winter 2013]**

Solution

Since all the limits are constant and integrand (function) is explicit in x, y, and z, the integral can be written as

$$\int_0^1\int_0^\pi\int_0^\pi y\sin z\,dx\,dy\,dz = \int_0^1 \sin z\,dz \int_0^\pi y\,dy \int_0^\pi dx$$
$$= |-\cos z|_0^1 \cdot \left|\frac{y^2}{2}\right|_0^\pi \cdot |x|_0^\pi$$

$$= (-\cos 1 + \cos 0)\left(\frac{\pi^2}{2}\right)(\pi)$$

$$= \frac{\pi^3}{2}(1-\cos 1)$$

Example 4

Evaluate $\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} (r^2\cos^2\theta + z^2) r\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}z.$ **[Winter 2016]**

Solution

$$\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} (r^2\cos^2\theta + z^2) r\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}z = \int_0^1 \int_0^{\sqrt{z}} \left\{\int_0^{2\pi} [r^2\cos^2\theta + z^2]\,\mathrm{d}\theta\right\} r\,\mathrm{d}r\,\mathrm{d}z$$

$$= \int_0^1 \int_0^{\sqrt{z}} \left\{\int_0^{2\pi} \left[r^2\left(\frac{1+\cos 2\theta}{2}\right) + z^2\right]\mathrm{d}\theta\right\} r\,\mathrm{d}r\,\mathrm{d}z$$

$$= \int_0^1 \int_0^{\sqrt{z}} \left\{ r\left[\frac{1}{2}r^2\left(\theta + \frac{\sin 2\theta}{2}\right) + z^2\theta\right]_0^{2\pi}\right\}\mathrm{d}r\,\mathrm{d}z$$

$$= 2\pi \int_0^1 \int_0^{\sqrt{z}} \left[\frac{1}{2}r^3 + z^2 r\right]\mathrm{d}r\,\mathrm{d}z$$

$$= 2\pi \int_0^1 \left| \frac{r^4}{8} + z^2\frac{r^2}{2} \right|_0^{\sqrt{2}} \mathrm{d}z$$

$$= 2\pi \int_0^1 \left| \frac{z^2}{8} + \frac{z^3}{2} \right| \mathrm{d}z$$

$$= 2\pi \left[\frac{z^3}{24} + \frac{z^4}{8}\right]_0^1$$

$$= 2\pi \left[\frac{1}{24} + \frac{1}{8}\right]$$

$$= 2\pi \left[\frac{3+1}{24}\right]$$

$$= 2\pi \cdot \frac{4}{24}$$
$$= \frac{\pi}{3}$$

Example 5

Evaluate $\int_0^1 \int_0^{1-x} \int_0^{x+y} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$

Solution

$$\int_0^1 \int_0^{1-x} \int_0^{x+y} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^1 \int_0^{1-x} \left[\int_0^{x+y} \mathrm{d}z \right] \mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^1 \int_0^{1-x} \left| z \right|_0^{x+y} \mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^1 \int_0^{1-x} (x+y)\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^1 \left| xy + \frac{y^2}{2} \right|_0^{1-x} \mathrm{d}x$$
$$= \int_0^1 \left[x(1-x) + \frac{(1-x)^2}{2} \right] \mathrm{d}x$$
$$= \int_0^1 \left[x - x^2 + \frac{(1-x)^2}{2} \right] \mathrm{d}x$$
$$= \left| \frac{x^2}{2} - \frac{x^3}{3} + \frac{1}{2} \cdot \frac{(1-x)^3}{(-3)} \right|_0^1$$
$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{6}$$
$$= \frac{1}{3}$$

Example 6

Evaluate $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$

Solution

$$\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^1 \int_0^{1-x} \left[\int_0^{x+y} e^z\,\mathrm{d}z \right] \mathrm{d}y\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_0^1 \int_0^{1-x} \left| e^z \right|_0^{x+y} \mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \left[\int_0^{1-x} (e^{x+y} - e^0)\mathrm{d}y \right] \mathrm{d}x \\
&= \int_0^1 \left| e^{x+y} - y \right|_0^{1-x} \mathrm{d}x \\
&= \int_0^1 \left[(e - e^x) - (1 - x) \right] \mathrm{d}x \\
&= \left| (e-1)x + \frac{x^2}{2} - e^x \right|_0^1 \\
&= (e-1) + \frac{1}{2} - e + e^0 \\
&= \frac{1}{2}
\end{aligned}
$$

Example 7

Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$

Solution

$$
\begin{aligned}
\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x &= \int_0^a e^x \int_0^x e^y \left[\int_0^{x+y} e^z \mathrm{d}z \right] \mathrm{d}y\,\mathrm{d}x \\
&= \int_0^a e^x \int_0^x e^y \left| e^z \right|_0^{x+y} \mathrm{d}y\,\mathrm{d}x \\
&= \int_0^a e^x \int_0^x e^y \left(e^{x+y} - e^0 \right) \mathrm{d}y\,\mathrm{d}x \\
&= \int_0^a e^x \left[\int_0^x \left(e^x e^{2y} - e^y \right) \mathrm{d}y \right] \mathrm{d}x \\
&= \int_0^a e^x \left| e^x \cdot \frac{e^{2y}}{2} - e^y \right|_0^x \mathrm{d}x \\
&= \int_0^a e^x \left[\left(e^x \cdot \frac{e^{2x}}{2} - e^x \cdot \frac{1}{2} \right) - (e^x - e^0) \right] \mathrm{d}x \\
&= \int_0^a \left(\frac{1}{2} e^{4x} - \frac{3}{2} e^{2x} + e^x \right) \mathrm{d}x \\
&= \left| \frac{1}{2} \cdot \frac{e^{4x}}{4} - \frac{3}{2} \cdot \frac{e^{2x}}{2} + e^x \right|_0^a \\
&= \frac{1}{8}\left(e^{4a} - e^0 \right) - \frac{3}{4}\left(e^{2a} - e^0 \right) + \left(e^a - e^0 \right) \\
&= \frac{1}{8} e^{4a} - \frac{3}{4} e^{2a} + e^a - \frac{3}{8}
\end{aligned}
$$

Example 8

Evaluate the integral $\int_0^2 \int_1^2 \int_0^{yz} xyz \, dx\, dy\, dz.$ **[Summer 2014]**

Solution

$$\int_0^2 \int_1^2 \int_0^{yz} xyz \, dx\, dy\, dz = \int_0^2 \int_1^2 yz\left[\int_0^{yz} x\, dx\right] dy\, dz$$

$$= \int_0^2 \int_1^2 yz \left|\frac{x^2}{2}\right|_0^{yz} dy\, dz$$

$$= \int_0^2 \int_1^2 \frac{1}{2} y^3 z^3 dy\, dz$$

$$= \frac{1}{2}\left|\frac{y^4}{4}\right|_1^2 \left|\frac{z^4}{4}\right|_0^2$$

$$= \frac{1}{2}\left(\frac{1}{4}\right)[16-1]\frac{1}{4}[16-0]$$

$$= \frac{15}{2}$$

Example 9

Evaluate $\int_1^3 \int_1^{\frac{1}{x}} \int_0^{\sqrt{xy}} xy\, dz\, dy\, dx.$

Solution

$$\int_1^3 \int_1^{\frac{1}{x}} \int_0^{\sqrt{xy}} xy\, dz\, dy\, dx = \int_1^3 \int_1^{\frac{1}{x}} \left[\int_0^{\sqrt{xy}} dz\right] xy\, dy\, dx$$

$$= \int_1^3 \int_1^{\frac{1}{x}} |z|_0^{\sqrt{xy}} dy\, dx$$

$$= \int_1^3 \left[\int_1^{\frac{1}{x}} \sqrt{xy}\, dy\right] dx$$

$$= \int_1^3 \sqrt{x} \left|\frac{2y^{\frac{3}{2}}}{3}\right|_1^{\frac{1}{x}} dx$$

$$= \frac{2}{3}\int_1^3 \sqrt{x}\left(\frac{1}{x^{\frac{3}{2}}} - 1\right) dx$$

$$= \frac{2}{3}\int_1^3 \left(\frac{1}{x} - x^{\frac{1}{2}}\right) dx$$

$$= \frac{2}{3}\left| \log x - \frac{2x^{\frac{3}{2}}}{3} \right|_1^3$$

$$= \frac{2}{3}\left[(\log 3 - \log 1) - \frac{2}{3}\left(3^{\frac{3}{2}} - 1\right) \right]$$

$$= \frac{2}{3}\left[\log 3 - 2\sqrt{3} + \frac{2}{3} \right]$$

Example 10

Evaluate $\int_0^2 \int_1^z \int_0^{yz} xyz \, dx \, dy \, dz.$ **[Summer 2017]**

Solution

The innermost limits depend on y and z. Hence, integrating first w.r.t. x,

$$\int_0^2 \int_1^z \int_0^{yz} xyz \, dx \, dy \, dz = \int_0^2 \int_1^z \left| \frac{x^2}{2} \right|_0^{yz} yz \, dy \, dz$$

$$= \frac{1}{2}\int_0^2 \int_1^z (y^2 z^2) yz \, dy \, dz$$

$$= \frac{1}{2}\int_0^2 z^3 \left[\int_1^z y^3 \, dy \right] dz$$

$$= \frac{1}{2}\int_0^2 z^3 \left| \frac{y^4}{4} \right|_1^z dz$$

$$= \frac{1}{8}\int_0^2 z^3 (z^4 - 1) dz$$

$$= \frac{1}{8}\left| \frac{z^8}{8} - \frac{z^4}{4} \right|_0^2$$

$$= \frac{1}{8}(32 - 4)$$

$$= \frac{7}{2}$$

Example 11

Evaluate $\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}\,dx\,dy\,dz.$

Solution

The innermost limits depend on x and y. Hence, integrating first w.r.t. z,

$$\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}\,dx\,dy\,dz$$

$$=\int_0^1\int_0^{1-x}\left[\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}\,dz\right]dy\,dx$$

$$=\int_0^1\int_0^{1-x}\left|\frac{1}{-2(x+y+z+1)^2}\right|_0^{1-x-y}dy\,dx$$

$$=-\frac{1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{\{x+y+(1-x-y)+1\}^2}-\frac{1}{(x+y+1)^2}\right]dy\,dx$$

$$=-\frac{1}{2}\int_0^1\left[\int_0^{1-x}\left\{\frac{1}{4}-\frac{1}{(x+y+1)^2}\right\}dy\right]dx$$

$$=-\frac{1}{2}\int_0^1\left|\frac{y}{4}+\frac{1}{x+y+1}\right|_0^{1-x}dx$$

$$=-\frac{1}{2}\int_0^1\left[\frac{1-x}{4}+\frac{1}{x+(1-x)+1}-\frac{1}{x+1}\right]dx$$

$$=-\frac{1}{2}\int_0^1\left(\frac{1-x}{4}+\frac{1}{2}-\frac{1}{x+1}\right)dx$$

$$=-\frac{1}{2}\left|\frac{x}{4}-\frac{x^2}{8}+\frac{x}{2}-\log(x+1)\right|_0^1$$

$$=-\frac{1}{2}\left(\frac{5}{8}-\log 2\right)$$

Example 12

Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx.$ **[Winter 2014]**

Solution

$$\int_0^1\int_0^{\sqrt{1-x^2}} xy\left[\int_0^{\sqrt{1-x^2-y^2}} z\,dz\right]dx\,dy = \int_0^1\int_0^{\sqrt{1-x^2}} xy\left|\frac{z^2}{z}\right|_0^{\sqrt{1-x^2-y^2}} dx\,dy$$

$$= \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} xy(1-x^2-y^2)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{2}\int_0^1 x\left[\int_0^{\sqrt{1-x^2}}\left[(1-x^2)y-y^3\right]\right]\mathrm{d}y\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^1 x\left|(1-x^2)\frac{y^2}{2}-\frac{y^4}{4}\right|_0^{\sqrt{1-x^2}}\mathrm{d}x$$

$$= \frac{1}{2}\int_0^1 x\left[(1-x^2)\frac{(1-x^2)}{2}-\left(\frac{1-x^2)^2}{4}\right)\right]\mathrm{d}x$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{(1-x^2)^2}{4}\right]\mathrm{d}x$$

$$= \frac{1}{8}\int_0^1 x(1-x^2)^2\,\mathrm{d}x$$

$$= \frac{1}{8}\int_0^1 \{x(1-2x^2+x^4)\}\,\mathrm{d}x$$

$$= \frac{1}{8}\int_0^1 (x-2x^3+x^5)\,\mathrm{d}x$$

$$= \frac{1}{8}\left|\frac{x^2}{2}-\frac{2x^4}{4}+\frac{x^6}{6}\right|_0^1$$

$$= \frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right)$$

$$= \frac{1}{48}$$

Example 13

Evaluate $\int_1^e\int_1^{\log y}\int_1^{e^x} \log z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$ **[Summer 2016]**

Solution

The inner most limit depends on x and middle limit depends on y. Hence, integrating first w.r.t. z,

$$\int_1^e\int_1^{\log y}\int_1^{e^x} \log z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_1^e\int_1^{\log y}\int_1^{e^x} \log z\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_1^e \int_1^{\log y} \left(\left| z \log z \right|_1^{e^x} - \int_1^{e^x} z \cdot \frac{1}{z} \mathrm{d}z \right) \mathrm{d}x\, \mathrm{d}y$$

$$= \int_1^e \int_1^{\log y} \left(e^x \log e^x - \log 1 - \left| z \right|_1^{e^x} \right) \mathrm{d}x\, \mathrm{d}y$$

$$= \int_1^e \left[\int_1^{\log y} \left(e^x x - e^x + 1 \right) \mathrm{d}x \right] \mathrm{d}y$$

$$= \int_1^e \left| xe^x - e^x - e^x + x \right|_1^{\log y} \mathrm{d}y$$

$$= \int_1^e \left[e^{\log y} (\log y - 2) + \log y - e(1-2) - 1 \right] \mathrm{d}y$$

$$= \int_1^e \left[y(\log y - 2) + \log y + e - 1 \right] \mathrm{d}y$$

$$= \int_1^e \left[(y+1) \log y - 2y + e - 1 \right] \mathrm{d}y$$

$$= \left| \log y \left(\frac{y^2}{2} + y \right) \right|_1^e - \int_1^e \frac{1}{y} \left(\frac{y^2}{2} + y \right) \mathrm{d}y - \left| y^2 \right|_1^e + \left| (e-1)y \right|_1^e$$

$$= \log e \left(\frac{e^2}{2} + e \right) - \log 1 \left(\frac{1}{2} + 1 \right) - \left| \frac{y^2}{4} + y \right|_1^e - (e^2 - 1) + \left[(e-1)(e-1) \right]$$

$$= \frac{e^2}{2} + e - \left[\frac{1}{4}(e^2 - 1) + (e - 1) \right] - e^2 + 1 + e^2 - 2e + 1$$

$$= \frac{e^2}{4} - 2e + \frac{13}{4}$$

Example 14

Evaluate $\int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z}{(1 + x^2 + y^2 + z^2)^2}$.

Solution

1. It is difficult to integrate this integral in cartesian form. Putting $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$ integral changes to spherical form.
2. Limits of $x : x = 0$ to $x \to \infty$

   Limits of $y : y = 0$ to $y \to \infty$

   Limits of $z : z = 0$ to $z \to \infty$

The region of integration is the positive octant of the plane.

Limits of $r : r = 0$ to $r \to \infty$

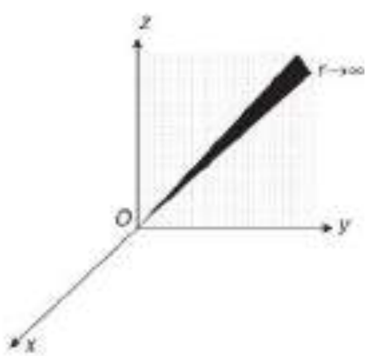

Fig. 9.128

Limits of $\theta = 0$ to $\theta = \frac{\pi}{2}$

Limits of $\phi : \phi = 0$ to $\phi = \frac{\pi}{2}$

Hence, the spherical form of the given integral is

$$I = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{(1+x^2+y^2+z^2)^2}$$

$$= \int_0^{\infty}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi}{(1+r^2)^2}$$

$$= \int_0^{\infty} \frac{r^2\mathrm{d}r}{(1+r^2)^2} \int_0^{\frac{\pi}{2}} \sin\theta\,\mathrm{d}\theta \int_0^{\frac{\pi}{2}} \mathrm{d}\phi$$

Putting $r = \tan t, dr = \sec^2 t \,\mathrm{d}t$

When $r = 0, t = 0$

When $r \to \infty, t = \frac{\pi}{2}$

$$I = \int_0^{\frac{\pi}{2}} \sin\theta\mathrm{d}\theta \int_0^{\frac{\pi}{2}} \frac{\tan^2 t}{\sec^4 t}\sec^2 t\,\mathrm{d}t \int_0^{\frac{\pi}{2}} \mathrm{d}\phi$$

$$= \left|-\cos\theta\right|_0^{\frac{\pi}{2}} \left(\int_0^{\frac{\pi}{2}} \sin^2 t\,\mathrm{d}t\right) |\phi|_0^{\frac{\pi}{2}}$$

$$= \left(-\cos\frac{\pi}{2} + \cos 0\right)\cdot\left(\int_0^{\frac{\pi}{2}} \frac{1-\cos 2t}{2}\mathrm{d}t\right)\cdot\frac{\pi}{2}$$

$$= \frac{1}{2}\left|t - \frac{\sin 2t}{2}\right|_0^{\frac{\pi}{2}} \frac{\pi}{2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{2}(\sin\pi - \sin 0)\right]\frac{\pi}{2}$$

$$= \frac{\pi^2}{8}$$

Example 15

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$

Solution

1. It is difficult to integrate this integral in cartesian form. Putting $x = r \sin\theta \cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$ integral changes to spherical form.

2. Limits of $z: z=0$ to $z=\sqrt{a^2-x^2-y^2}$

   Limits of $y: y=0$ to $y=\sqrt{a^2-x^2}$

   Limits of $x: x=0$ to $x=a$

The region of integration is the positive octant of the sphere $x^2+y^2+z^2=a^2$.

   Limits of $r: r=0$ to $r=a$

   Limits of $\theta: \theta=0$ to $\theta=\frac{\pi}{2}$

   Limits of $\phi: \phi=0$ to $\phi=\frac{\pi}{2}$

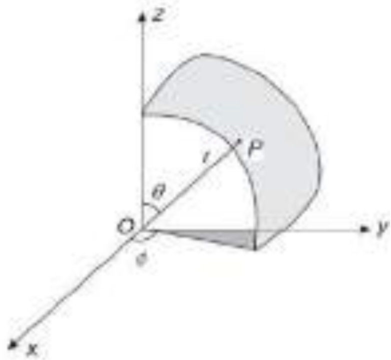

Fig. 9.129

Hence, the spherical form of the given integral is

$$I=\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} xyz\,dx\,dy\,dz$$

$$=\int_{\phi=0}^{\frac{\pi}{2}}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a} r^3\sin^2\theta\cos\theta\cdot\cos\phi\sin\phi\cdot r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin 2\phi}{2}\,d\phi\int_0^{\frac{\pi}{2}}\sin^3\theta\cos\theta\,d\theta\int_0^a r^5 dr$$

$$=\frac{1}{2}\left|\frac{-\cos 2\phi}{2}\right|_0^{\frac{\pi}{2}}\cdot\left|\frac{\sin^4\theta}{4}\right|_0^{\frac{\pi}{2}}\cdot\left|\frac{r^6}{6}\right|_0^a \qquad \left[\because \int [f(\theta)]^n f'(\theta)d\theta=\frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$=\frac{1}{2}\cdot\frac{1}{2}(-\cos\pi+\cos 0)\cdot\frac{1}{4}\left(\sin\frac{\pi}{2}-\sin 0\right)\cdot\frac{a^6}{6}$$

$$=\frac{1}{2}\cdot\frac{2}{2}\cdot\frac{1}{4}\cdot\frac{a^6}{6}$$

$$=\frac{a^6}{48}$$

Example 16

Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1}\frac{dz\,dy\,dx}{\sqrt{x^2+y^2+z^2}}$ *by transforming into spherical polar coordinates.*

Solution

1. Limits of $z: z=\sqrt{x^2+y^2}$ to $z=1$

   Limits of $y: y=0$ to $y=\sqrt{1-x^2}$

   Limits of $x: x=0$ to $x=1$

2. The region of integration is the part of the cone $z^2 = x^2 + y^2$ bounded above by the plane $z = 1$ in the positive octant (since all three limits are positive).
3. Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, spherical polar form of
   (i) the cone $z^2 = x^2 + y^2$ is

$$r^2\cos^2\theta = r^2\sin^2\theta(\cos^2\phi + \sin^2\phi)$$
$$= r^2\sin^2\theta$$
$$\cos\theta = \sin\theta$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4}$$

   (ii) the plane $z = 1$ is $r\cos\theta = 1$

$$r = \sec\theta$$

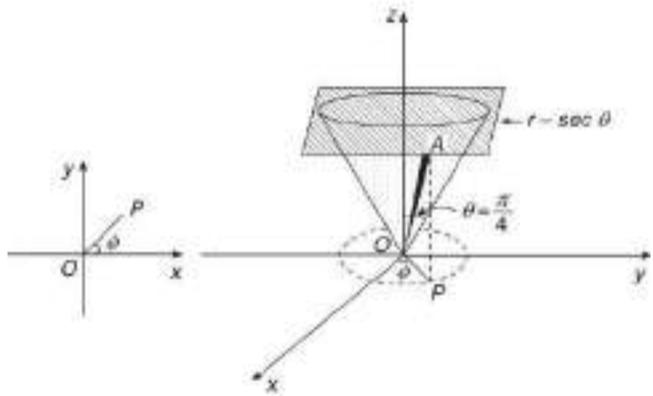

Fig. 9.130

4. Draw an elementary radius vector OA which starts from the origin and terminates on the plane $r = \sec\theta$.

Limits of r : $r = 0$ to $r = \sec\theta$

Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{4}$

Limits of ϕ : $\phi = 0$ to $\phi = \frac{\pi}{2}$ (in positive octant)

Hence, the spherical form of the given integral is

$$I = \int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^1 \frac{dz\,dy\,dx}{\sqrt{x^2+y^2+z^2}}$$
$$= \int_{\phi=0}^{\frac{\pi}{2}}\int_{\theta=0}^{\frac{\pi}{4}}\int_{r=0}^{\sec\theta} \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{r}$$
$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\left[\int_0^{\sec\theta} r\,dr\right]\sin\theta\,d\theta\,d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \left| \frac{r^2}{2} \right|_0^{\sec\theta} \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \left[\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{2} \sin\theta \, d\theta \right] d\phi$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{4}} \tan\theta \sec\theta \, d\theta$$

$$= \frac{1}{2} |\phi|_0^{\frac{\pi}{2}} |\sec\theta|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2} \cdot \frac{\pi}{2} \left(\sec\frac{\pi}{4} - \sec 0 \right)$$

$$= \frac{\pi}{4} \left(\sqrt{2} - 1 \right)$$

Type II Evaluation of Triple Integrals Over the Given Region

Example 1

Evaluate $\iiint x^2 yz \, dx \, dy \, dz$ *over the region bounded by the planes* $x = 0, y = 0, z = 0$ *and* $x + y + z = 1$.

Solution

1. Draw an elementary volume AB parallel to z-axis in the region. AB starts from xy-plane and terminates on the plane $x + y + z = 1$.

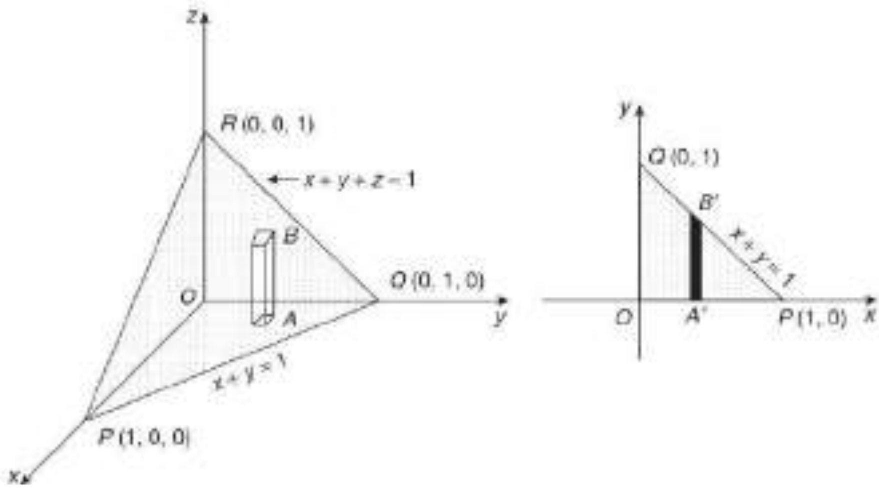

Fig. 9.131

Limits of z : $z = 0$ to $z = 1 - x - y$

2. Projection of the plane $x + y + z = 1$ in xy-plane is ΔOPQ. Putting $z = 0$ in $x + y + z = 1$, the equation of the line PQ is obtained as $x + y = 1$.
3. Draw a vertical strip $A'B'$ in the region OPQ. $A'B'$ starts from the x-axis and terminates on the line $x + y = 1$.

 Limits of y : $y = 0$ to $y = 1 - x$
 Limits of x : $x = 0$ to $x = 1$

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^2 yz \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} x^2 y \left| \frac{z^2}{2} \right|_0^{1-x-y} dy \, dx$$

$$= \frac{1}{2} \int_0^1 x^2 \left[\int_0^{1-x} y \left\{ (1-x)^2 + y^2 - 2y(1-x) \right\} dy \right] dx$$

$$= \frac{1}{2} \int_0^1 x^2 \left[\int_0^{1-x} \left\{ y(1-x)^2 + y^3 - 2y^2(1-x) \right\} dy \right] dx$$

$$= \frac{1}{2} \int_0^1 x^2 \left| (1-x)^2 \frac{y^2}{2} + \frac{y^4}{4} - 2(1-x) \frac{y^3}{3} \right|_0^{1-x} dx$$

$$= \frac{1}{2} \int_0^1 x^2 \left[(1-x)^2 \cdot \frac{(1-x)^2}{2} + \frac{(1-x)^4}{4} - 2(1-x) \cdot \frac{(1-x)^3}{3} \right] dx$$

$$= \frac{1}{2} \int_0^1 \frac{x^2}{12} (1-x)^4 \, dx$$

$$= \frac{1}{24} \left| \frac{(1-x)^5}{-5} \cdot x^2 - \frac{(1-x)^6}{30} \cdot 2x + \frac{(1-x)^7}{-210} \cdot 2 \right|_0^1$$

$$= \frac{1}{24} \left(0 + \frac{1}{105} \right)$$

$$= \frac{1}{2520}$$

Example 2

Evaluate $\iiint\limits_E 2x \, dV$, *where* E *is the region under the plane* $2x + 3y + z = 6$ *that lies in the first octant.* **[Winter 2015]**

Solution

1. Draw an elementary volume AB parallel to z-axis in the region. AB starts from xy-plane and terminates on the plane $2x + 3y + z = 6$.

 Limits of z : $z = 0$ to $z = 6 - 2x - 3y$

2. Projection of the plane $2x + 3y + z = 6$ in xy-plane is ΔOPQ. Putting $z = 0$ in $2x + 3y + z = 6$, the equation of the line PQ is obtained as $2x + 3y = 6$.

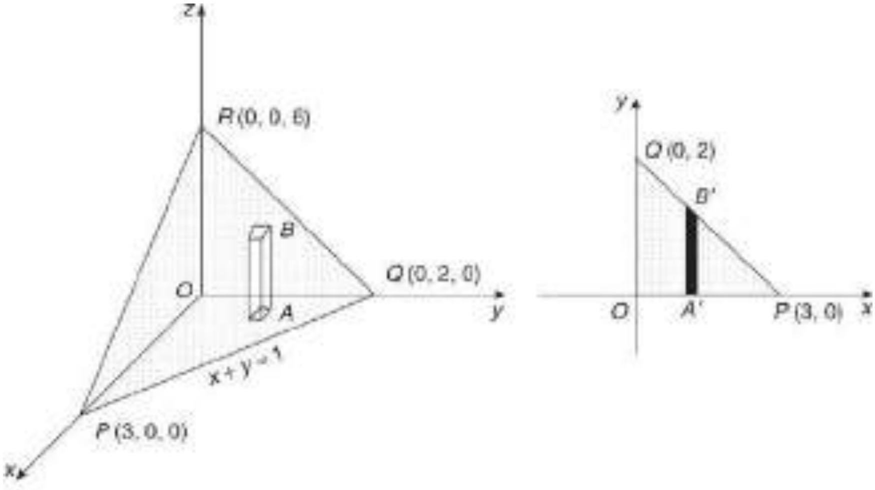

Fig. 9.132

3. Draw a vertical strip $A'B'$ in the region OPQ. $A'B'$ starts from the x-axis and terminates on the line $2x + 3y = 6$.

Limits of y : $y = 0$ to $y = \dfrac{6-2x}{3}$

Limits of x : $x = 0$ to $x = 3$

$$I = \iiint_E 2x \, \mathrm{d}V$$

$$= \int_0^3 \int_0^{\frac{6-2x}{3}} \int_0^{6-2x-3y} 2x \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x$$

$$= \int_0^3 \int_0^{\frac{6-2x}{3}} 2x \left| z \right|_0^{6-2x-3y} \mathrm{d}y \, \mathrm{d}x$$

$$= 2 \int_0^3 \int_0^{\frac{6-2x}{3}} x(6-2x-3y) \, \mathrm{d}y \, \mathrm{d}x$$

$$= 2 \int_0^3 x \left| (6-2x)y - \frac{3y^2}{2} \right|_0^{\frac{6-2x}{3}} \mathrm{d}x$$

$$= 2\int_0^3 x \left[(6-2x) \frac{(6-2x)}{3} - \frac{3}{2} \left(\frac{6-2x}{3} \right)^2 \right] \mathrm{d}x$$

$$= 2\int_0^3 x\frac{(6-2x)^2}{6}\mathrm{d}x$$

$$= \frac{4}{3}\int_0^3 x(9+x^2-6x)\mathrm{d}x$$

$$= \frac{4}{3}\int_0^3 (9x+x^3-6x^2)\,\mathrm{d}x$$

$$= \frac{4}{3}\left|9\frac{x^2}{2}+\frac{x^4}{4}-6\frac{x^3}{3}\right|_0^3$$

$$= 9$$

Example 3

Evaluate $\iiint xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ *over the positive octant of the sphere* $x^2 + y^2 + z^2 = 4$.

Solution

Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, the equation of the sphere $x^2+y^2+z^2=4$ reduces to $r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta = 4$, $r^2 = 4$, $r = 2$.

The region is the positive octant of the sphere $r = 2$.

Limits of r : $r = 0$ to $r = 2$

Limits of $\theta = 0$ to $\theta = \frac{\pi}{2}$

Limits of $\phi = 0$ to $\phi = \frac{\pi}{2}$

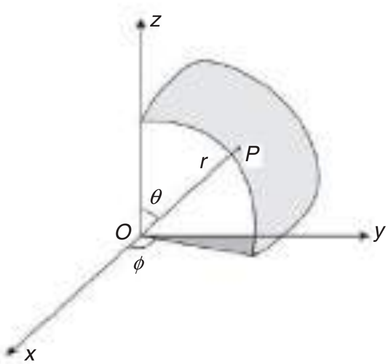

Fig. 9.133

Hence, the spherical form of the given integral is

$$I = \iiint xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^2 (r^3\sin^2\theta\cos\theta\cos\phi\sin\phi)r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= \int_0^{\frac{\pi}{2}}\sin^3\theta\cos\theta\,\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\frac{\sin 2\phi}{2}\mathrm{d}\phi\int_0^2 r^5\mathrm{d}r$$

$$= \left|\frac{\sin^4\theta}{4}\right|_0^{\frac{\pi}{2}}\left|\frac{-\cos 2\phi}{4}\right|_0^{\frac{\pi}{2}}\left|\frac{r^6}{6}\right|_0^2 \qquad \left[\because \int [f(\theta)]^n f'(\theta)d\theta = \frac{[f(\theta)]^{n+1}}{n+1}, n \neq 1\right]$$

$$= \frac{1}{4}\left(\sin^4\frac{\pi}{2} - \sin 0\right)\left[-\frac{1}{4}(\cos\pi - \cos 0)\right]\left(\frac{2^6}{6}\right)$$
$$= \frac{4}{3}$$

Example 4

Evaluate $\iiint \frac{dx\,dy\,dz}{\sqrt{a^2 - x^2 - y^2 - z^2}}$ *over the region bounded by the sphere* $x^2 + y^2 + z^2 = a^2$.

Solution

1. Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, the equation of the sphere $x^2 + y^2 + z^2 = a^2$ reduces to $r = a$.
2. For the complete sphere, limits of $r : r = 0$ to $r = a$
   limits of $\theta : \theta = 0$ to $\theta = \pi$
   limits of $\phi : \phi = 0$ to $\phi = 2\pi$

   Hence, the spherical form of the given integral is

$$I = \iiint \frac{dx\,dy\,dz}{\sqrt{a^2 - x^2 - y^2 - z^2}}$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{\sqrt{a^2 - r^2}}$$
$$= \int_0^{2\pi} d\phi \int_0^{\pi}\sin\theta\,d\theta \int_0^{a}\frac{r^2 + a^2 - a^2}{\sqrt{a^2 - r^2}}\,dr$$
$$= |\phi|_0^{2\pi}\cdot|-\cos\theta|_0^{\pi}\cdot\int_0^{a}\left(\frac{a^2}{\sqrt{a^2 - r^2}} - \sqrt{a^2 - r^2}\right)dr$$
$$= (2\pi)(-\cos\pi + \cos 0)\left|a^2\sin^{-1}\frac{r}{a} - \frac{r}{2}\sqrt{a^2 - r^2} - \frac{a^2}{2}\sin^{-1}\frac{r}{a}\right|_0^{a}$$
$$= 2\pi(2)\left(a^2\sin^{-1}1 - \frac{a^2}{2}\sin^{-1}1\right)$$
$$= 4\pi\left(\frac{a^2}{2}\sin^{-1}1\right)$$
$$= 4\pi\cdot\frac{a^2}{2}\cdot\frac{\pi}{2}$$
$$= \pi^2a^2$$

Example 5

Evaluate $\iiint \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{\frac{1}{2}}}$ *over the region bounded by the spheres* $x^2 + y^2 + z^2 = a^2$ *and* $x^2 + y^2 + z^2 = b^2$, $a > b > 0$.

Solution

1. Putting $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$, equations of the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$ reduce to $r = a$ and $r = b$ respectively.
2. Draw an elementary radius vector *OAB* from the origin in the region. This radius vector enters in the region from the sphere $r = b$ and terminates on the sphere $r = a$.
3. Limits of r : $r = b$ to $r = a$.
   For the complete sphere,
   limits of θ : $\theta = 0$ to $\theta = \pi$
   limits of ϕ : $\phi = 0$ to $\phi = 2\pi$
   Hence, the spherical form of the given integral is

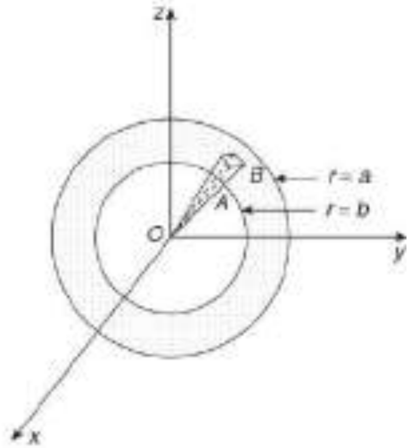

Fig. 9.134

$$I = \iiint \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{\frac{1}{2}}}$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_b^{a} \frac{r^2 \sin\theta}{r}\, dr\, d\theta\, d\phi$$
$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_b^{a} r\, dr$$
$$= \left|\phi\right|_0^{2\pi} \cdot \left|-\cos\theta\right|_0^{\pi} \cdot \left|\frac{r^2}{2}\right|_b^{a}$$
$$= 2\pi(-\cos\pi + \cos 0)\frac{(a^2 - b^2)}{2}$$
$$= 2\pi(a^2 - b^2).$$

Example 6

Evaluate $\iiint z^2\, dx\, dy\, dz$ *over the region common to the sphere* $x^2 + y^2 + z^2 = 4$ *and the cylinder* $x^2 + y^2 = 2x$.

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, the equation of
   (i) the sphere $x^2 + y^2 + z^2 = 4$ reduces to
   $$r^2 + z^2 = 4$$
   $$z^2 = 4 - r^2.$$
   (ii) the cylinder $x^2 + y^2 = 2x$ reduces to
   $r^2 = 2r\cos\theta$, $r = 2\cos\theta$.

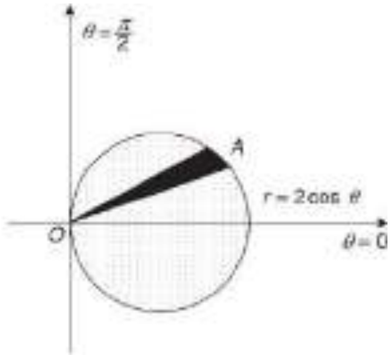

Fig. 9.135

2. Draw an elementary volume parallel to z-axis in the region. This elementary volume starts from the part of the sphere $z^2 = 4 - r^2$, below xy-plane and terminates on the part of the sphere $z^2 = 4 - r^2$, above xy-plane.

   Limits of $r : z = -\sqrt{4-r^2}$ to $z = \sqrt{4-r^2}$
3. Projection of the region in $r\theta$-plane is the circle $r = 2\cos\theta$.
4. Draw an elementary radius vector OA in the region ($r = 2\cos\theta$) which starts from the origin and terminates on the circle $r = 2\cos\theta$

   Limits of $r : r = 0$ to $r = 2\cos\theta$

   Limits of $\theta : \theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$

   Hence, the cylindrical form of the given integral is

$$I = \iiint z^2 \,dx\,dy\,dz$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} z^2 r \,dz\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \left| \frac{z^3}{3} \right|_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dr\,d\theta$$

$$= \frac{1}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} 2(4-r^2)^{\frac{3}{2}} r\,dr\,d\theta$$

$$= \frac{1}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \left[-(4-r^2)^{\frac{3}{2}} (-2r)\,dr \right] d\theta$$

$$= -\frac{1}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \frac{2(4-r^2)^{\frac{5}{2}}}{5} \right|_0^{2\cos\theta} d\theta \qquad \left[\because \int [f(r)]^n f'(r)\,dr = \frac{[f(r)]^{n+1}}{n+1} \right]$$

$$= -\frac{2}{15} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[(4 - 4\cos^2\theta)^{\frac{5}{2}} - (4)^{\frac{5}{2}} \right] d\theta$$

$$= -\frac{2}{15}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2^5\sin^5\theta - 2^5)\,\mathrm{d}\theta$$

$$= -\frac{2}{15}\left[0 - 2^5\,|\theta|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right] \quad \left[\begin{array}{l}\because \int_{-a}^{a} f(\theta)\mathrm{d}\theta = 0, \text{if } f(-\theta) = -f(\theta)\\ \text{Here } \sin^5(-\theta) = -\sin^5\theta\end{array}\right]$$

$$= \frac{2^6\pi}{15}$$

$$= \frac{64\pi}{15}$$

Example 7

Evaluate $\iiint xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ *over the region bounded by the planes* $x = 0, y = 0, z = 0, z = 1$ *and the cylinder* $x^2 + y^2 = 1$.

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, equation of the cylinder $x^2 + y^2 = 1$ reduces to $r^2 = 1$, $r = 1$.

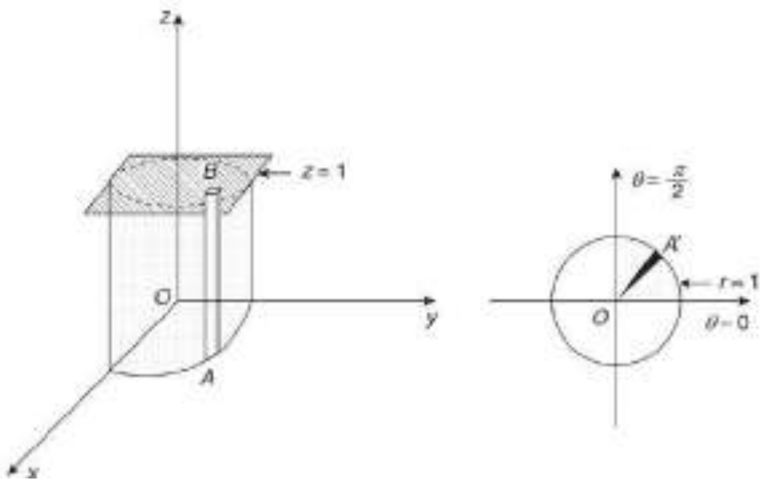

Fig. 9.136

2. Draw an elementary volume AB parallel to z-axis in the region. This elementary volume AB starts from xy-plane and terminates on the plane $z = 1$.
   Limits of z : $z = 0$ to $z = 1$

3. Projection of the region in $r\theta$-plane is the part of the circle $r = 1$ in the first quadrant.
4. Draw an elementary radius vector OA' in the region in the $r\theta$-plane which starts from the origin and terminates on the circle $r = 1$.

   Limits of $r : r = 0 \quad \text{to} \quad r = 1$

   Limits of $\theta : \theta = 0 \quad \text{to} \quad \theta = \frac{\pi}{2}$

   Hence, the cylindrical form of the given integral is

$$
\begin{aligned}
I &= \iiint xyz \, dx \, dy \, dz \\
&= \int_{z=0}^{1} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{1} r^2 \cos\theta \sin\theta \cdot zr \, dz \, dr \, d\theta \\
&= \int_0^1 z \, dz \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{2} d\theta \int_0^1 r^3 dr \\
&= \left| \frac{z^2}{2} \right|_0^1 \left| -\frac{\cos 2\theta}{4} \right|_0^{\frac{\pi}{2}} \left| \frac{r^4}{4} \right|_0^1 \\
&= \frac{1}{16}
\end{aligned}
$$

Example 8

Evaluate $\iiint \sqrt{x^2 + y^2} \, dx \, dy \, dz$ *over the region bounded by the right circular cone* $x^2 + y^2 = z^2$, $z > 0$ *and the planes* $z = 0$ *and* $z = 1$.

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, the equation of the cone $x^2 + y^2 = z^2$ reduces to $r^2 = z^2$, $r = z$.
2. Draw an elementary volume AB parallel to z-axis in the region, which starts from the cone $r = z$ and terminates on the plane $z = 1$.

   Limits of $z : z = r \quad \text{to} \quad z = 1$.

3. Projection of the region in $r\theta$-plane is the curve of intersection of the cone $r = z$ and the plane $z = 1$ which is obtained as $r = 1$, a circle with centre at the origin and radius 1.
4. Draw an elementary radius vector OA' in the region which starts from the origin and terminates on the circle $r = 1$.

   Limits of $r : r = 0 \quad \text{to} \quad r = 1$

   Limits of $\theta : \theta = 0 \quad \text{to} \quad \theta = 2\pi$

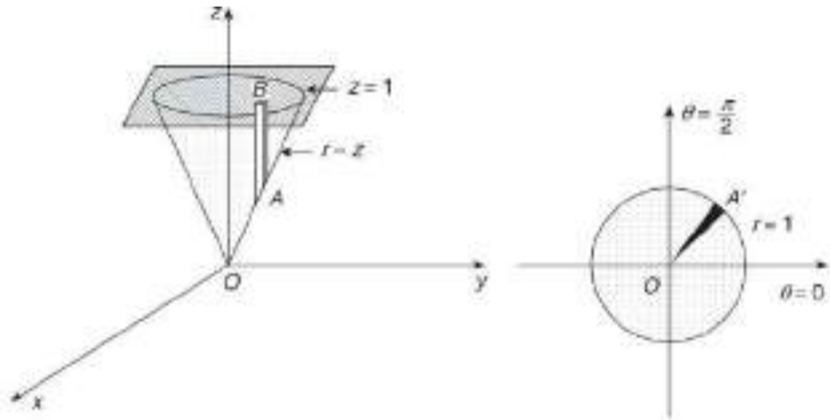

Fig. 9.137

Hence, the cylindrical form of the given integral is

$$
\begin{aligned}
I &= \iiint \sqrt{x^2+y^2}\,dx\,dy\,dz \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=r}^{1} r\cdot r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 r^2 |z|_r^1\,dr\,d\theta \\
&= \int_0^{2\pi} d\theta \int_0^1 r^2(1-r)\,dr \\
&= |\theta|_0^{2\pi}\left|\frac{r^3}{3}-\frac{r^4}{4}\right|_0^1 \\
&= 2\pi\cdot\frac{1}{12} \\
&= \frac{\pi}{6}
\end{aligned}
$$

Example 9

Evaluate $\iiint (x^2+y^2)\,dx\,dy\,dz$ *over the region bounded by the paraboloid* $x^2 + y^2 = 3z$ *and the plane* $z = 3$.

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, the equation of the paraboloid $x^2 + y^2 = 3z$ reduces to $r^2 = 3z$.
2. Draw an elementary volume AB parallel to z-axis in the region which starts from the paraboloid $r^2 = 3z$ and terminates on the plane $z = 3$.

Limits of $z : z = \frac{r^2}{3}$ to $z = 3$

3. Projection of the region in $r\theta$-plane is the curve of intersection of the paraboloid $r^2 = 3z$ and the plane $z = 3$ which is obtained as $r^2 = 9$, $r = 3$, a circle with centre at the origin and radius 1.

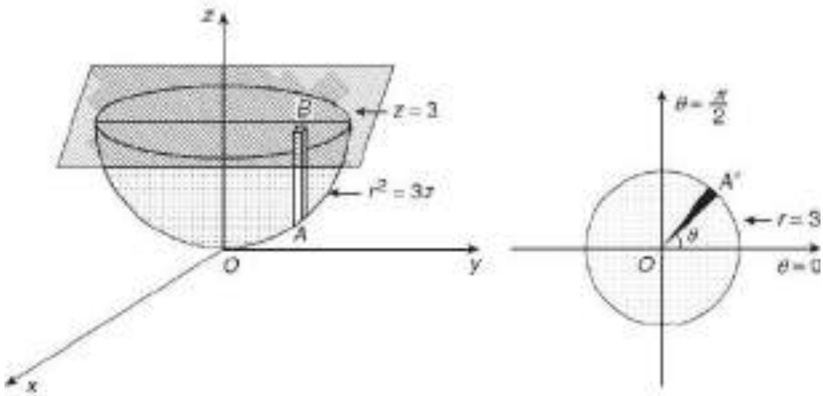

Fig. 9.138

4. Draw an elementary radius vector OA' in the region (circle $r = 3$) which starts from origin and terminates on the circle $r = 3$.
   Limits of $r : r = 0$ to $r = 3$
   Limits of $\theta : \theta = 0$ to $\theta = 2\pi$

Hence, the cylindrical form of the given integral is

$$
\begin{aligned}
I &= \iiint (x^2 + y^2)\,dx\,dy\,dz \\
&= \int_0^{2\pi}\int_0^3\int_{\frac{r^2}{3}}^3 r^2 \cdot r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^3 r^3 \left| z \right|_{\frac{r^2}{3}}^3 dr\,d\theta \\
&= \int_0^{2\pi} d\theta \int_0^3 r^3\left(3 - \frac{r^2}{3}\right) dr \\
&= \left|\theta\right|_0^{2\pi} \left|\frac{3r^4}{4} - \frac{r^6}{18}\right|_0^3 \\
&= 2\pi\left(\frac{3^5}{4} - \frac{3^6}{18}\right) \\
&= \frac{81\pi}{2}
\end{aligned}
$$

Example 10

Evaluate $\iiint_V \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dx\,dy\,dz,$ *where V is the volume of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Solution

It is difficult to integrate this integral in cartesian form. Therefore, transforming the ellipsoid into a sphere using following change of variables.

Putting $\frac{x}{a}=u, \frac{y}{b}=v, \frac{z}{c}=w,$ equation of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ reduces to $u^2+v^2+w^2=1,$ which is a sphere of radius 1 and centre at the origin,

$$dx\,dy\,dz = |J|\,du\,dv\,dw$$

where,

$$J=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}$$

$$=\begin{vmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{vmatrix}=abc$$

Therefore, $$dx\,dy\,dz = abc\,du\,dv\,dw$$

New form of the integral is

$$I=\iiint\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dx\,dy\,dz$$

$$=\iiint\sqrt{1-u^2-v^2-w^2}\cdot abc\,du\,dv\,dw$$

Since in the new coordinate system u, v, w, the region of integration is a sphere, therefore using spherical coordinates $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$ and $du\,dv\,dw = r^2\sin\theta\,dr\,d\theta\,d\phi$, the equation of the sphere $u^2+v^2+w^2 = 1$ reduce to $r^2 = 1$, $r = 1$.

For complete sphere limits of $r: r=0$ to $r=1$ (radius of sphere)

limits of $\theta: \theta=0$ to $\theta=\pi$

limits of $\phi: \phi=0$ to $\phi=2\pi$

Hence, the spherical form of the given integral is

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^{1}\sqrt{1-r^2}\,abc\cdot r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= abc\int_0^{2\pi} d\phi\int_0^{\pi}\sin\theta\,d\theta\int_0^{1} r^2\sqrt{1-r^2}\,dr$$

Putting $r = \sin t$, $dr = \cos t\,dt$

When $r = 0$, $t = 0$

When $r = 1$, $t = \frac{\pi}{2}$

$$\therefore I = abc\left|\phi\right|_0^{2\pi}\left|-\cos\theta\right|_0^{\pi}\int_0^{\frac{\pi}{2}}\sin^2 t\cdot\cos t\cdot\cos t\,dt$$

$$= abc(2\pi)(2)\cdot\frac{1}{2}B\left(\frac{3}{2},\frac{3}{2}\right)$$

$$= 2\pi\,abc\,\frac{\left\lceil\frac{3}{2}\right.\left\lceil\frac{3}{2}\right.}{\left\lceil 3\right.}$$

$$= 2\pi\,abc\,\frac{\left(\frac{1}{2}\left\lceil\frac{1}{2}\right.\right)^2}{2}$$

$$= \frac{\pi^2 abc}{4}$$

Example 11

Evaluate $\iiint x^2y^2z^2\,dx\,dy\,dz$ *over the region bounded by the surfaces* $xy = 4$, $xy = 9$, $yz = 1$, $yz = 4$, $zx = 25$, $zx = 49$.

Solution

Evaluation of integral becomes easier by changing the variables. Under the transformation $xy = u$, $yz = v$, $zx = w$, the surfaces get transformed to $u = 4$, $u = 9$, $v = 1$, $v = 4$, $w = 25$, $w = 49$.

These equations represent the planes parallel to vw, wu and uv planes in the new coordinate system.

It is easier to find partial derivatives of u, v, w w.r.t. x, y and z.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}$$

$$= \begin{vmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{vmatrix}$$

$$= y(zx - 0) - x(0 - yz)$$

$$= 2xyz$$

$$du\,dv\,dw = |J|\,dx\,dy\,dz = 2xyz\,dx\,dy\,dz$$

$$dx\,dy\,dz = \frac{1}{2xyz}\,du\,dv\,dw$$

$$= \frac{1}{2\sqrt{uvw}}\,du\,dv\,dw \qquad [\because\ x^2y^2z^2 = uvw]$$

Limits of u : $u = 4$ to $u = 9$

Limits of v : $v = 1$ to $v = 4$

Limits of w : $w = 25$ to $w = 49$

Hence, the new form of the integral is

$$I = \iiint x^2y^2z^2\,dx\,dy\,dz$$

$$= \int_{w=25}^{49}\int_{v=1}^{4}\int_{u=4}^{9} uvw \cdot \frac{1}{2\sqrt{uvw}}\,du\,dv\,dw$$

$$= \frac{1}{2}\int_{25}^{49} w^{\frac{1}{2}}dw \int_1^4 v^{\frac{1}{2}}dv \int_4^9 u^{\frac{1}{2}}du$$

$$= \frac{1}{2}\left|\frac{2w^{\frac{3}{2}}}{3}\right|_{25}^{49}\left|\frac{2v^{\frac{3}{2}}}{3}\right|_{1}^{4}\left|\frac{2u^{\frac{3}{2}}}{3}\right|_{4}^{9}$$

$$= \frac{4}{27}(343 - 125)(8 - 1)(27 - 8)$$

$$= \frac{115976}{27}$$

EXERCISE 9.7

(I) Evaluate the following integrals:

1. $\int_0^1 dx \int_0^2 dy \int_1^2 x^2yz\,dz$

$\left[\text{Ans.: } 1\right]$

2. $\int_0^{\log 2}\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

$\left[\text{Ans.: } \frac{5}{8}\right]$

3. $\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$

$\left[\textbf{Ans.: } \frac{a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)\right]$

4. $\int_0^{\pi}\int_0^{a(1+\cos\theta)}\int_0^{h} 2\left[1-\frac{r}{a(1+\cos\theta)}\right] r\,dz\,dr\,d\theta$

$\left[\textbf{Ans.: } \frac{\pi a^2 h}{2}\right]$

5. $\int_0^4\int_0^{2\sqrt{z}}\int_0^{\sqrt{4z-x^2}} dy\,dx\,dz$

$\left[\textbf{Ans.: } 8\pi\right]$

6. $\int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} r\,dz\,dr\,d\theta$

$\left[\textbf{Ans.: } \frac{5a^3}{64}\right]$

7. $\int_0^2\int_0^y\int_{x-y}^{x+y}(x+y+z)\,dz\,dx\,dy$

$\left[\textbf{Ans.: } 16\right]$

8. $\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} xyz\,dz\,dy\,dx.$

$\left[\textbf{Ans.: } \frac{a^6}{48}\right]$

(II) Evaluate the following integrals over the given region of integration:

1. $\iiint (x+y+z)\,dx\,dy\,dz$ over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

$\left[\textbf{Ans.: } \frac{1}{8}\right]$

2. $\iiint \frac{dx\,dy\,dz}{(1+x+y+z)^3}$ over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

$\left[\textbf{Ans.: } \frac{1}{2}\left(\log 2-\frac{5}{8}\right)\right]$

3. $\iiint xyz\,dx\,dy\,dz$ over the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$.

$\left[\textbf{Ans.: } \frac{a^6}{48}\right]$

4. $\iiint xyz(x^2+y^2+z^2)\,dx\,dy\,dz$ over the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$.

$\left[\textbf{Ans.: } \frac{a^8}{64}\right]$

5. $\iiint (y^2z^2 + z^2x^2 + x^2y^2)\,dx\,dy\,dz$ over the sphere of radius a and centre at the origin.

$$\left[\text{Ans.}: \frac{4\pi a^7}{35}\right]$$

6. $\iiint \frac{z^2}{x^2+y^2+z^2}\,dx\,dy\,dz$ over the sphere $x^2 + y^2 + z^2 = 2$.

$$\left[\text{Ans.}: \frac{8\pi\sqrt{2}}{9}\right]$$

7. $\iiint \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{\frac{3}{2}}}$ over the region bounded by the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, $a > b > 0$.

$$\left[\text{Ans.}: 4\pi \log\left(\frac{a}{b}\right)\right]$$

8. $\iiint z^2\,dx\,dy\,dz$ over the region common to the spheres $x^2 + y^2 + z^2 = a^2$ and cylinder $x^2 + y^2 = ax$.

$$\left[\text{Ans.}: \frac{2\pi a^5}{15}\right]$$

9. $\iiint (x^2 + y^2)\,dx\,dy\,dz$ over the region bounded by the paraboloid $x^2 + y^2 = 2z$ and the plane $z = 2$.

$$\left[\text{Ans.}: \frac{16\pi}{3}\right]$$

10. $\iiint x^2yz\,dx\,dy\,dz$ over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

$$\left[\text{Ans.}: \frac{a^3b^2c^2}{2520}\right]$$

11. $\iiint xyz\,dx\,dy\,dz$ over the positive octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1$.

$$\left[\text{Ans.}: \frac{a^2b^2c^2}{48}\right]$$

12. $\iiint \sqrt{\frac{x^2}{1}+\frac{y^2}{4}+\frac{z^2}{9}}\,dx\,dy\,dz$ over the region bounded by the ellipsoid $\frac{x^2}{1}+\frac{y^2}{4}+\frac{z^2}{9}=1$.

$$[\text{Ans.}: 8\pi]$$

9.7 AREA BY DOUBLE INTEGRALS

9.7.1 Area in Cartesian Coordinates

(i) The area A bounded by the curves $y = y_1(x)$ and $y = y_2(x)$ intersecting at the points $P(a, b)$ and $Q(c, d)$ is

$$A = \int_a^c \int_{y_1(x)}^{y_2(x)} dy\,dx$$

(ii) If equation of the curves are represented as $x = x_1(y)$ and $x = x_2(y)$ then

$$A = \int_b^d \int_{x_1(y)}^{x_2(y)} dx\,dy$$

Note: Consider the symmetricity of the region while calculating area.

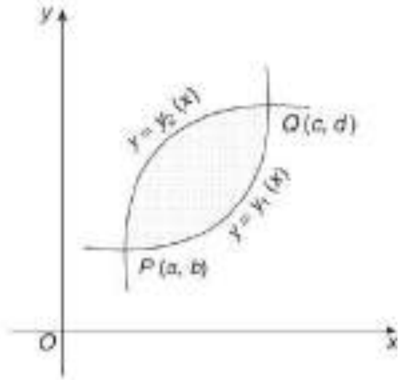

Fig. 9.139

Example 1

Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *above x-axis.*

Solution

1. The region is symmetric about y-axis. Total area = 2 (area bounded by the ellipse in the first quadrant)
2. Draw a vertical strip AB in the region which lies in the first quadrant. AB starts from the x-axis and terminates on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

   Limits of y : $y = 0$ to $y = b\sqrt{1 - \frac{x^2}{a^2}}$

   Limits of x : $x = 0$ to $x = a$

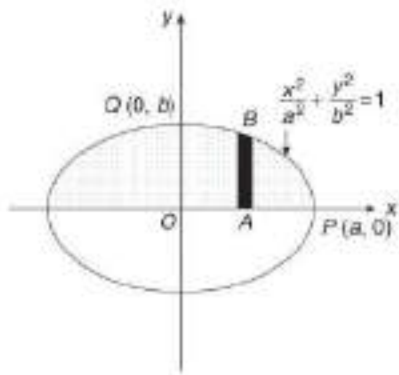

Fig. 9.140

$$A = 2\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dy\,dx$$

$$= 2\int_0^a |y|_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= 2\int_0^a b\sqrt{1 - \frac{x^2}{a^2}}\,dx$$

$$= \frac{2b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= \frac{2b}{a}\left|\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right|_0^a$$

$$= \frac{2b}{a}\left(\frac{a^2}{2}\sin^{-1}1\right)$$

$$= \frac{2b}{a}\left(\frac{a^2}{2}\cdot\frac{\pi}{2}\right)$$

$$= \frac{\pi ab}{2}$$

Example 2

Find the area bounded by the parabola $y^2 = 4x$ and the line $2x - 3y + 4 = 0$.

Solution

1. The points of intersection of the parabola $y^2 = 4x$ and the line $2x - 3y + 4 = 0$ are obtained as

$$\left(\frac{2x+4}{3}\right)^2 = 4x$$

$$(x+2)^2 = 9x$$

$$x^2 - 5x + 4 = 0$$

$$x = 1, 4$$

$$\therefore y = 2, 4$$

The points of intersection are $P(1, 2)$ and $Q(4, 4)$.

2. Draw a vertical strip AB which starts from the line $2x - 3y + 4 = 0$ and terminates on the parabola $y^2 = 4x$.

Limits of y : $y = \frac{2x+4}{3}$ to $y = 2\sqrt{x}$

Limits of x : $x = 1$ to $x = 4$

$$A = 2\int_1^4 \int_{\frac{2x+4}{3}}^{2\sqrt{x}} \mathrm{d}y\,\mathrm{d}x$$

$$= \int_1^4 |y|_{\frac{2x+4}{3}}^{2\sqrt{x}}\,\mathrm{d}x$$

$$= \int_1^4 \left(2\sqrt{x} - \frac{2x+4}{3}\right)\mathrm{d}x$$

$$= \left|2\cdot 2\frac{x^{\frac{3}{2}}}{3} - \frac{x^2}{3} - \frac{4x}{3}\right|_1^4$$

$$= \frac{4}{3}(8-1) - \frac{1}{3}(16-1) - \frac{4}{3}(4-1)$$

$$= \frac{1}{3}$$

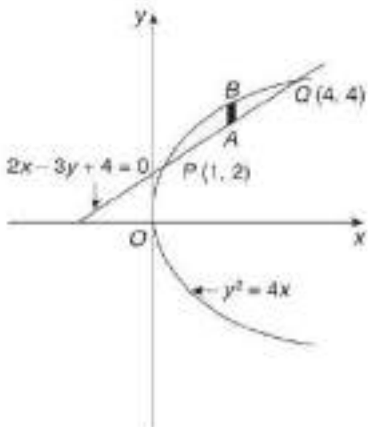

Fig. 9.141

Example 3

Find the area enclosed by the curves $y = x^2$ and $y = x$.

Solution

1. The points of intersection of the parabola $y = x^2$ and the line $y = x$ are obtained as

$$x = x^2$$
$$x = 0, 1$$
$$\therefore \quad y = 0, 1$$

   The points of intersection are O (0, 0) and P (1, 1).

2. Draw a vertical strip AB which starts from the parabola $y = x^2$ and terminates on the line $y = x$.

   Limits of y : $y = x^2$ to $y = x$

   Limits of x : $x = 0$ to $x = 1$

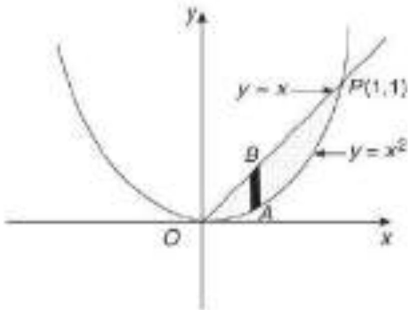

Fig. 9.142

$$A = \int_0^1 \int_{x^2}^{x} dy\,dx$$
$$= \int_0^1 \left| y \right|_{x^2}^{x} dx$$
$$= \int_0^1 (x - x^2)\,dx$$
$$= \left| \frac{x^2}{2} - \frac{x^3}{3} \right|_0^1$$
$$= \frac{1}{2} - \frac{1}{3}$$
$$= \frac{1}{6}$$

Example 4

Find the area enclosed by the parabola $y^2 = 4ax$ and the lines $x + y = 3a$, $y = 0$ in the first quadrant.

Solution

1. The points of intersection of the parabola $y^2 = 4ax$ and the line $x + y = 3a$ are obtained as

$$y^2 = 4a(3a - y)$$
$$y^2 + 4ay - 12a^2 = 0$$
$$y = 2a, -6a$$
$$\therefore \quad x = a, 9a$$

   The point of intersection is Q (a, $2a$) which lies in the first quadrant.

2. Area enclosed in the first quadrant is OPQ.

Draw a horizontal strip AB which starts from the parabola $y^2 = 4ax$ and terminates on the line $x + y = 3a$.

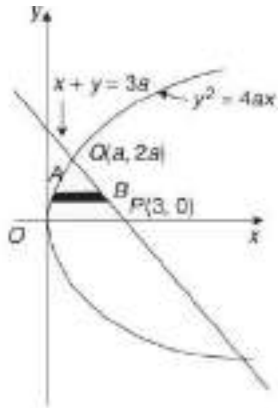

Fig. 9.143

Limits of $x : x = \frac{y^2}{4a}$ to $x = 3a - y$

Limits of $y : y = 0$ to $y = 2a$

$$A = \int_0^{2a}\int_{\frac{y^2}{4a}}^{3a-y} dx\,dy$$

$$= \int_0^{2a} \left| x \right|_{\frac{y^2}{4a}}^{3a-y} dy$$

$$= \int_0^{2a}\left(3a - y - \frac{y^2}{4a}\right)dy$$

$$= \left| 3ay - \frac{y^2}{2} - \frac{1}{4a}\cdot\frac{y^3}{3}\right|_0^{2a}$$

$$= 6a^2 - 2a^2 - \frac{1}{4a}\cdot\frac{8a^3}{3}$$

$$= \frac{10}{3}a^2$$

Note: In case of vertical strip, two vertical strips are required to cover the entire region. Therefore one horizontal strip is preferred over vertical strip.

Example 5

Find the area bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Solution

1. The points of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ are obtained as

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$x^4 = 16a^2(4ax)$$

$$x(x^3 - 64a^3) = 0$$

$$x = 0, \quad x = 4a$$

$$\therefore \ y = 0, \quad y = 4a$$

The points of intersection are O (0, 0) and P ($4a$, $4a$).

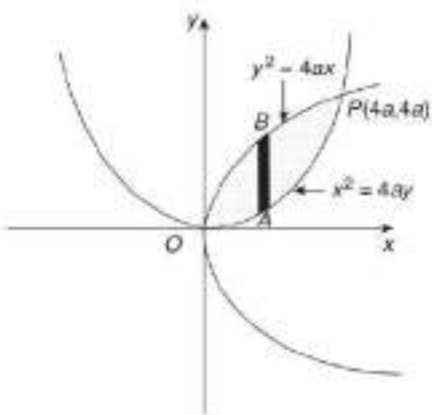

Fig. 9.144

2. Draw a vertical strip AB which starts from the parabola $x^2 = 4ay$ and terminates on the parabola $y^2 = 4ax$.

Limits of $y : y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$

Limits of $x : x = 0$ to $x = 4a$

$$A = \int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$$

$$= \int_0^{4a} |y|_{\frac{x^2}{4a}}^{2\sqrt{ax}} dx$$

$$= \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx$$

$$= \left|2\sqrt{a} \cdot \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{4a} \cdot \frac{x^3}{3}\right|_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}(4a)^{\frac{3}{2}} - \frac{1}{12a}(4a)^3$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2$$

$$= \frac{16}{3}a^2$$

Example 6

Find the area enclosed by the curves $y = 2 - x$ and $y^2 = 2(2 - x)$.

Solution

1. The points of intersection of the line $y = 2 - x$ and the parabola $y^2 = 2(2 - x)$ are obtained as

$$(2 - x)^2 = 2(2 - x)$$
$$(2 - x)(2 - x - 2) = 0$$
$$(2 - x)(-x) = 0$$
$$x = 2, 0$$
$$y = 0, 2$$

The points of intersection are P (2, 0) and Q (0, 2).

2. Draw a vertical strip AB which starts from the line $y = 2 - x$ and terminates on the parabola $y^2 = 2(2 - x)$.

Limits of y: $y = 2 - x$ to $y = \sqrt{2(2-x)}$

Limits of x: $x = 0$ to $x = 2$

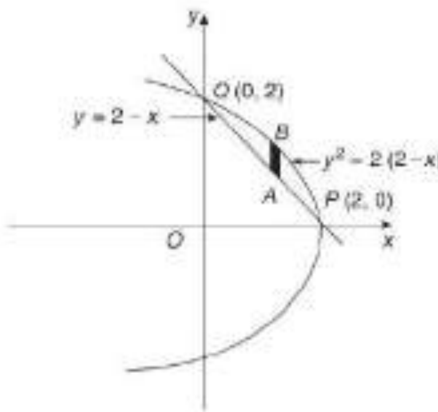

Fig. 9.145

$$A = \int_0^2 \int_{2-x}^{\sqrt{2(2-x)}} dy\,dx$$

$$= \int_0^2 |y|_{2-x}^{\sqrt{2(2-x)}} dx$$

$$= \int_0^2 \left[\sqrt{2}(2-x)^{\frac{1}{2}} - (2-x)\right] dx$$

$$= \left| \sqrt{2} \cdot \frac{2(2-x)^{\frac{3}{2}}}{-3} - 2x + \frac{x^2}{2} \right|_0^2$$

$$= \left(0 + \frac{8}{3}\right) - 2(2-0) + \frac{1}{2}(4-0)$$

$$= \frac{2}{3}$$

Example 7

Find the area bounded between the parabolas $x^2 = 4ay$ and $x^2 = -4a(y - 2a)$.

Solution

1. The parabola $x^2 = 4ay$ has vertex (0, 0) and the parabola $x^2 = -4a(y - 2a)$ has vertex (0, 2a). Both the parabolas are symmetric about the y-axis.
2. The points of intersection of $x^2 = 4ay$ and $x^2 = -4a(y - 2a)$ are obtained as

$$4ay = -4a(y - 2a)$$
$$8ay = 8a^2$$
$$y = a$$
$$\therefore x = \pm 2a$$

The points of intersection are $P(2a, a)$ and $R(-2a, a)$.

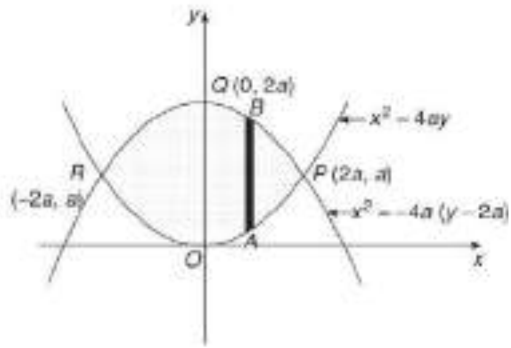

Fig. 9.146

3. The region is symmetric about y-axis.

Total area = 2 (Area in the first quadrant)

4. Draw a vertical strip AB in the region which lies in the first quadrant. AB starts from the parabola $x^2 = 4ay$ and terminates on the parabola $x^2 = -4a(y - 2a)$.

Limit of y : $y = \frac{x^2}{4a}$ to $y = 2a - \frac{x^2}{4a}$

Limits of x : $x = 0$ to $x = 2a$

$$A = 2\int_0^{2a} \int_{\frac{x^2}{4a}}^{2a - \frac{x^2}{4a}} dy\, dx$$

$$= 2\int_0^{2a} |y|_{\frac{x^2}{4a}}^{2a - \frac{x^2}{4a}} dx$$

$$= 2\int_0^{2a} \left(2a - \frac{x^2}{4a} - \frac{x^2}{4a}\right) dx$$

$$= 2\left|2ax - \frac{x^3}{6a}\right|_0^{2a}$$

$$= 2\left(4a^2 - \frac{4}{3}a^2\right)$$

$$= \frac{16}{3}a^2$$

Example 8

Find smaller of the area enclosed by the curves $y = 2 - x$ and $x^2 + y^2 = 4$.

Solution

1. The points of intersection of the line $y = 2 - x$ and the circle $x^2 + y^2 = 4$ are obtained as

$$x^2 + (2 - x)^2 = 4$$
$$x^2 + 4 - 4x + x^2 = 4$$
$$2x^2 = 4x$$
$$x = 2, 0$$
$$\therefore \quad y = 0, 2$$

The points of intersection are P (2, 0) and Q (0, 2).

2. Draw a vertical strip AB which starts from the line $y = 2 - x$ and terminates on the circle $x^2 + y^2 = 4$.

Limits of y : $y = 2 - x$ to $y = \sqrt{4 - x^2}$

Limits of x : $x = 0$ to $x = 2$

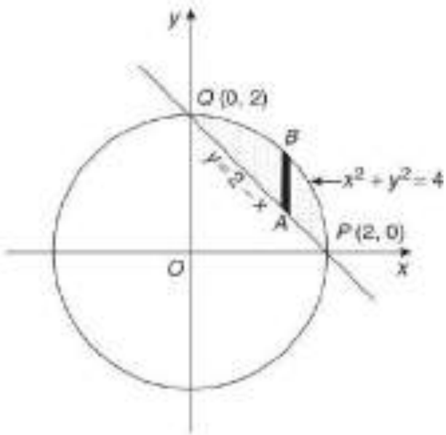

Fig. 9.147

$$A = \int_0^2 \int_{2-x}^{\sqrt{4-x^2}} dy\, dx$$

$$= \int_0^2 |y|_{2-x}^{\sqrt{4-x^2}} dx$$

$$= \int_0^2 \left[\sqrt{4-x^2} - (2-x)\right] dx$$

$$= \left|\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2} - 2x + \frac{x^2}{2}\right|_0^2$$

$$= 2\sin^{-1}1 - 2$$

$$= \pi - 2$$

Example 9

Find the area of the loop of the curve $x(x^2 + y^2) = a(x^2 - y^2)$.

Solution

The equation of the curve can be rewritten as $y^2 = x^2\left(\frac{a-x}{a+x}\right)$

1. The points of intersection of the curve with x-axis ($y = 0$) are obtained as

$$x^2(a - x) = 0$$
$$x = 0, x = a.$$

The loop of the curve lies between the points O (0, 0) and P (a, 0).

2. The region is symmetric about x-axis

Total area = 2 (Area above x-axis)

3. Draw a vertical strip AB in the region above x-axis. AB starts from x-axis and terminates on the curve $y^2 = x^2\left[\frac{a-x}{a+x}\right]$.

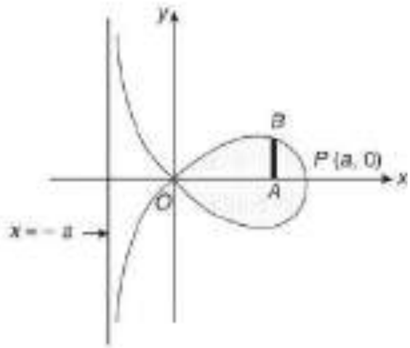

Fig. 9.148

Limits of y : $y = 0$ to $y = x\sqrt{\frac{a-x}{a+x}}$

Limits of x : $x = 0$ to $x = a$

$$A = 2\int_0^a \int_0^{x\sqrt{\frac{a-x}{a+x}}} dy\,dx$$

$$= 2\int_0^a |y|_0^{x\sqrt{\frac{a-x}{a+x}}} dx$$

$$= 2\int_0^a x\sqrt{\frac{a-x}{a+x}}dx$$

Putting $x = a\cos\theta$, $dx = -a\sin\theta\,d\theta$

When $x = 0, \theta = \frac{\pi}{2}$

When $x = a, \theta = 0$

$$A = 2\int_{\frac{\pi}{2}}^0 a\cos\theta\sqrt{\frac{a-a\cos\theta}{a+a\cos\theta}}(-a\sin\theta)d\theta$$

$$= 2a^2\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}d\theta$$

$$= 2a^2\int_0^{\frac{\pi}{2}} \cos\theta\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}d\theta$$

$$= 2a^2\int_0^{\frac{\pi}{2}} \cos\theta(1-\cos\theta)d\theta$$

$$= 2a^2 \int_0^{\frac{\pi}{2}} \left(\cos\theta - \frac{1+\cos 2\theta}{2} \right) d\theta$$

$$= 2a^2 \left| \sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4} \right|_0^{\frac{\pi}{2}}$$

$$= 2a^2 \left[\left(\sin\frac{\pi}{2} - \sin 0 \right) - \frac{1}{2}\left(\frac{\pi}{2} - 0 \right) - \frac{1}{4}(\sin\pi - \sin 0) \right]$$

$$= 2a^2 \left(1 - \frac{\pi}{4} \right)$$

Example 10

Find the area included between the curve $y^2(2a - x) = x^3$ and its asymptote. **[Summer 2017]**

Solution

The equation of the curve can be rewritten as

$$y^2 = \frac{x^3}{2a - x}$$

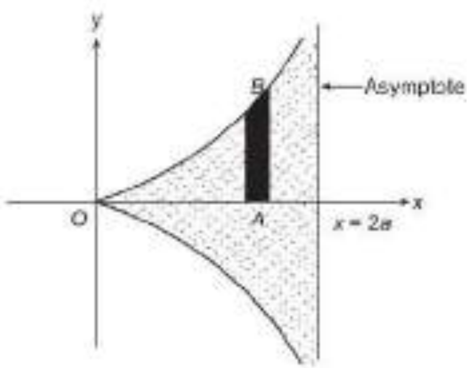

Fig. 9.149

1. The point of intersection of the curve with x-axis ($y = 0$) is $x = 0$.
2. The region is symmetric about x-axis.
   Total area = 2 (Area above x-axis)
3. Draw a vertical strip AB in the region above x-axis. AB starts from x-axis and terminates on the curve $y^2 = \frac{x^3}{2a - x}$.

   Limits of y: $y = 0$ to $y = x\sqrt{\frac{x}{2a - x}}$

   Limits of x: $x = 0$ to $x = 2a$

$$A = 2\int_0^{2a} \int_0^{x\sqrt{\frac{x}{2a-x}}} dy\, dx$$

$$= 2\int_0^{2a} \left| y \right|_0^{x\sqrt{\frac{x}{2a-x}}} dx$$

$$= 2\int_0^{2a} x\sqrt{\frac{x}{2a - x}}\, dx$$

Putting $x = 2a\sin^2\theta$,

$dx = 2a\,(2\sin\theta\cos\theta\,d\theta)$

When $x = 0$, $\theta = 0$

When $x = 2a$, $\theta = \frac{\pi}{2}$

$$A = 2\int_0^{\frac{\pi}{2}} 2a\sin^2\theta\,\sqrt{\frac{2a\sin^2\theta}{2a\cos^2\theta}}\cdot 4a\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 2a\sin^2\theta\cdot\frac{\sin\theta}{\cos\theta}\cdot 4a\sin\theta\cos\theta\,d\theta$$

$$= 16a^2\int_0^{\frac{\pi}{2}}\sin^4\theta\,d\theta$$

$$= 16a^2\cdot\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{\pi}{4}$$

$$= 3\pi a^2$$

Example 11

Find the area between the rectangular hyperbola $3xy = 2$ *and the line* $12x + y = 6$.

Solution

1. The points of intersection of the rectangular hyperbola $3xy = 2$ and the line $12x + y = 6$ are obtained as

$$3x(6-12x) = 2$$

$$18x^2 - 9x + 1 = 0$$

$$x = \frac{1}{3}, \frac{1}{6}$$

$$\therefore y = 2, 4$$

The points of intersection are

$P\left(\frac{1}{3}, 2\right)$ and $Q\left(\frac{1}{6}, 4\right)$.

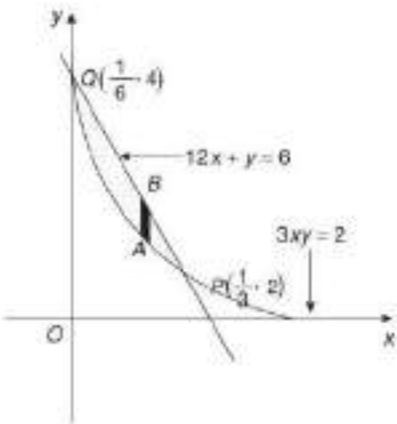

Fig. 9.150

2. Draw a vertical strip AB in the region which starts from the rectangular hyperbola $3xy = 2$ and terminates on the line $12x + y = 6$.

Limits of y : $y = \frac{2}{3x}$ to $y = 6 - 12x$

Limits of x : $x = \frac{1}{6}$ to $x = \frac{1}{3}$

$$A = \int_{\frac{1}{6}}^{\frac{1}{3}} \int_{\frac{2}{3x}}^{6-12x} dy\, dx$$

$$= \int_{\frac{1}{6}}^{\frac{1}{3}} |y|_{\frac{2}{3x}}^{6-12x} dx$$

$$= \int_{\frac{1}{6}}^{\frac{1}{3}} \left(6 - 12x - \frac{2}{3x}\right) dx$$

$$= \left|6x - 6x^2 - \frac{2}{3}\log x\right|_{\frac{1}{6}}^{\frac{1}{3}}$$

$$= (2-1) - 6\left(\frac{1}{9} - \frac{1}{36}\right) - \frac{2}{3}\left(\log\frac{1}{3} - \log\frac{1}{6}\right)$$

$$= \frac{1}{2} - \frac{2}{3}\log 2$$

Example 12

Find the area bounded by the hypocycloid $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$.

Solution

1. The hypocycloid is symmetric in all the quadrants.

   Total area = 4 (area in the first quadrant)

2. Draw a vertical strip *AB* parallel to *y*-axis in the region which lies in the first quadrant. *AB* starts from *x*-axis and terminates on the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$.

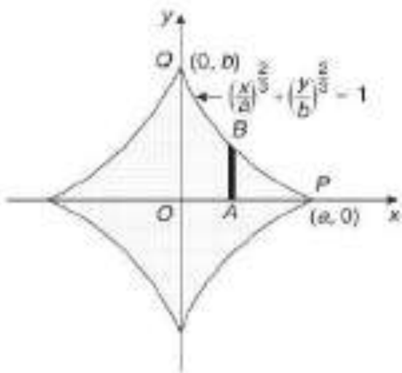

Fig. 9.151

   Limits of y : $y = 0$ to $y = b\left[1 - \left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}$

   Limits of x : $x = 0$ to $x = a$

$$A = 4\int_0^a \int_0^{b\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}} dy\, dx$$

$$= 4\int_0^a |y|_0^{b\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}} dx$$

$$= 4\int_0^a b\left[1 - \left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}} dx$$

Putting $x = a\cos^3 t,\ dx = 3a\cos^2 t\,(-\sin t)\,dt$

When $x = 0,\ t = \frac{\pi}{2}$

When $x = a,\ t = 0$

$$A = 4\int_{\frac{\pi}{2}}^{0} b(1-\cos^2 t)^{\frac{1}{2}}(-3a\cos^2 t\sin t)\,dt$$

$$= 12ab\int_0^{\frac{\pi}{2}} \sin^4 t\cos^2 t\,dt$$

$$= 12ab\,\frac{1}{2}B\left(\frac{5}{2},\frac{3}{2}\right)$$

$$= 6ab\,\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(4)}$$

$$= 6ab\,\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{3!}$$

$$= \frac{3}{8}\pi ab$$

EXERCISE 9.8

1. Find the area bounded by y-axis, the line $y = 2x$ and the line $y = 4$.

[Ans.: 4]

2. Find the area bounded by the lines $y = 2 + x$, $y = 2 - x$ and $x = 5$.

[Ans.: 25]

3. Find the area bounded by the parabola $y^2 + x = 0$, and the line $y = x + 2$.

$\left[\textbf{Ans.:}\ \frac{9}{2}\right]$

4. Find the area bounded by the parabola $x = y - y^2$ and the line $x + y = 0$.

$\left[\textbf{Ans.:}\ \frac{4}{3}\right]$

5. Find the area bounded by the curves $y^2 = 4x$ and $2x - 3y + 4 = 0$.

$\left[\textbf{Ans.:}\ \frac{1}{3}\right]$

6. Find the area bounded by the parabola $y = x^2 - 3x$ and the line $y = 2x$.

$$\left[\textbf{Ans.:} \frac{125}{6}\right]$$

7. Find the area bounded by the parabolas $y^2 = x$, $x^2 = -8y$.

$$\left[\textbf{Ans.:} \frac{8}{3}\right]$$

8. Find the area bounded by the parabolas $y = ax^2$ and $y = 1 - \frac{x^2}{a}$, where $a > 0$.

$$\left[\textbf{Ans.:} \frac{4}{3}\sqrt{\frac{a}{a^2+1}}\right]$$

9. Find the area of the loop of the curve $y^2 = x^2\left(\frac{a+x}{a-x}\right)$

$$\left[\textbf{Ans.:} 2a^2\left(\frac{\pi}{4} - 1\right)\right]$$

10. Find the area of one of the loops of $x^4 + y^4 = 2a^2xy$.

$$\left[\textbf{Ans.:} \frac{\pi a^2}{4}\right]$$

11. Find the area enclosed by the curve $9xy = 4$ and the line $2x + y = 2$.

$$\left[\textbf{Ans.:} \frac{1}{3} - \frac{4}{9}\log 2\right]$$

12. Find the area of the smaller region bounded by the circle $x^2 + y^2 = 9$ and a straight line $x = 3 - y$.

$$\left[\textbf{Ans.:} 4\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)\right]$$

13. Find the area bounded by the x-axis, circle $x^2 + y^2 = 16$ and the line $y = x$.
[**Ans.:** 2π]

14. Find the area bounded between the curves $y = 3x^2 - x - 3$ and $y = -2x^2 + 4x + 7$.

$$\left[\textbf{Ans.:} \frac{45}{2}\right]$$

15. Find the area bounded by the asteroid $(x)^{\frac{2}{3}} + (y)^{\frac{2}{3}} = (a)^{\frac{2}{3}}$.

$$\left[\textbf{Ans.:} \frac{3}{8}\pi a^2\right]$$

9.7.2 Area in Polar Coordinates

The area A bounded by the curves $r = r_1(\theta)$, $r = r_2(\theta)$ and the lines $\theta = \theta_1$ and $\theta = \theta_2$ is

$$A = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} r\,dr\,d\theta$$

Note: Consider the symmetricity of the region while calculating the area.

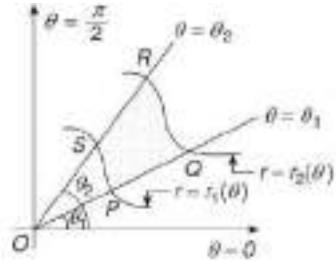

Fig. 9.152

Example 1

Find the area between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Solution

1. The region is symmetric about the line $\theta = \frac{\pi}{2}$.

   Total area = 2 (area in the first quadrant)

2. Draw an elementary radius vector OAB from the origin in the region which lies in the first quadrant. OAB enters in the region from the circle $r = 2\sin\theta$ and terminates on at the circle $r = 4\sin\theta$.

   Limits of $r: r = 2\sin\theta$ to $r = 4\sin\theta$

   Limits of $\theta: \theta = 0$ to $\theta = \frac{\pi}{2}$

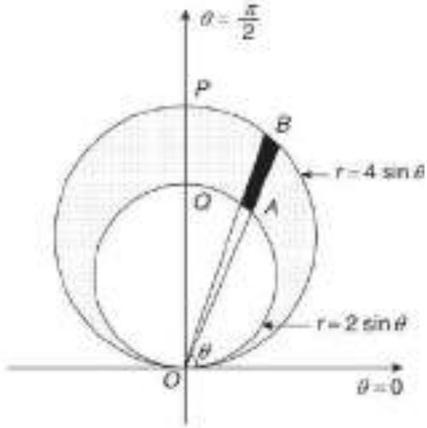

Fig. 9.153

$$A = 2\int_0^{\frac{\pi}{2}} \int_{2\sin\theta}^{4\sin\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left| \frac{r^2}{2} \right|_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} (16\sin^2\theta - 4\sin^2\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} 12\sin^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} 6(1 - \cos 2\theta)\,d\theta$$

$$= 6\left| \theta - \frac{\sin 2\theta}{2} \right|_0^{\frac{\pi}{2}} d\theta$$

$$= 6\left(\frac{\pi}{2} - \frac{\sin\pi - \sin 0}{2} \right)$$

$$= 3\pi$$

Example 2

Use double integral in polar form to find the area enclosed by the three petalled rose $r = \sin 3\theta$. **[Winter 2015]**

Solution

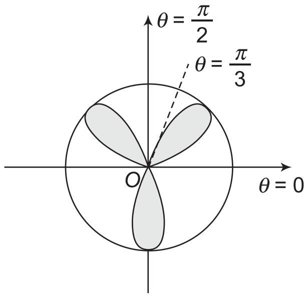

Fig. 9.154

1. This curve consists of three similar loops. Total area = 3 (area of the loop in the first quadrant)
2. When $r = 0$, $\sin 3\theta = 0$

$$3\theta = 0, \pi, 2\pi, 3\pi, \ldots$$

$$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \ldots$$

Since, in the first quadrant,

$$r = 0 \text{ at } \theta = 0, \frac{\pi}{3}$$

loop exists between $\theta = 0$ and $\theta = \frac{\pi}{3}$.

3. Draw an elementary radius vector *OA* from the origin in the loop which lies in the first quadrant. *OA* starts from the origin and terminates on the curve = sin θ.

Limits of r : $r = 0$ to $r = \sin 3\theta$

Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{3}$

$$A = 3\int_0^{\frac{\pi}{3}}\int_0^{\sin 3\theta} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 3\int_0^{\frac{\pi}{3}} \left|\frac{r^2}{2}\right|_0^{\sin 3\theta} \mathrm{d}\theta$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{3}} \sin^2 3\theta\,\mathrm{d}\theta$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{3}} \left(\frac{1-\cos 6\theta}{2}\right)\mathrm{d}\theta$$

$$= \frac{3}{4}\left|\theta - \frac{\sin 6\theta}{6}\right|_0^{\frac{\pi}{3}}$$

$$= \frac{3}{4}\left(\frac{\pi}{3} - \frac{1}{6}\sin 2\pi\right)$$

$$= \frac{3}{4}\left(\frac{\pi}{3}\right)$$

$$= \frac{\pi}{4}$$

Example 3

Find the area of the crescent bounded by the circles $r = \sqrt{3}$ *and* $r = 2\cos\theta$.

Solution

1. The points of intersection of $r = \sqrt{3}$ and $r = 2\cos\theta$ are obtained as

$$\sqrt{3} = 2\cos\theta$$

$$\cos\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \pm\frac{\pi}{6}$$

Hence, $\theta = \frac{\pi}{6}$ at P.

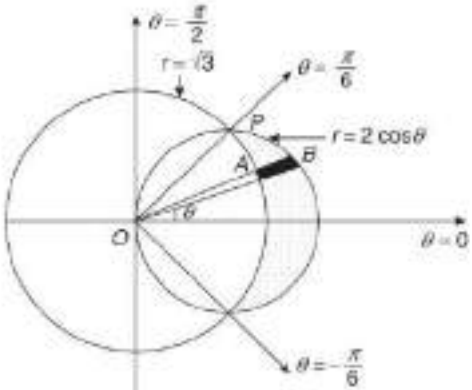

Fig. 9.155

2. The region is symmetric about the initial line, $\theta = 0$.
   Area of the crescent = 2 (area above the initial line, $\theta = 0$)

3. Draw an elementary radius vector OAB from the origin in the region above the initial line. OAB enters in the region from the circle $r = \sqrt{3}$ and terminates on at the circle $r = 2\cos\theta$.
   Limits of $r: r = \sqrt{3}$ to $r = 2\cos\theta$
   Limits of $\theta: \theta = 0$ to $\theta = \frac{\pi}{6}$

$$A = 2\int_0^{\frac{\pi}{6}}\int_{\sqrt{3}}^{2\cos\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{6}}\left|\frac{r^2}{2}\right|_{\sqrt{3}}^{2\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{6}}(4\cos^2\theta - 3)d\theta$$

$$= \int_0^{\frac{\pi}{6}}[2(1+\cos 2\theta) - 3]d\theta$$

$$= \left|2\frac{\sin 2\theta}{2} - \theta\right|_0^{\frac{\pi}{6}}$$

$$= \sin\frac{\pi}{3} - \frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

Example 4

Find the area which lies inside the circle $r = 3a\cos\theta$ and outside the cardioid $r = a(1 + \cos\theta)$.

Solution

1. The points of intersection of the circle $r = 3a\cos\theta$ and the cardioid $r = a(1 + \cos\theta)$ are obtained as

$$3a\cos\theta = a(1+\cos\theta)$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \pm\frac{\pi}{3}$$

Hence, $\theta = \frac{\pi}{3}$ at R.

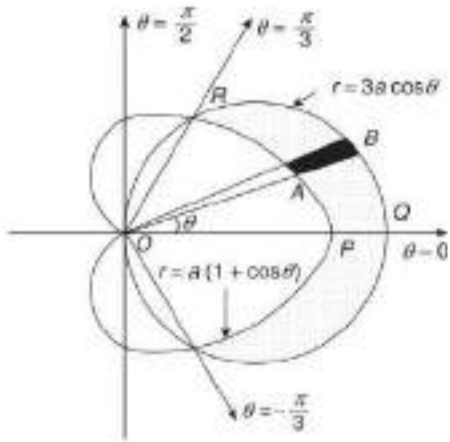

Fig. 9.156

2. The region is symmetric about the initial line $\theta = 0$.

   Total area = 2 (area above the initial line)

3. Draw an elementary radius vector OAB from the origin in the region above the initial line. OAB enters in the region from the cardioid $r = a(1 + \cos\theta)$ and terminates on the circle $r = 3a\cos\theta$.

   Limits of r : $r = a(1 + \cos\theta)$ to $r = 3a\cos\theta$

   Limits of $\theta : \theta = 0$ to $\theta = \frac{\pi}{3}$

$$A = 2\int_0^{\frac{\pi}{3}}\int_{a(1+\cos\theta)}^{3a\cos\theta} r\,dr\,d\theta$$
$$= 2\int_0^{\frac{\pi}{3}}\left|\frac{r^2}{2}\right|_{a(1+\cos\theta)}^{3a\cos\theta} d\theta$$
$$= \int_0^{\frac{\pi}{3}}\left[9a^2\cos^2\theta - a^2(1+\cos\theta)^2\right]d\theta$$
$$= \int_0^{\frac{\pi}{3}}\left[8a^2\cos^2\theta - a^2 - 2a^2\cos\theta\right]d\theta$$
$$= a^2\int_0^{\frac{\pi}{3}}\left[4(1+\cos 2\theta) - 1 - 2\cos\theta\right]d\theta$$
$$= a^2\left|3\theta + \frac{4\sin 2\theta}{2} - 2\sin\theta\right|_0^{\frac{\pi}{3}}$$

$$= a^2\left(3\frac{\pi}{3} + 2\sin\frac{2\pi}{3} - 2\sin\frac{\pi}{3}\right)$$

$$= \pi a^2$$

Example 5

Find the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1 - \cos\theta)$. **[Summer 2016]**

Solution

1. The points of intersection of circle $r = a\sin\theta$ and the cardioid $r = a(1 - \cos\theta)$ are obtained as

$$a\sin\theta = a(1-\cos\theta)$$

$$2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\sin^2\frac{\theta}{2}$$

$$\sin\frac{\theta}{2} = 0,\ \tan\frac{\theta}{2} = 1$$

$$\frac{\theta}{2} = 0,\ \frac{\theta}{2} = \frac{\pi}{4}$$

$$\theta = 0,\ \theta = \frac{\pi}{2}$$

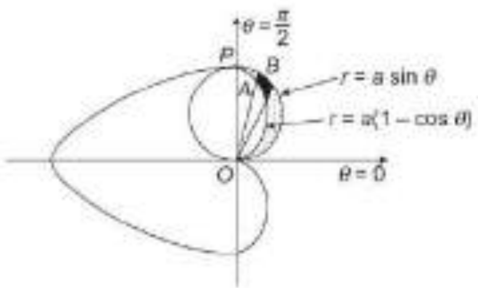

Fig. 9.157

Hence, $\theta = 0$ at origin and $\theta = \frac{\pi}{2}$ at P.

2. Draw an elementary radius vector OAB from origin in the region. OAB enters in the region from the cardioid $r = a(1 - \cos\theta)$ and terminates on the circle $r = a\sin\theta$.

Limit of r: $r = a(1 - \cos\theta)$ to $r = a\sin\theta$

Limit of θ: $\theta = 0$ to $\theta = \frac{\pi}{2}$

$$A = \int_0^{\frac{\pi}{2}}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left|\frac{r^2}{2}\right|_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\left[a^2\sin^2\theta - a^2(1-\cos\theta)^2\right]d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}}\left[\sin^2\theta - (1 - 2\cos\theta + \cos^2\theta)\right]d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}}\left[\sin^2\theta - \cos^2\theta + 2\cos\theta - 1\right]d\theta$$

$$= \frac{a^2}{2}\left[\int_0^{\frac{\pi}{2}}(-\cos 2\theta)d\theta + 2\int_0^{\frac{\pi}{2}}\cos\theta\, d\theta - \int_0^{\frac{\pi}{2}}d\theta\right]$$

$$= \frac{a^2}{2}\left[-\left|\frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{2}} + 2\left|\sin\theta\right|_0^{\frac{\pi}{2}} - \left|\theta\right|_0^{\frac{\pi}{2}}\right]$$

$$= \frac{a^2}{2}\left[-\frac{1}{2}(\sin\pi - \sin\theta) + 2(\sin\frac{\pi}{2} - \sin 0) - \frac{\pi}{2}\right]$$

$$= \frac{a^2}{2}\left[2 - \frac{\pi}{2}\right]$$

$$= a^2\left(1 - \frac{\pi}{4}\right)$$

Example 6

Find the area common to the cardioids $r = a(1 + \cos\theta)$ and $r = a(1 - \cos\theta)$.

Solution

1. The points of intersection of the cardioids

   $r = a(1+\cos\theta)$ and $r = a(1-\cos\theta)$

   are obtained as

   $$a(1+\cos\theta) = a(1-\cos\theta)$$
   $$\cos\theta = 0$$
   $$\theta = \pm\frac{\pi}{2}$$

   Hence, $\theta = \frac{\pi}{2}$ at *P*.

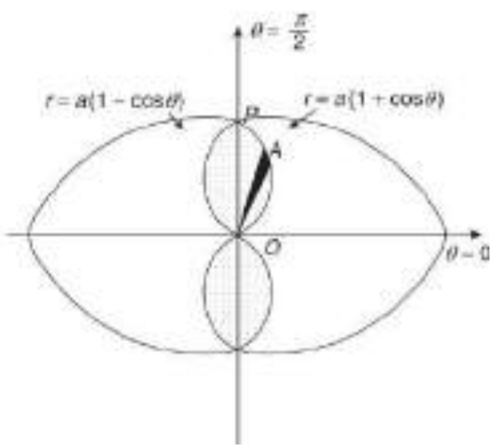

Fig. 9.158

2. The region is symmetric in all the quadrants

   Total area = 4 (area in the first quadrant)

3. Draw an elementary radius vector *OA* from the origin in the region which lies in the first quadrant. *OA* starts from the origin and terminates on the cardioid $r = a(1 - \cos\theta)$.

Limits of $r: r = 0$ to $r = a(1-\cos\theta)$

Limits of $\theta: \theta = 0$ to $\theta = \frac{\pi}{2}$

$$A = 4\int_0^{\frac{\pi}{2}}\int_0^{a(1-\cos\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 4\int_0^{\frac{\pi}{2}}\left|\frac{r^2}{2}\right|_0^{a(1-\cos\theta)}\mathrm{d}\theta$$

$$= 2\int_0^{\frac{\pi}{2}} a^2(1-\cos\theta)^2\,\mathrm{d}\theta$$

$$= 2a^2\int_0^{\frac{\pi}{2}}(1-2\cos\theta+\cos^2\theta)\mathrm{d}\theta$$

$$= 2a^2\int_0^{\frac{\pi}{2}}\left(1-2\cos\theta+\frac{1+\cos 2\theta}{2}\right)\mathrm{d}\theta$$

$$= 2a^2\left|\frac{3}{2}\theta-2\sin\theta+\frac{\sin 2\theta}{4}\right|_0^{\frac{\pi}{2}}$$

$$= 2a^2\left(\frac{3\pi}{4}-2\right)$$

Example 7

Find the area inside the cardioid $r = 3(1+\cos\theta)$ and outside the parabola $r = \dfrac{3}{1+\cos\theta}$.

Solution

1. The points of intersection of the cardioid $r = 3(1+\cos\theta)$ and the parabola $r = \dfrac{3}{1+\cos\theta}$ are obtained as

$$3(1+\cos\theta) = \frac{3}{(1+\cos\theta)}$$

$$(1+\cos\theta)^2 = 1$$

$$\cos\theta = 0$$

$$\theta = \pm\frac{\pi}{2}$$

Hence, $\theta = \frac{\pi}{2}$ at P.

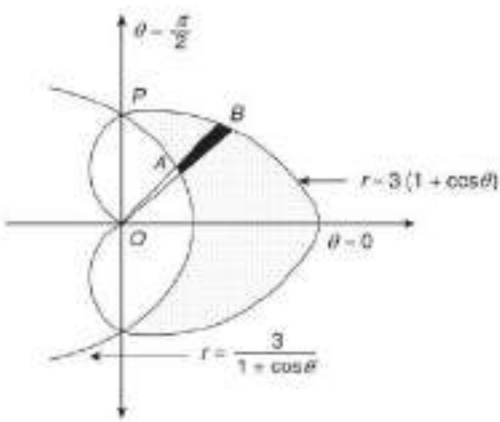

Fig. 9.159

2. The region is symmetric about the initial line $\theta = 0$.

   Total area = 2 (area above the initial line)

3. Draw an elementary radius vector OAB from the origin in the region above the initial line $\theta = 0$. OAB enters in the region from the parabola $r = \frac{3}{1+\cos\theta}$ and terminates on the cardioid $r = 3(1+\cos\theta)$.

   Limits of $r : r = \frac{3}{1+\cos\theta}$ to $r = 3(1+\cos\theta)$

   Limits of $\theta : \theta = 0$ to $\theta = \frac{\pi}{2}$

$$A = 2\int_0^{\frac{\pi}{2}} \int_{\frac{3}{1+\cos\theta}}^{3(1+\cos\theta)} r\, dr\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left|\frac{r^2}{2}\right|_{\frac{3}{1+\cos\theta}}^{3(1+\cos\theta)} d\theta$$

$$= \int_0^{\frac{\pi}{2}} 9\left[(1+\cos\theta)^2 - \frac{1}{(1+\cos\theta)^2}\right] d\theta$$

$$= 9\int_0^{\frac{\pi}{2}} \left[1 + 2\cos\theta + \frac{1+\cos 2\theta}{2} - \frac{1}{\left(2\cos^2\frac{\theta}{2}\right)^2}\right] d\theta$$

$$= 9\int_0^{\frac{\pi}{2}} \left[\frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2} - \frac{1}{4}\left(1+\tan^2\frac{\theta}{2}\right)\sec^2\frac{\theta}{2}\right] d\theta$$

$$= 9\int_0^{\frac{\pi}{2}} \left[\frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2} - \frac{1}{4}\sec^2\frac{\theta}{2} - \frac{1}{2}\cdot\tan^2\frac{\theta}{2}\left(\frac{1}{2}\sec^2\frac{\theta}{2}\right)\right] d\theta$$

$$= 9\left|\frac{3\theta}{2} + 2\sin\theta + \frac{\sin 2\theta}{4} - \frac{1}{4}\cdot 2\tan\frac{\theta}{2} - \frac{1}{2}\frac{\tan^3\frac{\theta}{2}}{3}\right|_0^{\frac{\pi}{2}}$$

$$\left[\because \int [f(\theta)]^n f'(\theta)\, d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= 9\left(\frac{3\pi}{4} + 2\sin\frac{\pi}{2} + \frac{\sin\pi}{4} - \frac{1}{2}\tan\frac{\pi}{4} - \frac{1}{6}\tan^3\frac{\pi}{4}\right)$$

$$= 9\left(\frac{3\pi}{4} + \frac{4}{3}\right)$$

Example 8

Find the area common to both the circles $r = \cos\theta$ and $r = \sin\theta$.

[Winter 2013]

Solution

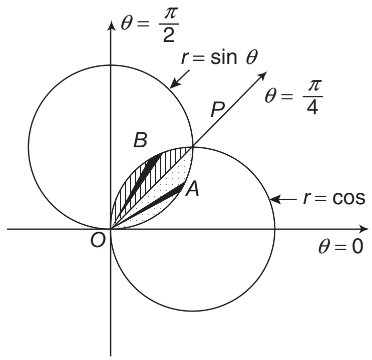

Fig. 9.160

1. So the point of intersection of the circles $r = \cos\theta$ and $r = \sin\theta$ is obtained as

$$\cos\theta = \sin\theta$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4}$$

   Hence, $\theta = \frac{\pi}{4}$ is the point of intersection
2. The point of intersection divides the region into two subregions *OAP* and *OBP*.
3. Draw an elementary radius in each subregion.
   (i) In subregion *OAP*, radius vector *OA* starts from the origin and terminates on the circle $r = \sin\theta$.
   Limit of r: $r = 0$ to $r = \sin\theta$
   Limit of θ: $\theta = 0$ to $\theta = \frac{\pi}{4}$
   (ii) In the subregion *OBP*, the radius vector *OB* starts from the origin and terminates on the circle $r = \cos\theta$.
   Limit of r: $r = 0$ to $r = \cos\theta$
   Limit of θ: $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$

$$A = \int_0^{\frac{\pi}{4}}\int_0^{\sin\theta} r\,\mathrm{d}r\,\mathrm{d}\theta + \int_0^{\frac{\pi}{2}}\int_0^{\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{4}} \left|\frac{r^2}{2}\right|_0^{\sin\theta} \mathrm{d}\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left|\frac{r^2}{2}\right|_0^{\cos\theta} \mathrm{d}\theta$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \sin^2\theta\,\mathrm{d}\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta\right]$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \left(\frac{1-\cos 2\theta}{2}\right)\mathrm{d}\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)\mathrm{d}\theta\right]$$

$$= \frac{1}{4}\left[\int_0^{\frac{\pi}{4}}(1-\cos 2\theta)\,d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta\right]$$

$$= \frac{1}{4}\left[\left|\theta - \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{4}} + \left|\theta + \frac{\sin 2\theta}{2}\right|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right]$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{4} - \frac{1}{2}\right) + \left(\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{4}\left(\frac{\pi}{2} - 1\right)$$

$$= \frac{\pi}{8} - \frac{1}{4}$$

Example 9

Find the area common to the circles $r = \cos\theta$ and $r = \sqrt{3}\sin\theta$.

Solution

1. The point of intersection of the circles $r = \cos\theta$ and $r = \sqrt{3}\sin\theta$ is obtained as

$$\sqrt{3}\sin\theta = \cos\theta$$

$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6}$$

Hence, $\theta = \frac{\pi}{6}$ at P.

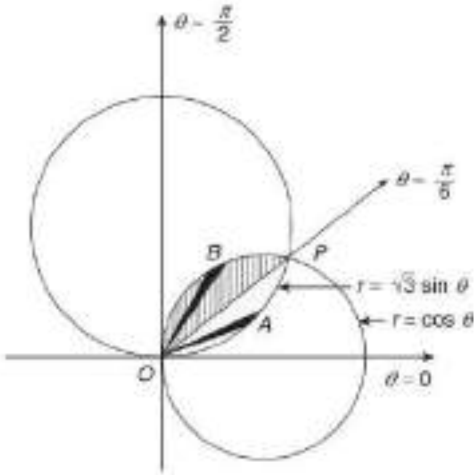

Fig. 9.161

2. Divide the region *OAPBO* into two subregions *OAP* and *OBP*. Draw an elementary radius vector in each subregion.

   (i) In subregion *OAP*, radius vector *OA* starts from the origin and terminates on the circle $r = \sqrt{3}\sin\theta$.

   Limits of $r: r = 0$ to $r = \sqrt{3}\sin\theta$

   Limits of $\theta: \theta = 0$ to $\theta = \frac{\pi}{6}$

(ii) In the subregion *OBP*, the radius vector *OB* starts from the origin and terminates on the circle $r = \cos\theta$.

Limits of $r : r = 0$ to $r = \cos\theta$

Limits of $\theta : \theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{2}$

$$A = \int_0^{\frac{\pi}{6}}\int_0^{\sqrt{3}\sin\theta} r\,dr\,d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{\cos\theta} r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{6}}\left|\frac{r^2}{2}\right|_0^{\sqrt{3}\sin\theta} d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left|\frac{r^2}{2}\right|_0^{\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} 3\sin^2\theta\,d\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{6}}\left(\frac{1-\cos 2\theta}{2}\right)d\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{3}{4}\left|\theta - \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{6}} + \frac{1}{4}\left|\theta + \frac{\sin 2\theta}{2}\right|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{3}{4}\left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right) + \frac{1}{4}\left(\frac{\pi}{2} - \frac{\pi}{6} + \frac{1}{2}\sin\pi - \frac{1}{2}\sin\frac{\pi}{3}\right)$$

$$= \frac{5\pi}{24} - \frac{\sqrt{3}}{4}$$

Example 10

Find the area common to the circle r = a and the cardioid r = a(1 + cos θ).

Solution

1. The points of intersection of the circle $r = a$ and the cardioid $r = a(1 + \cos\theta)$ are obtained as

$$a = a(1+\cos\theta)$$
$$\cos\theta = 0$$
$$\theta = \pm\frac{\pi}{2}$$

Hence, $\theta = \frac{\pi}{2}$ at Q.

2. The region is symmetric about the initial line $\theta = 0$.

   Total area = 2 (area above the initial line)

3. Divide the region *OPQO* above the initial line into two subregions *OPQ* and *OBQ*. Draw an elementary radius vector in each subregion.
   (i) In the subregion *OPQ* the radius vector *OA* starts from the origin and terminates on the circle $r = a$.
   Limits of $r: r = 0 \quad \text{to} \quad r = a$

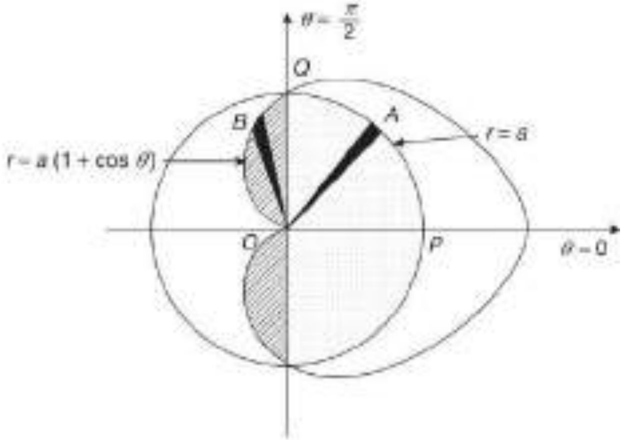

Fig. 9.162

   Limits of $\theta: \theta = 0 \quad \text{to} \quad \theta = \frac{\pi}{2}$

   (ii) In the subregion *OBQ*, radius vector *OB* starts from the origin and terminates on the cardioid $r = a\,(1 + \cos\theta)$.
   Limits of $r: r = 0 \quad \text{to} \quad r = a(1+\cos\theta)$

   Limits of $\theta: \theta = \frac{\pi}{2} \quad \text{to} \quad \theta = \pi$

$$A = 2\left(\int_0^{\frac{\pi}{2}}\int_0^a r\,\mathrm{d}r\,\mathrm{d}\theta + \int_{\frac{\pi}{2}}^{\pi}\int_0^{a(1+\cos\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta\right)$$

$$= 2\left(\int_0^{\frac{\pi}{2}}\left|\frac{r^2}{2}\right|_0^a \mathrm{d}\theta + \int_{\frac{\pi}{2}}^{\pi}\left|\frac{r^2}{2}\right|_0^{a(1+\cos\theta)} \mathrm{d}\theta\right)$$

$$= \int_0^{\frac{\pi}{2}} a^2\,\mathrm{d}\theta + \int_{\frac{\pi}{2}}^{\pi} a^2(1+\cos\theta)^2\,\mathrm{d}\theta$$

$$= a^2 |\theta|_0^{\frac{\pi}{2}} + a^2 \int_{\frac{\pi}{2}}^{\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= a^2 . \frac{\pi}{2} + a^2 \left| \frac{3\theta}{2} + 2\sin\theta + \frac{\sin 2\theta}{4} \right|_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{\pi a^2}{2} + \frac{3a^2}{2}\left(\pi - \frac{\pi}{2}\right) + 2a^2\left(\sin\pi - \sin\frac{\pi}{2}\right) + \frac{a^2}{4}(\sin 2\pi - \sin\pi)$$

$$= a^2\left(\frac{5\pi}{4} - 2\right)$$

Example 11

Find the area between the curve $r = a(\sec\theta + \cos\theta)$ *and its asymptote* $r = a \sec\theta$.

Solution

1. The region is symmetric about the initial line $\theta = 0$
   Total area = 2(area above the initial line)
2. Draw an elementary radius vector *OAB* in the region above the initial line.
   OAB enters in the region from the line $r = a \sec\theta$ and terminates on the curve $r = a(\sec\theta + \cos\theta)$.
   Limits of r : $r = a \sec\theta$ to $r = a(\sec\theta + \cos\theta)$
   Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{2}$

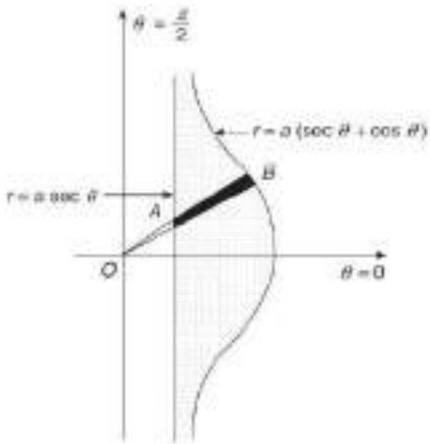

Fig. 9.163

$$A = 2\int_0^{\frac{\pi}{2}} \int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\, dr\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left| \frac{r^2}{2} \right|_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$

$$= \int_0^{\frac{\pi}{2}} [a^2(\sec\theta + \cos\theta)^2 - a^2\sec^2\theta] d\theta$$

$$= a^2 \int_0^{\frac{\pi}{2}} (\cos^2\theta + 2) d\theta$$

$$= a^2\left[\frac{1}{2}B\left(\frac{3}{2},\frac{1}{2}\right)+|2\theta|_0^{\frac{\pi}{2}}\right]$$

$$= a^2\left[\frac{1}{2}\frac{\overline{|\frac{3}{2}}\,\overline{|\frac{1}{2}}}{\overline{|2}}+\frac{2\pi}{2}\right]$$

$$= \frac{5\pi}{4}a^2$$

Example 12

Find the area of the loop of the curve $x^4+y^4=8xy$.

Solution

1. The equation of the curve in polar form is $r^4(\cos^4\theta+\sin^4\theta)=8r^2\cos\theta\sin\theta$

$$r^2=\frac{8\cos\theta\sin\theta}{\cos^4\theta+\sin^4\theta}$$

2. Draw an elementary radius vector *OA* from the origin in the region which lies in the first quadrant. *OA* starts from the origin and terminates on the curve $r^2=\frac{8\cos\theta\sin\theta}{\cos^4\theta+\sin^4\theta}$.

   Limits of

   $r: r=0$ to $r=\sqrt{\frac{8\cos\theta\sin\theta}{\cos^4\theta+\sin^4\theta}}$

   Limits of $\theta:\theta=0$ to $\theta=\frac{\pi}{2}$

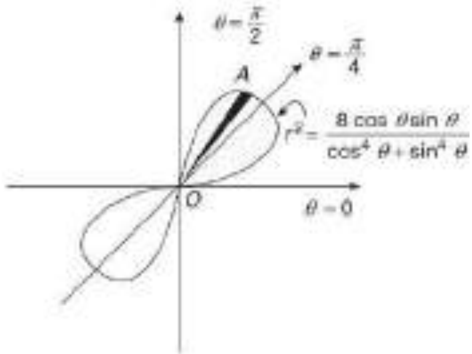

Fig. 9.164

$$A=\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{\frac{8\cos\theta\sin\theta}{\cos^4\theta+\sin^4\theta}}} r\,dr\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\left|\frac{r^2}{2}\right|_0^{\sqrt{\frac{8\cos\theta\sin\theta}{\cos^4\theta+\sin^4\theta}}}d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(\frac{8\cos\theta\sin\theta}{\cos^4\theta+\sin^4\theta}\right)d\theta$$

$$=4\int_0^{\frac{\pi}{2}}\left(\frac{\tan\theta\sec^2\theta}{1+\tan^4\theta}\right)d\theta$$

Putting $\tan^2\theta = t$, $2\tan\theta\sec^2\theta\, d\theta = dt$

When $\theta = 0, t = 0$

When $\theta = \frac{\pi}{2}, t \to \infty$

$$A = 2\int_0^{\infty} \frac{dt}{1+t^2}$$

$$= 2\left|\tan^{-1} t\right|_0^{\infty}$$

$$= 2\left(\frac{\pi}{2}\right)$$

$$= \pi$$

EXERCISE 9.9

1. Find the area common to the circles $r = a$ and $r = 2a\cos\theta$.

$$\left[\textbf{Ans.}: a^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)\right]$$

2. Find the area of the crescent bounded by the circles $r = \sqrt{2}$ and $r = 2\cos\theta$.

[**Ans.**: 1]

3. Find the area which lies inside the cardioid $r = 2a(1 + \cos\theta)$ and outside the parabola $r = \frac{2a}{1+\cos\theta}$.

$$\left[\textbf{Ans.}: 3\pi a^2 + \frac{16a^2}{3}\right]$$

4. Find the area bounded between the circles $r = 2a\sin\theta$, $r = 2b\sin\theta$ $(b > a)$.

$$\left[\textbf{Ans.}: \pi(b^2 - a^2)\right]$$

5. Find the area outside the circle $r = a$ and inside the cardioid $r = a(1 + \cos\theta)$.

$$\left[\textbf{Ans.}: \frac{a^2}{4}(\pi + 8)\right]$$

Points to Remember

Double Integrals

The double integral of a function $f(x, y)$ over the region R is denoted by $\iint_R f(x, y)\,dx\,dy$.

Double Integrals over Rectangles and General Regions

Method-I: When the region R is bounded by the curves $y = y_1(x)$, $y = y_2(x)$ and $x = a$, $x = b$,

$$\iint_R f(x, y)\,dx\,dy = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x, y)\,dy\right] dx$$

Method-II: When the region R is bounded by the curves $x = x_1(y)$, $x = x_2(y)$ and $y = c$, $y = d$,

$$\iint_R f(x, y)\,dx\,dy = \int_c^d \left[\int_{x_1(y)}^{x_2(y)} f(x, y)\,dx\right] dy$$

If all the four limits are constant and $f(x, y)$ is explicit, then the $f(x, y)$ can be integrated w.r.t. any variable first and also can be written as product of two single integrals.

Change of Order of Integration

Sometimes evaluation of double integral becomes easier by changing the order of integration. To change the order of integration,

1. Draw the region of integration with the help of the given limits.
2. Draw vertical or horizontal strip as per the required order of integration
3. Find the limits of integration

$$\int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x, y)\,dy\right] dx = \int_c^d \left[\int_{x_1(y)}^{x_2(y)} f(x, y)\,dx\right] dy$$

Double Integrals in Polar Coordinates

Putting $x = r\cos\theta$, $y = r\sin\theta$,

$$\iint f(x, y)\,dy\,dx = \iint f(r\cos\theta, r\sin\theta)|J|\,dr\,d\theta$$

where Jacobian, $J = r$

Hence, $$\iint f(x, y)\,dy\,dx = \iint f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Triple Integrals

The triple integral of a continuous function $f(x, y, z)$ over a region V is denoted by $\iiint_V f(x, y, z)\,dx\,dy\,dz$.

Triple Integrals in Cartesian Coordinates

If the region V is bounded below by a surface $z = z_1(x, y)$ and above by a surface $z = z_2(x, y)$ and if the projection of region V in xy-plane is R which is bounded by the curves $y = y_1(x)$, $y = y_2(x)$ and $x = a$, $x = b$ then

$$\iiint_V f(x, y, z)\,dx\,dy\,dz = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} \left\{ \int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z)\,dz \right\} dy \right] dx$$

Triple Integrals in Cylindrical Coordinates

Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$,

$$\iiint f(x, y, z)\,dx\,dy\,dz = \iiint f(r\cos\theta, r\sin\theta, z)\,|J|\,dz\,dr\,d\theta$$

where Jacobian, $J = r$

Hence, $\iiint f(x, y, z)\,dx\,dy\,dz = \iiint f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$

Triple Integrals in Spherical Coordinates

Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$,

$$\iiint f(x, y, z)\,dx\,dy\,dz = \iiint f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\,|J|\,dr\,d\theta\,d\phi$$

where Jacobian, $J = r^2\sin\theta$

Hence, $\iiint f(x, y, z)\,dx\,dy\,dz$

$$= \iiint f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\cdot r^2 \sin\theta\,dr\,d\theta\,d\phi$$

Area by Double Integrals

Area in Cartesian Coordinates

(i) The area bounded by the curves $y = y_1(x)$ and $y = y_2(x)$ intersecting at the points $P(a, b)$ and $Q(c, d)$ is

$$A = \int_a^c \int_{y_1(x)}^{y_2(x)} dy\,dx$$

(ii) The area bounded by the curves $x = x_1(y)$ and $x = x_2(y)$ and intersecting at the points $P(a, b)$ and $Q(c, d)$ is

$$A = \int_a^c \int_{y_1(x)}^{y_2(x)} dy\,dx$$

Area in Polar Coordinates

The area bounded by the curves $r = r_1(\theta)$, $r = r_2(\theta)$ and the lines $\theta = \theta_1$ and $\theta = \theta_2$ is

$$A = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} r\,dr\,d\theta$$

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. To evaluate $\int_0^\infty \int_0^x xe^{-\frac{x^2}{y}}\, dx\, dy$ by change of order of integration, the lower limit for the variable x is equal to

(a) y^2 (b) 0 (c) ∞ (d) y

2. $\int_0^4 \int_0^3 \int_0^2 dx\, dy\, dz =$

(a) 9 (b) 24 (c) 1 (d) 0

3. By changing the order of integration, the integral $\int_0^2 \int_1^{e^x} dy\, dx$ is equivalent to the double integral_______.

(a) $\int_1^e \int_{\log y}^2 dx\, dy$ (b) $\int_1^{e^2} \int_{\log y}^2 dx\, dy$

(c) $\int_{e^2}^1 \int_2^{\log y} dx\, dy$ (d) $\int_1^{e^2} \int_2^{\log y} dx\, dy$

4. By changing to spherical polar co-ordinates, $\iiint_R dy\, dx\, dz,$ where R is the region of hemisphere $x^2 + y^2 + z^2 = a^2$ is equivalent to triple integral ______

(a) $\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^a r^2 \sin\theta\, dr\, d\theta\, d\phi$ (b) $\int_0^{2\pi} \int_0^{\pi} \int_0^a r^2 \sin\theta\, dr\, d\theta\, d\phi$

(c) $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a r^2 \sin\theta\, dr\, d\theta\, d\phi$ (d) $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a r^2 \cos\theta\, dr\, d\theta\, d\phi$

5. $\int_0^a \int_0^x \int_0^y xyz\, dz\, dy\, dx =$

(a) $\frac{a^6}{24}$ (b) $\frac{a^6}{48}$ (c) $\frac{a^4}{48}$ (d) $\frac{a^4}{24}$

6. In evaluating $\iint xy(x + y)\, dx\, dy$ over the region between $y = x^2$ and $y = x$, the limits are

(a) $x = 0$ to 1, $xy = 0$ to 1 (b) $x = 0$ to 1, $y = 0$ to x

(c) $x = 0$ to 1, $y = 0$ to x^2 (d) $x = 0$ to 1, $y = x^2$ to x

7. $\int_0^a \int_0^{\sqrt{a^2 - y^2}} (a^2 - x^2 - y^2)\, dx\, dy =$

(a) $\frac{\pi a^4}{8}$ (b) $\frac{\pi a^4}{4}$ (c) $\frac{\pi a^4}{2}$ (d) πa^4

8. After transforming to polar co-ordinates $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy =$

(a) $\int_0^{\frac{\pi}{2}} \int_0^1 e^{-r^2} dr\, d\theta$ (b) $\int_0^{\frac{\pi}{2}} \int_0^1 e^{-r^2} r\, dr\, d\theta$

(c) $\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r\, dr\, d\theta$ (d) $\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-y^2} r\, dr\, d\theta$

9. $\int_0^\pi \int_0^{a\cos\theta} r\sin\theta\, dr\, d\theta =$

(a) $\frac{a^2}{4}$ (b) $\frac{a^2}{3}$ (c) $\frac{a^2}{2}$ (d) $\frac{a^2}{6}$

10. $\int_0^1 \int_0^2 xy^2\, dy\, dx =$

(a) $\frac{5}{3}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{4}{3}$

11. $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin(x+y)\, dx\, dy$ is

(a) 0 (b) π (c) $\frac{\pi}{2}$ (d) 2

12. The value of the integral $\iint xy\, dx\, dy$ over the region bounded by the x-axis, ordinate at $x = 2a$ and the parabola $x^2 = 4ay$ is

(a) $\frac{a^4}{3}$ (b) $\frac{a^4}{5}$ (c) $\frac{a^4}{7}$ (d) $\frac{a^4}{9}$

13. The triple integral $\iiint_R dx\, dy\, dz$ gives

(a) volume (b) area (c) surface area (d) density

14. The value of $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dz\, dx\, dy$ is

(a) $\frac{4}{35}$ (b) $\frac{3}{35}$ (c) $\frac{8}{35}$ (d) $\frac{6}{35}$

15. The value of the integral $\int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 r^2 \sin\theta\, dr\, d\theta\, d\phi$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{4}$

16. The value of the integral $\int_0^\infty \int_0^\infty e^{-x^2(1+y^2)} x\, dx\, dy$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

17. Changing the order of integration in the double integral $\int_0^\infty \int_{\frac{x}{4}}^{2} f(x,y)\,dy\,dx$ leads to $\int_r^s \int_p^q f(x,y)\,dx\,dy$, then q is

(a) $4y$ (b) $16y^2$ (c) x (d) 8

18. The limits of integration of $\iint (x^2 + y^2)\,dx\,dy$ over the domain bounded by $y = x^2$ and $y^2 = x$ are

(a) $x = 0$ to 1, $y = x^2$ to $\sqrt{x}$ (b) $x = 0$ to 1, $y = 0$ to 1

(c) $x = y^2$ to $\sqrt{y}$, $y = 0$ to 1 (d) $x = 0$ to y, $y = \sqrt{x}$ to x^2

19. $\iint \frac{xy}{\sqrt{1-y^2}}\,dx\,dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$ is

(a) $\frac{1}{6}$ (b) $\frac{2}{3}$ (c) $\frac{5}{6}$ (d) $\frac{5}{3}$

20. $\iint r^3\,dr\,d\theta$ over the region included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$ is

(a) $\int_0^{\pi} \int_{2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta$ (b) $\int_0^{\frac{\pi}{2}} \int_{2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta$

(c) $\int_{-\pi}^{\pi} \int_{2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta$ (d) $\int_0^{\frac{\pi}{2}} \int_{\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta$

21. On converting into polar co-ordinates $\int_0^a \int_0^{\sqrt{a^2 - x^2}} (x^2 + y^2)\,dy\,dx =$

(a) $\int_0^a \int_0^{\frac{\pi}{2}} r^2\,dr\,d\theta$ (b) $\int_0^a \int_0^{\frac{\pi}{2}} r^3\,dr\,d\theta$

(c) $\int_0^a \int_0^{\frac{\pi}{4}} r^3\,dr\,d\theta$ (d) $\int_0^a \int_0^{\frac{\pi}{4}} r^2\,dr\,d\theta$

22. In spherical co-ordinates, $dx\,dy\,dz$ is equal to

(a) $r\,d\theta\,d\phi\,dr$ (d) $r\sin\theta\,d\theta\,d\phi\,dr$ (c) $r^2\sin\theta\,d\theta\,d\phi\,dr$ (d) $r^2\,d\theta\,d\phi\,dr$

23. The value of the integral $\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2)\,dz\,dy\,dx$ is

(a) 1 (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) 3

24. The value of $\int_0^a dx \int_0^{\sqrt{a^2 - x^2}} dy \int_0^{\sqrt{a^2 - x^2 - y^2}} dz$ is

(a) $4\pi a^2$ (b) $\frac{\pi a^3}{6}$ (c) $4\pi a^3$ (d) $\frac{\pi}{3} a^2$

25. The transformations $x + y = u$, $y = uv$ transform the area element $\mathrm{d}y\,\mathrm{d}x$ into $|J|\,\mathrm{d}u\,\mathrm{d}v$, where $|J|$ is equal to

(a) 1 (b) u (c) –1 (d) u^2

26. The value of $\iint_R x^3 y\,\mathrm{d}x\,\mathrm{d}y$, where R is region enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the first quadrant is

(a) $\frac{b^2a^4}{24}$ (b) $\frac{b^3a^4}{24}$ (c) $\frac{ba^4}{24}$ (d) $\frac{b^2a^2}{24}$

27. By changing the order of integration, $\int_0^1\int_0^x f(x,y)\mathrm{d}y\,\mathrm{d}x =$

(a) $\int_1^0\int_1^y f(x,y)\mathrm{d}x\,\mathrm{d}y$ (b) $\int_1^0\int_y^1 f(x,y)\mathrm{d}x\,\mathrm{d}y$

(c) $\int_0^1\int_1^y f(x,y)\mathrm{d}x\,\mathrm{d}y$ (d) $\int_0^1\int_y^1 f(x,y)\mathrm{d}x\,\mathrm{d}y$

28. $\int_0^{2\pi}\mathrm{d}\theta\int_0^1 e^{2r}\,\mathrm{d}r$ is equal to

(a) $e^2 - 1$ (b) $\frac{\pi}{2}(e^2-1)$ (c) $\pi(e^2 - 1)$ (d) $2\pi(e^2 - 1)$

29. The value of $\iint_R x^2y^3\,\mathrm{d}x\,\mathrm{d}y$, where R is the region bounded by the rectangle $0 \le x \le 1$ and $0 \le y \le 3$ is

(a) $\frac{27}{4}$ (b) $\frac{27}{8}$ (c) $\frac{29}{4}$ (d) $\frac{29}{8}$

30. The value of $\iint 3y\,\mathrm{d}x\,\mathrm{d}y$ over the triangle with vertices (–1, 1), (0, 0) and (1, 1) is **[Winter 2015]**

(a) 0 (b) 1 (c) 2 (d) 3

31. The area of the curve $y = x^2 + 1$ bounded by the x-axis and the line $x = 1$ and $x = 2$ is **[Summer 2014]**

(a) $\frac{3}{10}$ (b) $\frac{10}{3}$ (c) 6 (d) $\frac{1}{6}$

32. The equation of a cylindrical surface $x^2 + y^2 = 9$ becomes _____ when converted to cylindrical polar coordinates. **[Summer 2016]**

(a) $r = 9$ (b) $r^2 = 9$ (c) $r = \pm 3$ (d) $r = 3$

33. $\int_0^2\int_0^{x^2} e^{\frac{y}{x}}\,\mathrm{d}y\mathrm{d}x$ is equal to **[Summer 2016]**

(a) $e^2 - 1$ (b) e^2 (c) $e^2 + 1$ (d) e^{-2}

34. $\int_1^2 \int_1^2 \frac{1}{xy} dxdy =$ **[Winter 2016]**

(a) 0 (b) $(\log 2)^2$ (c) 1 (d) log 2

35. The region of $\int_1^4 \int_2^6 dx\,dy$ represents **[Winter 2016]**

(a) rectangle (b) square (c) circle (d) triangle

36. The region $\int_1^2 \int_1^2 dx\,dy$ represents **[Summer 2017]**

(a) rectangle (b) square (c) circle (d) triangle

37. The value of $\int_0^1 \int_0^1 (3x^2 - 2y^2)\,dx\,dy$ is **[Summer 2017]**

(a) 0 (b) 1 (c) –1 (d) $\frac{1}{3}$

Answers

1. (d)	2. (b)	3. (b)	4. (a)	5. (b)	6. (d)	7. (a)	8. (c)	9. (b)
10. (d)	11. (d)	12. (a)	13. (a)	14. (a)	15. (a)	16. (c)	17. (a)	18. (a)
19. (a)	20. (a)	21. (b)	22. (c)	23. (a)	24. (b)	25. (b)	26. (a)	27. (d)
28. (c)	29. (a)	30. (c)	31. (b)	32. (d)	33. (a)	34. (b)	35. (a)	36. (a)
37. (d)								

UNIT-6

Chapter 10. Matrices

CHAPTER 10

Matrices

Chapter Outline

10.1 INTRODUCTION

A matrix is a rectangular table of elements which may be numbers or any abstract quantities that can be added and multiplied. Matrices are used to describe linear equations, keep track of the coefficients of linear transformation and record data that depend on multiple parameters. There are many applications of matrices in mathematics, viz. graph theory, probability theory, statistics, computer graphics, geometrical optics, etc.

10.2 MATRIX

A set of mn elements (real or complex) arranged in a rectangular array of m rows and n columns is called a matrix of order m by n, written as $m \times n$.

An $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

The matrix can also be expressed in the form $A = [a_{ij}]_{m \times n}$, where a_{ij} is the element in the i^{th} row and j^{th} column, written as $(i, j)^{\text{th}}$ element of the matrix.

10.3 SOME DEFINITIONS ASSOCIATED WITH MATRICES

1. Row Matrix

A matrix having only one row and any number of columns, is called a row matrix or row vector, e.g.

$$[2 \quad 5 \quad -3 \quad 4]$$

2. Column Matrix

A matrix, having only one column and any number of rows, is called a column matrix or column vector, e.g.

$$\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

3. Zero or Null Matrix

A matrix, whose all the elements are zero, is called zero or null matrix and is denoted by **0**, e.g.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4. Square Matrix

A matrix, in which the number of rows is equal to the number of columns, is called a square matrix, e.g.

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 2 \\ -1 & 4 & -5 \\ 2 & 6 & 8 \end{bmatrix}$$

5. Diagonal Matrix

A square matrix, whose all non-diagonal elements are zero and at least one diagonal element is non-zero, is called a diagonal matrix. e.g.

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

6. Unit or Identity Matrix

A diagonal matrix, whose all diagonal elements are unity, is called a unit or identity matrix and is denoted by *I*, e.g.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7. Scalar Matrix

A square matrix, whose all diagonal elements are equal, is called a scalar matrix, e.g.

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

8. Upper Triangular Matrix

A square matrix, in which all the elements below the diagonal are zero, is called an upper triangular matrix, e.g.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & -5 \\ 0 & 0 & 8 \end{bmatrix}$$

9. Lower Triangular Matrix

A square matrix, in which all the elements above the diagonal are zero, is called a lower triangular matrix, e.g.

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ 2 & 6 & 8 \end{bmatrix}$$

10. Trace of a Matrix

The sum of all the diagonal elements of a square matrix is called the trace of a matrix,

e.g.
$$A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 6 & -2 \\ -1 & 0 & 3 \end{bmatrix}$$

$$\text{Trace of } A = 2 + 6 + 3 = 11$$

11. Transpose of a Matrix

A matrix obtained by interchanging rows and columns of a matrix is called transpose of a matrix and is denoted by A' or A^T, e.g.

If
$$A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 6 \\ -4 & 1 & 5 \end{bmatrix} \text{ then } \quad A^T = \begin{bmatrix} 1 & 0 & -4 \\ -1 & 2 & 1 \\ 3 & 6 & 5 \end{bmatrix}$$

i.e.,if $A = [a_{ij}]_{m\times n}$, then $A^T = [a_{ji}]_{n\times m}$

12. Symmetric Matrix

A square matrix $A = [a_{ij}]_{m\times m}$ is called symmetric if $a_{ij} = a_{ji}$ for all i and j, i.e., $A = A^T$, e.g.

$$\begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} 1 & i & -3i \\ i & -2 & 4 \\ -3i & 4 & 3 \end{bmatrix}$$

13. Skew-Symmetric Matrix

A square matrix $A = [a_{ij}]_{m\times m}$ is called skew-symmetric if $a_{ij} = -a_{ji}$ for all i and j, i.e., $A = -A^T$. Thus, the diagonal elements of a skew-symmetric matrix are all zero, e.g.

$$\begin{bmatrix} 0 & -3i & -4 \\ 3i & 0 & 8 \\ 4 & -8 & 0 \end{bmatrix}$$

14. Conjugate of a Matrix

A matrix obtained from any given matrix A, on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\bar{A}$, e.g.

$$A = \begin{bmatrix} 1+3i & 2+5i & 8 \\ -i & 6 & 9-i \end{bmatrix}, \qquad \bar{A} = \begin{bmatrix} 1-3i & 2-5i & 8 \\ i & 6 & 9+i \end{bmatrix}$$

15. Transposed Conjugate of a Matrix

The conjugate of the transpose of a matrix A is called the conjugate transpose or transposed conjugate of A and is denoted by A^{θ}, e.g.

$$A^{\theta} = (\bar{A})^{T} = (\overline{A^{T}})$$

For example, If $A = \begin{bmatrix} 1-2i & 2+3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$, $\quad A^{T} = \begin{bmatrix} 1-2i & 4-5i & 8 \\ 2+3i & 5+6i & 7+8i \\ 3+4i & 6-7i & 7 \end{bmatrix}$

Then, $\quad A^{\theta} = \begin{bmatrix} 1+2i & 4+5i & 8 \\ 2-3i & 5-6i & 7-8i \\ 3-4i & 6+7i & 7 \end{bmatrix}$

16. Hermitian Matrix

A square matrix $A = [a_{ij}]$ is called Hermitian if $a_{ij} = \overline{a_{ji}}$ for all i and j, i.e., $A = A^{\theta}$, e.g.

$$\begin{bmatrix} 1 & 2+3i & 3-4i \\ 2-3i & 0 & 2-7i \\ 3+4i & 2+7i & 2 \end{bmatrix}$$

17. Skew-Hermitian Matrix

A square matrix $A = [a_{ij}]$ is called skew-Hermitian if $a_{ij} = -\overline{a_{ji}}$ for all i and j, i.e., $A = -A^{\theta}$. Hence, diagonal elements of a skew-Hermitian matrix must be either purely imaginary or zero, e.g.

$$\begin{bmatrix} i & 2+3i \\ -2+3i & 0 \end{bmatrix}$$

18. Orthogonal Matrix

A square matrix A is called orthogonal if $AA^{T} = I$.

19. Unitary Matrix

A square matrix A is called unitary if $AA^{\theta} = I$.

20. Determinant of a Matrix

If A is a square matrix, determinant of A is represented as $|A|$.

If $\quad A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 0 \end{bmatrix}$, then $|A| = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 0 \end{vmatrix}$

21. Singular and Non-Singular Matrices

A square matrix A is called singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

22. Minor of an Element of a Determinant

If $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, minor of an element of a determinant is a determinant obtained by leaving the row and column passing through that element, e.g.,

minor of element $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, minor of element $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$,

minor of element $a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

23. Cofactor of an Element of a Determinant

Cofactor of an element a_{ij} of a determinant is the minor multiplied by $(-1)^{i+j}$, e.g.,

Cofactor of the element $a_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Cofactor of the element $a_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Cofactor of the element $a_{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

24. Inverse of a Matrix

If A is a square matrix and $|A| \neq 0$,

$$AA^{-1} = I = A^{-1}A$$

where, A^{-1} is called inverse of the matrix A.

10.4 ELEMENTARY ROW OPERATIONS IN MATRIX

Elementary transformation is any one of the following operations on a matrix.

(i) The interchange of any two rows (or columns)

(ii) The multiplication of the elements of any row (or column) by any non-zero number

(iii) The addition or subtraction of k items the elements of a row (or column) to the corresponding elements of another row (or column), where $k \neq 0$

Symbols to be used for elementary transformation:

(i) R_{ij} : Interchange of i^{th} and j^{th} row

(ii) kR_i : Multiplication of i^{th} row by a non zero number k

(iii) $R_i + kR_j$: Addition of k times the j^{th} row to the i^{th} row

The corresponding column transformations are denoted by C_{ij}, kC_i and $C_i + kC_j$ respectively.

10.4.1 Elementary Matrices

A matrix obtained from a unit matrix by subjecting it to any row or column transformation is called an elementary matrix.

10.4.2 Equivalence of Matrices

If B be an $m \times n$ matrix obtained from an $m \times n$ matrix by elementary transformation of A, then A is called the equivalent to B. Symbolically, we can write $A \sim B$.

10.5 ROW ECHELON AND REDUCED ROW ECHELON FORMS OF A MATRIX

A matrix A is said to be in row echelon form if it satisfies the following properties:

(i) Every zero row of the matrix A occurs below a non-zero row.

(ii) The first non-zero number from the left of a non-zero row is a 1. This is called a leading 1.

(iii) For each non-zero row, the leading 1 appears to the right and below any leading 1 in the preceding rows.

The following matrices are in row echelon form.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A matrix A is said to be in reduced row echelon form if each column that contains a leading 1 in row echelon form of the matrix A has zeros everywhere else in that column.

The following matrices are in reduced row echelon form.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -4 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 1

In each part determine whether the matrix is in row echelon form, reduced row echelon form, both or neither.

(i) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 7 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -6 & 4 & 3 \\ 0 & 1 & 3 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Solution

(i) The given matrix is in reduced row echelon form and row echelon form since it satisfies properties (i), (ii), (iii) and columns containing leading 1 have zero everywhere else.

(ii) The given matrix is neither in row echelon form nor in reduced row echelon form since it does not satisfy the property (iii).

(iii) The given matrix is in row echelon form since it satisfies properties (i), (ii) and (iii).

(iv) The given matrix is neither in row echelon form nor in reduced row echelon form since it does not satisfy the property (i).

Example 2

Find a row echelon form of the following matrix:

$$\begin{bmatrix} 0 & -1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & -1 & 2 \\ 3 & 2 & 4 & 1 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 0 & -1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & -1 & 2 \\ 3 & 2 & 4 & 1 \end{bmatrix}$$

$$R_{13}$$

$$\sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 2 & 3 & 4 & 5 \\ 0 & -1 & 2 & 3 \\ 3 & 2 & 4 & 1 \end{bmatrix}$$

$$\overset{R_2-2R_1,\ R_4-3R_1}{\sim}\begin{bmatrix}1 & 3 & -1 & 2\\ 0 & -3 & 6 & 1\\ 0 & -1 & 2 & 3\\ 0 & -7 & 7 & -5\end{bmatrix}$$

$$\overset{R_{23}}{\sim}\begin{bmatrix}1 & 3 & -1 & 2\\ 0 & -1 & 2 & 3\\ 0 & -3 & 6 & 1\\ 0 & -7 & 7 & -5\end{bmatrix}$$

$$\overset{(-1)R_2}{\sim}\begin{bmatrix}1 & 3 & -1 & 2\\ 0 & 1 & -2 & -3\\ 0 & -3 & 6 & 1\\ 0 & -7 & 7 & -5\end{bmatrix}$$

$$\overset{R_3+3R_2,\ R_4+7R_2}{\sim}\begin{bmatrix}1 & 3 & -1 & 2\\ 0 & 1 & -2 & -3\\ 0 & 0 & 0 & -8\\ 0 & 0 & -7 & -26\end{bmatrix}$$

$$\overset{R_{34}}{\sim}\begin{bmatrix}1 & 3 & -1 & 2\\ 0 & 1 & -2 & -3\\ 0 & 0 & -7 & -26\\ 0 & 0 & 0 & -8\end{bmatrix}$$

$$\overset{\left(-\frac{1}{7}\right)R_3}{\sim}\begin{bmatrix}1 & 3 & -1 & 2\\ 0 & 1 & -2 & -3\\ 0 & 0 & 1 & \frac{26}{7}\\ 0 & 0 & 0 & -8\end{bmatrix}$$

$$\left(-\frac{1}{8}\right)R_4$$

$$\sim\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & \frac{26}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 3

Find a row echelon form of the following matrix:

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ -1 & 0 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & 0 & -3 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ -1 & 0 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & 0 & -3 \end{bmatrix}$$

$$R_2 + R_1,\ R_4 - 2R_1$$

$$\sim\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 2 & 0 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & 6 & -5 \end{bmatrix}$$

$$R_{23}$$

$$\sim\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 0 & 5 \\ 0 & -1 & 6 & -5 \end{bmatrix}$$

$$R_3 - 2R_2,\ R_4 + R_2$$

$$\sim\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -4 & 7 \\ 0 & 0 & 8 & -6 \end{bmatrix}$$

$$\left(-\frac{1}{4}\right)R_3$$

$$\sim\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -\frac{7}{4} \\ 0 & 0 & 8 & -6 \end{bmatrix}$$

$$R_4 - 8R_3$$

$$\sim\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -\frac{7}{4} \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$\left(\frac{1}{8}\right)R_4$$

$$\sim\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -\frac{7}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 4

Find the reduced row echelon form of the following matrix:

$$\begin{bmatrix} 0 & -1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & -1 & 2 \\ 3 & 2 & 4 & 1 \end{bmatrix}$$

Solution

The row echelon form of the matrix is

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & \frac{26}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Beginning with the last non-zero row and working upward, we add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

$$R_3 - \frac{26}{7}R_4,\ R_2 + 3R_4,\ R_1 - 2R_4$$

$$\sim \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 + 2R_3,\ R_1 + R_3$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 5

Find the reduced row echelon form of the following matrix:

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ -1 & 0 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & 0 & -3 \end{bmatrix}$$

Solution

The row echelon form of the matrix is

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -\frac{7}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Beginning with the last non-zero row and working upward, we add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

$$R_3 + \frac{7}{4}R_4, \; R_2 + R_4, \; R_1 - R_4$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 - 2R_3, \; R_1 + 3R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

10.6 RANK OF A MATRIX

The positive integer r is said to be the rank of a matrix A if it possesses the following properties:

(i) There is at least one minor of order r which is nonzero.

(ii) Every minor of order greater than r is zero.

Rank of matrix A is denoted by $\rho(A)$.

Note (1): The rank of a matrix remains unchanged by elementary transformations.

Note (2): The rank of the transpose of a matrix is same as that of the original matrix.

Note (3): The rank of the product of two matrices cannot exceed the rank of either matrix.

$$\rho(AB) \le \rho(A) \quad \text{or} \quad \rho(AB) \le \rho(B)$$

10.6.1 Rank of a Matrix by Echelon Form

The rank of a matrix in echelon form is equal to the number of nonzero rows of the matrix,

e.g.,

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix A is in echelon form and the number of nonzero rows is two. Hence, rank of the matrix is two,

i.e., $\rho(A) = 2$

Example 1

Find the rank of the following matrix by reducing it to echelon form:

$$\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$$

Solution

Let

$$A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$$

Reducing the matrix A to echelon form,

$$\underset{R_{13}}{\sim} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \end{bmatrix}$$

$$\underset{R_3 - 5R_1}{\sim} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{bmatrix}$$

$$R_3 - 8R_2$$
$$\sim\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \end{bmatrix}$$

$$\left(-\frac{1}{12}\right)R_3$$
$$\sim\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

The equivalent matrix is in echelon form.
Number of nonzero rows = 3

$$\rho(A) = 3$$

Example 2

Find the rank of the following matrix by reducing it to echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

Solution

Let
$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

Reducing the matrix A to echelon form,

$$R_2 + 2R_1,\ R_3 - R_1$$
$$\sim\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\overset{R_{24}}{\sim}\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 2 \\ 0 & 3 & 3 & -3 \end{bmatrix}$$

$$\overset{R_3+2R_2,\ R_4-3R_2}{\sim}\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equivalent matrix is in echelon form.
Number of nonzero rows = 2

$$\rho(A) = 2$$

Example 3

Find the rank of the following matrix by reducing it to echelon form:

$$\begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Solution

Let

$$A = \begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Reducing the matrix A to echelon form,

$$\overset{R_{13}}{\sim}\begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 2 & 2 & 1 \\ 3 & -2 & 0 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\overset{R_3-3R_1,\ R_{24}}{\sim}\begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 4 & 9 & -7 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

$R_3 - 4R_2,\ R_4 - 2R_2$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & -2 & -1 \end{bmatrix}$$

$R_4 + 2R_3$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & -23 \end{bmatrix}$$

$\left(-\dfrac{1}{23}\right) R_4$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The equivalent matrix is in echelon form.

Number of nonzero rows = 4

$$\rho(A) = 4$$

EXERCISE 10.2

1. Find the ranks of the following matrices by reducing them to echelon forms:

(i) $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -1.5 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix}$

(v) $\begin{bmatrix} 3 & -2 & 0 & -1 & -7 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & 6 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & 7 \end{bmatrix}$

[**Ans. :** (i) 2 (ii) 1 (iii) 4 (iv) 2 (v) 4 (vi) 2]

10.7 INVERSE OF A MATRIX BY GAUSS–JORDAN METHOD

Let A be any non-singular matrix. Then $A = IA$. Applying suitable elementary row transformation to A on the L.H.S and to I on the R.H.S, of a matrix A reduces to I and I reduces to any matrix B.

Hence, $I = BA$

$B = A^{-1}$

Example 1

Find the inverse of the following matrix by elementary transformations (Gauss–Jordan method):

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

Solution

Let

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$A = I_3 A$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Reducing the matrix A to reduced row echelon form,

$$R_{13}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

$$R_2 - 4R_1,\ R_3 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -15 \\ 0 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & 0 & -2 \end{bmatrix} A$$

$\left(-\frac{1}{5}\right)R_2$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{5} & \frac{4}{5} \\ 1 & 0 & -2 \end{bmatrix} A$$

$R_3 + R_2$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{5} & \frac{4}{5} \\ 1 & -\frac{1}{5} & -\frac{6}{5} \end{bmatrix} A$$

$(-1)R_3$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{5} & \frac{4}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix} A$$

$R_2 - 3R_3, R_1 - 4R_3$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -\frac{4}{5} & -\frac{19}{5} \\ 3 & -\frac{4}{5} & -\frac{14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix} A$$

$R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & \frac{4}{5} & \frac{9}{5} \\ 3 & -\frac{4}{5} & -\frac{14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix} A$$

$$I_3 = A^{-1}A$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & \frac{4}{5} & \frac{9}{5} \\ 3 & -\frac{4}{5} & -\frac{14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$$

Example 2

Find the inverse of the following matrix by elementary transformations (Gauss–Jordan method):

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & -2 & 1 & 6 \end{bmatrix}$$

Solution

Let

$$A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & -2 & 1 & 6 \end{bmatrix}$$

$$A = I_4 A$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & -2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Reducing the matrix A to reduced row echelon form,

$R_3 - 2R_1, R_4 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 3 & 2 & -3 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix} A$$

$R_3 - 3R_2, R_4 - R_2$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -3 & 1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix} A$$

$(-1)R_3$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix} A$$

$R_2 + R_4,\ R_3 - 2R_4$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 0 & -2 \\ -3 & 0 & 0 & 1 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix} A$$

$R_2 - R_3$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 0 & -2 \\ -5 & -3 & 1 & 1 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix} A$$

$R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 & -1 \\ -5 & -3 & 1 & 1 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix} A$$

$$I_4 = A^{-1}A$$

$$\therefore A^{-1} = \begin{bmatrix} 2 & -1 & 1 & -1 \\ -5 & -3 & 1 & 1 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix}$$

EXERCISE 10.3

1. Using elementary row transformations, find the inverses of the following matrices:

(i) $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & -1 & 5 \\ 0 & 2 & 0 \\ -5 & 3 & -15 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & -3 \\ 1 & -4 & 9 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ (vi) $\begin{bmatrix} 2 & 4 & 3 & 2 \\ 3 & 6 & 5 & 2 \\ 2 & 5 & 2 & -3 \\ 4 & 5 & 14 & 14 \end{bmatrix}$

(vii) $\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$ (viii) $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

(ix) $\begin{bmatrix} 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ -1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ (x) $\begin{bmatrix} 2 & 0 & 0 & -4 \\ 2 & 6 & 0 & -16 \\ 1 & 0 & 3 & -5 \\ -2 & 0 & 0 & 10 \end{bmatrix}$

Ans.:

(i) $\frac{1}{3}\begin{bmatrix} 1 & -2 & -1 \\ 1 & -5 & 2 \\ -3 & 12 & 0 \end{bmatrix}$ (ii) $\frac{1}{5}\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & 3 \end{bmatrix}$

(iv) $\frac{1}{17}\begin{bmatrix} 6 & 5 & 1 \\ -21 & 8 & 5 \\ -10 & 3 & 4 \end{bmatrix}$ (v) $\frac{1}{4}\begin{bmatrix} 6 & -1 & -9 \\ -4 & 2 & 6 \\ 2 & -1 & -1 \end{bmatrix}$ (vi) $\begin{bmatrix} -23 & 29 & -\frac{64}{5} & -\frac{18}{5} \\ 10 & -12 & \frac{26}{5} & \frac{7}{5} \\ 1 & -2 & \frac{6}{5} & \frac{2}{5} \\ 2 & -2 & \frac{3}{5} & \frac{1}{5} \end{bmatrix}$

(vii) $\begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix}$ (viii) $\begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$ (ix) $\begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix}$

(x) $\begin{bmatrix} 5 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

10.8 SYSTEM OF NON-HOMOGENEOUS LINEAR EQUATIONS

A system of m non-homogeneous linear equations in n variables $x_1, x_2, ..., x_n$ or simply a linear system, is a set of m linear equations, each in n variables. A linear system is represented by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad \vdots \quad\quad\quad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Writing these equations in matrix form,

$$A\mathbf{x} = B$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is called coefficient matrix of order $m \times n$,

$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is any vector of order $n \times 1$.

$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ is any vector of order $m \times 1$.

10.8.1 Solutions of System of Linear Equations: Gauss Elimination and Gauss–Jordan Methods

For a system of m linear equations in n variables, there are three possibilities of the solutions to the system:

(i) The system has unique solution.
(ii) The system has infinite solutions.
(iii) The system has no solution.

When the system of linear equations has one or more solutions, the system is said to be consistent, otherwise it is inconsistent.

The matrix $[A : B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$

is called the augmented matrix of the given system of linear equations.

To solve a system of linear equations, elementary transformations are used to reduce the augmented matrix to either row echelon form or reduced row echelon form.

Reducing the augmented matrix to row echelon form is called Gauss elimination method. Reducing the augmented matrix to reduced row echelon form is called Gauss–Jordan method.

The Gauss elimination method for solving the linear system is as follows:

Step 1: Write the augmented matrix.
Step 2: Obtain the row echelon form of the augmented matrix by using elementary row operations.
Step 3: Write the corresponding linear system of equations from row echelon form.
Step 4: Solve the corresponding linear system of equations by back substitution.

The Gauss–Jordan method for solving the linear system is as follows:

Step 1: Write the augmented matrix.
Step 2: Obtain the reduced row echelon form of the augmented matrix by using elementary row operations.
Step 3: For each non-zero row of the matrix, solve the corresponding system of equations for the variables associated with the leading one in that row.

Note: The linear system has a unique solution if $\det(A) \neq 0$

Example 1

Solve the following system by Gauss elimination method:

$$\begin{aligned} x+\ \ y+2z&=9 \\ 2x+4y-3z&=1 \\ 3x+6y-5z&=0 \end{aligned}$$

Solution

The matrix form of the system is

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\underset{\sim}{R_2 - 2R_1,\ R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 2 & \vdots & 9 \\ 0 & 2 & -7 & \vdots & -17 \\ 0 & 3 & -11 & \vdots & -27 \end{bmatrix}$$

$$\underset{\sim}{\left(\frac{1}{2}\right)R_2} \begin{bmatrix} 1 & 1 & 2 & \vdots & 9 \\ 0 & 1 & -\frac{7}{2} & \vdots & -\frac{17}{2} \\ 0 & 3 & -11 & \vdots & -27 \end{bmatrix}$$

$$\underset{\sim}{R_3 - 3R_2} \begin{bmatrix} 1 & 1 & 2 & \vdots & 9 \\ 0 & 1 & -\frac{7}{2} & \vdots & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & \vdots & -\frac{3}{2} \end{bmatrix}$$

$$\underset{\sim}{(-2)R_3} \begin{bmatrix} 1 & 1 & 2 & \vdots & 9 \\ 0 & 1 & -\frac{7}{2} & \vdots & -\frac{17}{2} \\ 0 & 0 & 1 & \vdots & 3 \end{bmatrix}$$

The corresponding system of equations is

$$x + y + 2z = 9$$
$$y - \frac{7}{2}z = -\frac{17}{2}$$
$$z = 3$$

Solving these equations,

$$x = 1,\ y = 2$$

Hence, $x = 1$, $y = 2$, $z = 3$ is the solution of the system.

Example 2

Solve the following system by Gauss elimination method:

$$\begin{aligned} 4x-2y+6z &= 8 \\ x+y-3z &= -1 \\ 15x-3y+9z &= 21 \end{aligned}$$

Solution

The matrix form of the system is

$$\begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_{12}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

$$R_2 - 4R_1, \; R_3 - 15R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{array}\right]$$

$$\left(-\frac{1}{6}\right)R_2, \; \left(-\frac{1}{18}\right)R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 1 & -3 & -2 \end{array}\right]$$

$$R_3 - R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x + y - 3z &= -1 \\ y - 3z &= -2 \end{aligned}$$

The leading ones are in columns 1 and 2. Hence, the variables x and y are called leading variables whereas the variable z is called a free variable. Assigning the free variable z an arbitrary value t,

$$\begin{aligned} y &= 3t - 2 \\ x &= -1 - 3t + 2 + 3t = 1 \end{aligned}$$

Hence, $x = 1$, $y = 3t - 2$, $z = t$ is the solution of the system where t is a parameter.

Example 3

Solve the following system by Gauss elimination method:

$$\begin{aligned} 3x + y - 3z &= 13 \\ 2x - 3y + 7z &= 5 \\ 2x + 19y - 47z &= 32 \end{aligned}$$

Solution

The matrix form of the system is

$$\begin{bmatrix} 3 & 1 & -3 \\ 2 & -3 & 7 \\ 2 & 19 & -47 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 5 \\ 32 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 3 & 1 & -3 & 13 \\ 2 & -3 & 7 & 5 \\ 2 & 19 & -47 & 32 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\left(\frac{1}{3}\right)R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -1 & \frac{13}{3} \\ 2 & -3 & 7 & 5 \\ 2 & 19 & -47 & 32 \end{array}\right]$$

$$\xrightarrow{R_2-2R_1,\ R_3-2R_1} \sim \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -1 & \frac{13}{3} \\ 0 & -\frac{11}{3} & 9 & -\frac{11}{3} \\ 0 & \frac{55}{3} & -45 & \frac{70}{3} \end{array}\right]$$

$$\xrightarrow{\left(-\frac{3}{11}\right)R_2} \sim \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -1 & \frac{13}{3} \\ 0 & 1 & -\frac{27}{11} & 1 \\ 0 & \frac{55}{3} & -45 & \frac{70}{3} \end{array}\right]$$

$$\xrightarrow{R_3-\frac{55}{3}R_2} \sim \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -1 & \frac{13}{3} \\ 0 & 1 & -\frac{27}{11} & 1 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

From the last row of the augmented matrix,

$$0x+0y+0z=5$$

Hence, the system is inconsistent and has no solution.

Example 4

Solve the following system for x, y and z.

$$-\frac{1}{x}+\frac{3}{y}+\frac{4}{z}=30$$

$$\frac{3}{x}+\frac{2}{y}-\frac{1}{z}=9$$

$$\frac{2}{x}-\frac{1}{y}+\frac{2}{z}=10$$

Solution

The matrix form of the system is

$$\begin{bmatrix} -1 & 3 & 4 \\ 3 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \begin{bmatrix} 30 \\ 9 \\ 10 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} -1 & 3 & 4 & 30 \\ 3 & 2 & -1 & 9 \\ 2 & -1 & 2 & 10 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\begin{array}{l} (-1)R_1 \\ \sim \left[\begin{array}{ccc:c} 1 & -3 & -4 & -30 \\ 3 & 2 & -1 & 9 \\ 2 & -1 & 2 & 10 \end{array}\right] \end{array}$$

$$\begin{array}{l} R_2 - 3R_1,\ R_3 - 2R_1 \\ \sim \left[\begin{array}{ccc:c} 1 & -3 & -4 & -30 \\ 0 & 11 & 11 & 99 \\ 0 & 5 & 10 & 70 \end{array}\right] \end{array}$$

$$\begin{array}{l} \left(\frac{1}{11}\right)R_2,\ \left(\frac{1}{5}\right)R_3 \\ \sim \left[\begin{array}{ccc:c} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 1 & 2 & 14 \end{array}\right] \end{array}$$

$$\begin{array}{l} R_3 - R_2 \\ \sim \left[\begin{array}{ccc:c} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right] \end{array}$$

The corresponding system of equations is

$$\frac{1}{x}-\frac{3}{y}-\frac{4}{z}=-30$$
$$\frac{1}{y}+\frac{1}{z}=9$$
$$\frac{1}{z}=5$$

Solving these equations,

$$x=\frac{1}{2},\ y=\frac{1}{4},\ z=\frac{1}{5}$$

Hence, $x=\frac{1}{2}, y=\frac{1}{4}, z=\frac{1}{5}$ is the solution of the system.

Example 5

Solve the following system of non-linear equations for the unknown angles α, β and γ, where $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$ *and* $0 \le \gamma < \pi$.

$$\begin{aligned} 2\sin\alpha - \cos\beta + 3\tan\gamma &= 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma &= 2 \\ 6\sin\alpha - 3\cos\beta + \tan\gamma &= 9 \end{aligned}$$

Solution

The matrix form of the system is

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & -2 \\ 6 & -3 & 1 \end{bmatrix}\begin{bmatrix} \sin\alpha \\ \cos\beta \\ \tan\gamma \end{bmatrix}=\begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 4 & 2 & -2 & 2 \\ 6 & -3 & 1 & 9 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\left(\frac{1}{2}\right)R_1$$

$$\sim\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 4 & 2 & -2 & 2 \\ 6 & -3 & 1 & 9 \end{array}\right]$$

$$R_2 - 4R_1,\ R_3 - 6R_1$$

$$\sim\begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & \vdots & \frac{3}{2} \\ 0 & 4 & -8 & \vdots & -4 \\ 0 & 0 & -8 & \vdots & 0 \end{bmatrix}$$

$$\left(\frac{1}{4}\right)R_2,\ \left(-\frac{1}{8}\right)R_3$$

$$\sim\begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & \vdots & \frac{3}{2} \\ 0 & 1 & -2 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & 0 \end{bmatrix}$$

The corresponding system of equations is

$$\sin\alpha - \frac{1}{2}\cos\beta + \frac{3}{2}\tan\gamma = \frac{3}{2}$$
$$\cos\beta - 2\tan\gamma = -1$$
$$\tan\gamma = 0$$

Solving these equations,

$$\gamma = 0$$
$$\cos\beta = -1 \quad\Rightarrow\quad \beta = \pi$$
$$\sin\alpha = \frac{1}{2}\cos\beta - \frac{3}{2}\tan\gamma + \frac{3}{2}$$
$$= \frac{1}{2}(-1) - \frac{3}{2}(0) + \frac{3}{2} = 1$$
$$\alpha = \frac{\pi}{2}$$

Hence, $\alpha = \frac{\pi}{2}, \beta = \pi, \gamma = 0$ is the solution of the system.

Example 6

Investigate for what values of λ and μ the equations

$$x + 2y + z = 8$$
$$2x + 2y + 2z = 13$$
$$3x + 4y + \lambda z = \mu$$

have (i) no solution, (ii) a unique solution, and (iii) many solutions.

Solution

The matrix form of the system is

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \\ \mu \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 2 & 2 & 13 \\ 3 & 4 & \lambda & \mu \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - 2R_1,\ R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -2 & 0 & -3 \\ 0 & -2 & \lambda - 3 & \mu - 24 \end{array}\right]$$

$$\left(-\frac{1}{2}\right) R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & -2 & \lambda - 3 & \mu - 24 \end{array}\right]$$

$$R_3 + 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & \lambda - 3 & \mu - 21 \end{array}\right]$$

(i) If $\lambda = 3$ and $\mu \neq 21$, the system is inconsistent and has no solution.

(ii) If $\lambda \neq 3$ and μ has any value, the system is consistent and has a unique solution.

(iii) If $\lambda = 3$ and $\mu = 21$, the system is consistent and has infinite (many) solutions.

Example 7

Determine the values of λ for which the following equations are consistent. Also, solve the system for these values of λ.

$$\begin{aligned} x+2y+z&=3\\ x+y+z&=\lambda\\ 3x+y+3z&=\lambda^2 \end{aligned}$$

Solution

The matrix form of the system is

$$\begin{bmatrix} 1 & 2 & 1\\ 1 & 1 & 1\\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x\\ y\\ z \end{bmatrix}=\begin{bmatrix} 3\\ \lambda\\ \lambda^2 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3\\ 1 & 1 & 1 & \lambda\\ 3 & 1 & 3 & \lambda^2 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2-R_1,\ R_3-3R_1$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3\\ 0 & -1 & 0 & \lambda-3\\ 0 & -5 & 0 & \lambda^2-9 \end{array}\right]$$

$$(-1)R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3\\ 0 & 1 & 0 & 3-\lambda\\ 0 & -5 & 0 & \lambda^2-9 \end{array}\right]$$

$$R_3+5R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3\\ 0 & 1 & 0 & 3-\lambda\\ 0 & 0 & 0 & \lambda^2-5\lambda+6 \end{array}\right]$$

The equations will be consistent if $\lambda^2 - 5\lambda + 6 = 0$, i.e. $\lambda = 3$ or $\lambda = 2$.

Case I: When $\lambda = 3$,

$$\begin{aligned} x+2y+z&=3\\ y&=0 \end{aligned}$$

Assigning the free variable z any arbitrary value t,

$$x=3-2(0)-t=3-t$$

Hence, $x = 3 - t$, $y = 0$, $z = t$ is the solution of the system where t is a parameter.

Case II: When $\lambda = 2$,

$$\begin{aligned} x + 2y + z &= 3 \\ y \quad &= 1 \end{aligned}$$

Assigning the free variable z any arbitrary value t,

$$x = 3 - 2(1) - t = 1 - t$$

Hence, $x = 1 - t$, $y = 1$, $z = t$ is the solution of the system where t is a parameter.

Example 8

Show that the system of equations

$$\begin{aligned} 3x + 4y + 5z &= \alpha \\ 4x + 5y + 6z &= \beta \\ 5x + 6y + 7z &= \gamma \end{aligned}$$

is consistent only if α, β *and* λ *are in arithmetic progression (A.P.).*

Solution

The matrix form of the system is

$$\begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & \alpha \\ 4 & 5 & 6 & \beta \\ 5 & 6 & 7 & \gamma \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - R_1,\ R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 3 & 4 & 5 & \alpha \\ 1 & 1 & 1 & \beta - \alpha \\ 2 & 2 & 2 & \gamma - \alpha \end{array}\right]$$

$$R_{12}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & \beta - \alpha \\ 3 & 4 & 5 & \alpha \\ 2 & 2 & 2 & \gamma - \alpha \end{array}\right]$$

$$R_2 - 3R_1,\ R_3 - 2R_1$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & \beta-\alpha \\ 0 & 1 & 2 & 4\alpha-3\beta \\ 0 & 0 & 0 & \alpha-2\beta+\gamma \end{array}\right]$$

The system of equations is consistent if,

$$\alpha - 2\beta + \gamma = 0$$
$$\beta = \frac{\alpha+\gamma}{2}$$

i.e. α, β and γ are in arithmetic progression (A.P.)

Example 9

Show that if $\lambda \neq 0$, the system of equations

$$\begin{aligned} 2x + y \quad &= a \\ x + \lambda y - z &= b \\ y + 2z &= c \end{aligned}$$

has a unique solution for every value of a, b, c. If $\lambda = 0$, determine the relation satisfied by a, b, c such that the system is consistent. Find the solution by taking $\lambda = 0$, $a = 1$, $b = 1$, $c = -1$.

Solution

The matrix form of the system is

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & \lambda & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The system has a unique solution if $\det(A) \neq 0$

$$\det(A) = 2(2\lambda+1) - 1\ (2+0) \neq 0$$
$$4\lambda \neq 0$$
$$\lambda \neq 0$$

Hence, the system of equations has a unique solution if $\lambda \neq 0$ for any value of a, b, c.

If $\lambda = 0$, the system is either inconsistent or has an infinite number of solutions.

For $\lambda = 0$, the augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & a \\ 1 & 0 & -1 & b \\ 0 & 1 & 2 & c \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\stackrel{R_{12}}{\sim}\left[\begin{array}{ccc|c} 1 & 0 & -1 & b \\ 2 & 1 & 0 & a \\ 0 & 1 & 2 & c \end{array}\right]$$

$$\stackrel{R_2 - 2R_1}{\sim}\left[\begin{array}{ccc|c} 1 & 0 & -1 & b \\ 0 & 1 & 2 & a-2b \\ 0 & 1 & 2 & c \end{array}\right]$$

$$\stackrel{R_3 - R_2}{\sim}\left[\begin{array}{ccc|c} 1 & 0 & -1 & b \\ 0 & 1 & 2 & a-2b \\ 0 & 0 & 0 & c-a+2b \end{array}\right]$$

The system is consistent if $c - a + 2b = 0$.

The corresponding system of equations is

$$x - z = b$$
$$y + 2z = a - 2b$$

Assigning the free variable z any arbitrary value t,

$$y = a - 2b - 2t$$
$$x = b + t$$

Hence, $x = b + t$, $y = a - 2b - 2t$, $z = t$ is the solution of the system where t is a parameter.

When $a = 1$, $b = 1$, $c = -1$

$$x = 1 + t$$
$$y = -1 - 2t$$
$$z = t$$

Example 10

Solve the following system by Gauss–Jordan method:

$$\begin{aligned} x_1 + \ \ x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

Solution

The matrix form of the system is

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$R_2 + R_1,\ R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$$(-1)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$$R_3 + 10R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right]$$

$$\left(-\frac{1}{52}\right)R_3$$
$$\sim\begin{bmatrix}1 & 1 & 2 & \vdots & 8\\0 & 1 & -5 & \vdots & -9\\0 & 0 & 1 & \vdots & 2\end{bmatrix}$$

$$R_2+5R_3,\ R_1-2R_3$$
$$\sim\begin{bmatrix}1 & 1 & 0 & \vdots & 4\\0 & 1 & 0 & \vdots & 1\\0 & 0 & 1 & \vdots & 2\end{bmatrix}$$

$$R_1-R_2$$
$$\sim\begin{bmatrix}1 & 0 & 0 & \vdots & 3\\0 & 1 & 0 & \vdots & 1\\0 & 0 & 1 & \vdots & 2\end{bmatrix}$$

The corresponding system of equations is

$$x_1=3$$
$$x_2=1$$
$$x_3=2$$

Hence, $x_1=3, x_2=1, x_3=2$ is the solution of the system.

Example 11

Solve the following system by Gauss–Jordan method:

$$2x_1+2x_2+2x_3=0$$
$$-2x_1+5x_2+2x_3=1$$
$$8x_1+x_2+4x_3=-1$$

Solution

The matrix form of the system is

$$\begin{bmatrix}2 & 2 & 2\\-2 & 5 & 2\\8 & 1 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$\left(\frac{1}{2}\right)R_1$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right]$$

$$R_2+2R_1,\ R_3-8R_1$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{array}\right]$$

$$R_3+R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left(\frac{1}{7}\right)R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1-R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{7} & -\frac{1}{7} \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x_1 + \frac{3}{7}x_3 = -\frac{1}{7}$$
$$x_2 + \frac{4}{7}x_3 = \frac{1}{7}$$

Since leading ones are in columns 1 and 2, x_1 and x_2 are called leading variables, whereas x_3 is a free variable. Assigning the free variable x_3 any arbitrary value t,

$$x_1 = -\frac{1}{7} - \frac{3}{7}t$$
$$x_2 = \frac{1}{7} - \frac{4}{7}t$$

Hence, $x_1 = -\frac{1}{7} - \frac{3}{7}t, x_2 = \frac{1}{7} - \frac{4}{7}t, x_3 = t$ is the solution of the system where t is a parameter.

Example 12

Solve the following system by Gauss–Jordan method:

$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x - 3w &= -3 \end{aligned}$$

Solution

The matrix form of the system is

$$\begin{bmatrix} 1 & -1 & 2 & -1 \\ 2 & 1 & -2 & -2 \\ -1 & 2 & -4 & 1 \\ 3 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \\ -3 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$R_2 - 2R_1,\ R_3 + R_1,\ R_4 - 3R_1$$

$$\sim \left[\begin{array}{cccc:c} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right]$$

$$\left(\frac{1}{3}\right)R_2$$

$$\sim \left[\begin{array}{cccc:c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right]$$

$$R_3 - R_2,\ R_4 - 3R_2$$

$$\sim \left[\begin{array}{cccc:c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 + R_2$$

$$\sim \left[\begin{array}{cccc:c} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x - \qquad\quad w &= -1 \\ y - 2z \qquad &= 0 \end{aligned}$$

The leading ones are in columns 1 and 2. Hence, the variables x and y are called leading variables whereas the variables z and w are called free variables. Assigning the free variables z and w any arbitrary values t_1 and t_2 respectively,

$$x = -1 + t_2$$

and $$y = 2t_1$$

Hence, $x = -1 + t_2,\ y = 2t_1,\ z = t_1,\ w = t_2$ is the solution of the system where t_1 and t_2 are parameters.

Example 13

Solve the following system by Gauss–Jordan method:

$$\begin{aligned} -2y+3z &= 1 \\ 3x+6y-3z &= -2 \\ 6x+6y+3z &= 5 \end{aligned}$$

Solution

The matrix form of the system is

$$\begin{bmatrix} 0 & -2 & 3 \\ 3 & 6 & -3 \\ 6 & 6 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$\overset{R_{12}}{\sim}\left[\begin{array}{ccc:c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$$\overset{\left(\frac{1}{3}\right)R_1}{\sim}\left[\begin{array}{ccc:c} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$$\overset{R_3-6R_1}{\sim}\left[\begin{array}{ccc:c} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{array}\right]$$

$$\left(-\frac{1}{2}\right)R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -6 & 9 & 9 \end{array}\right]$$

$$R_3+6R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 6 \end{array}\right]$$

From the last row of the augmented matrix,

$$0x+0y+0z=6$$

Hence, the system is inconsistent and has no solution.

Example 14

Solve the following system by Gauss–Jordan method:

$$\begin{aligned} x_1-2x_2-\ x_3+3x_4&=1 \\ 2x_1-4x_2+\ x_3\qquad\ &=5 \\ x_1-2x_2+2x_3-3x_4&=4 \end{aligned}$$

Solution

The matrix form of the system is

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$R_2 - 2R_1,\ R_3 - R_1$$
$$\sim\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right]$$

$$R_3 - R_2$$
$$\sim\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left(\frac{1}{3}\right)R_2$$
$$\sim\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 + R_2$$
$$\sim\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x_1 - 2x_2 + \qquad x_4 &= 2 \\ x_3 - 2x_4 &= 1 \end{aligned}$$

The leading ones are in columns 1 and 3. Hence, the variables x_1 and x_3 are called leading variables whereas the variables x_2 and x_4 are called free variables. Assigning the free variables x_2 and x_4 any arbitrary values t_1 and t_2 respectively,

$$\begin{aligned} x_1 &= 2 + 2t_1 - t_2 \\ x_3 &= 1 + 2t_2 \end{aligned}$$

Hence, $x_1 = 2+2t_1 - t_2, x_2 = t_1, x_3 = 1+2t_2, x_4 = t_2$ is the solution of the system where t_1 and t_2 are the parameters.

Example 15

Solve the following system by Gauss–Jordan method:

$$\begin{aligned} 2x - y + z &= 9 \\ 3x - y + z &= 6 \\ 4x - y + 2z &= 7 \\ -x + y - z &= 4 \end{aligned}$$

Solution

The matrix form of the system is

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & -1 & 1 \\ 4 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 7 \\ 4 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 2 & -1 & 1 & 9 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \\ -1 & 1 & -1 & 4 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$\overset{R_{14}}{\sim} \left[\begin{array}{ccc:c} -1 & 1 & -1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \\ 2 & -1 & 1 & 9 \end{array}\right]$$

$$\overset{(-1)R_1}{\sim} \left[\begin{array}{ccc:c} 1 & -1 & 1 & -4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \\ 2 & -1 & 1 & 9 \end{array}\right]$$

$$R_2 - 3R_1,\ R_3 - 4R_1,\ R_4 - 2R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & -4 \\ 0 & 2 & -2 & 18 \\ 0 & 3 & -2 & 23 \\ 0 & 1 & -1 & 17 \end{array}\right]$$

$$R_{24}$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & -4 \\ 0 & 1 & -1 & 17 \\ 0 & 3 & -2 & 23 \\ 0 & 2 & -2 & 18 \end{array}\right]$$

$$R_3 - 3R_2,\ R_4 - 2R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & -4 \\ 0 & 1 & -1 & 17 \\ 0 & 0 & 1 & -28 \\ 0 & 0 & 0 & -16 \end{array}\right]$$

From the last row of the augmented matrix,

$$0x + 0y + 0z = -16$$

Hence, the system is inconsistent and has no solution.

EXERCISE 10.3

1. Solve the following systems of equations by Gauss elimination method:

(i) $\begin{aligned} 2x - 3y - z &= 3 \\ x + 2y - z &= 4 \\ 5x - 4y - 3z &= -2 \end{aligned}$

(ii) $\begin{aligned} x + 2y - z &= 1 \\ x + y + 2z &= 9 \\ 2x + y - z &= 2 \end{aligned}$

(iii) $\begin{aligned} 6x + y + z &= -4 \\ 2x - 3y - z &= 0 \\ -x - 7y - 2z &= 7 \end{aligned}$

(iv) $\begin{aligned} 2x - y - z &= 2 \\ x + 2y + z &= 2 \\ 4x - 7y - 5z &= 2 \end{aligned}$

(v) $\begin{aligned} 2x_1 + x_2 + 2x_3 + x_4 &= 6 \\ 6x_1 - 6x_2 + 6x_3 + 12x_4 &= 36 \\ 4x_1 + 3x_2 + 3x_3 - 3x_4 &= 1 \\ 2x_1 + 2x_2 - x_3 + x_4 &= 10 \end{aligned}$

Ans.:
(i) inconistent
(ii) consistent $x = 2, y = 1, z = 3$
(iii) consistent $x = -1, y = -2, z = -4$
(iv) consistent $x = \frac{6+t}{5}, y = \frac{2-3t}{5}, z = t$
(v) consistent $x_1 = 2, x_2 = 1, x_3 = -1, x_4 = 3$

2. Solve the following system of equations by Gauss–Jordan method:

(i)
$$\begin{aligned} x + 2y + z &= -1 \\ 6x + y + z &= -4 \\ 2x - 3y - z &= 0 \\ -x - 7y - 2z &= 7 \\ x - y &= 1 \end{aligned}$$

(ii)
$$\begin{aligned} x + y + z &= 6 \\ x - 2y + 2z &= 5 \\ 3x + y + z &= 8 \\ 2x - 2y + 3z &= 7 \end{aligned}$$

(iii)
$$\begin{aligned} 2x_1 + x_2 + 5x_4 &= 4 \\ 3x_1 - 2x_2 + 2x_3 &= 2 \\ 5x_1 - 8x_2 - 4x_3 &= 1 \end{aligned}$$

Ans.:
(i) consistent $x = -1, y = -2, z = 4$
(ii) consistent $x = -1, y = -2, z = 3$
(iii) inconsistent

3. Investigate for what values of λ and μ, the system of simultaneous equations

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + \lambda z &= \mu \end{aligned}$$

has (i) no solution, (ii) a unique solution, and (iii) infinite number of solutions.

Ans.:
(i) $\lambda = 3, \mu \neq 10$ (ii) $\lambda \neq 3$, any value of μ (iii) $\lambda = 3, \mu = 10$

4. Investigate for what values of k the equations

$$\begin{aligned} x+y+\ \ z &= 1 \\ 2x+y+\ 4z &= k \\ 4x+y+10z &= k^2 \end{aligned}$$

have infinite number of solutions.

[Ans.: $k = 1, 2$**]**

5. Determine the values of λ for which the following system of equations.

$$\begin{aligned} 3x-\ \ y+\lambda z &= 0 \\ 2x+\ \ y+\ \ z &= 2 \\ x-2y-\lambda z &= -1 \end{aligned}$$

will fail to have a unique solution. For this value of λ, are the equations consistent?

$$\left[\textbf{Ans.: } \lambda = -\frac{7}{2}, \text{ no solution}\right]$$

6. Find for what values λ, the set of equations

$$\begin{aligned} 2x-3y+6z-5t &= 3 \\ y-4z+\ \ t &= 1 \\ 4x-5y+8z-9t &= \lambda \end{aligned}$$

has (i) no solution, and (ii) infinite number of solutions and find the solutions of the equations when they are consistent.

$$\left[\begin{aligned} &\textbf{Ans.: } \text{(i) } \lambda \neq 7, \\ &\qquad \text{(ii) } \lambda = 7,\ x = 3k_1 + k_2 + 3, \\ &\qquad\qquad y = 4k_1 - k_2 + 1,\ z = k_1, \\ &\qquad\qquad t = k_2 \end{aligned}\right]$$

10.9 SYSTEM OF HOMOGENEOUS LINEAR EQUATIONS

A system of m homogeneous linear equations in n variables $x_1, x_2, ..., x_n$ or simply a linear system, is a set of m linear equations each in n variables. A linear system is represented by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\quad & \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Writing these equations in matrix form,

$$A\mathbf{x} = \mathbf{0}$$

where A is any matrix of order $m \times n$, $\mathbf{x}$ is a vector of order $n \times 1$ and $\mathbf{0}$ is a null vector of order $m \times 1$. The matrix A is called coefficient matrix of the system of equations.

10.9.1 Solutions of a System of Linear Equations

For a system of m linear equations in n variables, there are two possibilities of the solutions to the system.

(i) The system has exactly one solution, i.e. $x_1 = 0, x_2 = 0 \ldots, x_n = 0$. This solution is called the trivial solution.

(ii) The system has infinite solutions.

Note: The system of equations has a non-trivial solution if $\det(A) = 0$.

Example 1

Solve the following system of equations by the Gauss–Jordan method.

$$\begin{aligned} 3x - y - z &= 0 \\ x + y + 2z &= 0 \\ 5x + y + 3z &= 0 \end{aligned}$$

Solution

The matrix form of the system is

$$\begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & 2 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 1 & 1 & 2 & 0 \\ 5 & 1 & 3 & 0 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$\overset{R_{12}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 3 & -1 & -1 & 0 \\ 5 & 1 & 3 & 0 \end{array}\right]$$

$$\overset{R_2-3R_1,\ R_3-5R_1}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -4 & -7 & 0 \\ 0 & -4 & -7 & 0 \end{array}\right]$$

$$\overset{\left(-\frac{1}{4}\right)R_2,\ \left(-\frac{1}{4}\right)R_3}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & \frac{7}{4} & 0 \\ 0 & 1 & \frac{7}{4} & 0 \end{array}\right]$$

$$\overset{R_3-R_2}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & \frac{7}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\overset{R_1-R_2}{\sim}\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{7}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x+\frac{1}{4}z=0$$
$$y+\frac{7}{4}z=0$$

Solving for the leading variables,

$$x=-\frac{1}{4}z$$
$$y=-\frac{7}{4}z$$

Assigning the free variable z an arbitrary value t,

$$x=-\frac{1}{4}t$$
$$y=-\frac{7}{4}t$$

Hence, $x=-\frac{1}{4}t, y=-\frac{7}{4}t$ is the non-trivial solution of the system where t is a parameter.

Example 2

Solve the following system of equations by the Gauss–Jordan method.

$$\begin{aligned} x+y-z+w&=0\\ x-y+2z-w&=0\\ 3x+y\qquad+w&=0 \end{aligned}$$

Solution

The matrix form of the system is

$$\begin{bmatrix} 1 & 1 & -1 & 1\\ 1 & -1 & 2 & -1\\ 3 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x\\ y\\ z\\ w \end{bmatrix}=\begin{bmatrix} 0\\ 0\\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0\\ 1 & -1 & 2 & -1 & 0\\ 3 & 1 & 0 & 1 & 0 \end{array}\right]$$

Reducing the augmented matrix to the reduced row echelon form,

$$R_2-R_1,\ R_3-3R_1$$

$$\sim\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0\\ 0 & -2 & 3 & -2 & 0\\ 0 & -2 & 3 & -2 & 0 \end{array}\right]$$

$$\left(-\frac{1}{2}\right)R_2,\ \left(-\frac{1}{2}\right)R_3$$

$$\sim\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0\\ 0 & 1 & -\frac{3}{2} & 1 & 0\\ 0 & 1 & -\frac{3}{2} & 1 & 0 \end{array}\right]$$

$$\underset{R_3 - R_2}{\sim} \left[\begin{array}{cccc:c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{R_1 - R_2}{\sim} \left[\begin{array}{cccc:c} 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x + \frac{1}{2}z \qquad = 0$$
$$y - \frac{3}{2}z + w = 0$$

Solving for the leading variables,

$$x = -\frac{1}{2}z$$
$$y = \frac{3}{2}z - w$$

Assigning the free variables z and w arbitrary values t_1 and t_2 respectively,

$$x = -\frac{1}{2}t_1$$
$$y = \frac{3}{2}t_1 - t_2$$

Hence, $x = -\frac{1}{2}t_1, y = \frac{3}{2}t_1 - t_2, z = t_1, w = t_2$ is the non-trivial solution of the system where t_1 and t_2 are parameters.

Example 3

Solve the following system of equations by the Gauss–Jordan method.

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

Solution

The matrix form of the system is

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$\overset{R_{12}}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\overset{R_2 - 2R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\overset{\left(-\frac{1}{3}\right)R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\overset{R_3 - R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\left(\frac{1}{2}\right)R_3$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_2 + R_3$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_1 - 2R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x = 0$$
$$y = 0$$
$$z = 0$$

Hence, the system has a trivial solution, i.e. $x = 0$, $y = 0$, $z = 0$.

Example 4

Show that the following non-linear system has 18 solutions if $0 \le \alpha \le 2\pi, 0 \le \beta \le 2\pi$ *and* $0 \le \gamma < 2\pi$.

$$\sin\alpha + 2\cos\beta + 3\tan\gamma = 0$$
$$2\sin\alpha + 5\cos\beta + 3\tan\gamma = 0$$
$$-\sin\alpha - 5\cos\beta + 5\tan\gamma = 0$$

Solution

The matrix form of the system is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ -1 & -5 & 5 \end{bmatrix}\begin{bmatrix} \sin\alpha \\ \cos\beta \\ \tan\gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & -5 & 5 & 0 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$R_2 - 2R_1,\ R_3 + R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -3 & 8 & 0 \end{array}\right]$$

$$R_3 + 3R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$(-1)R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_2 + 3R_3,\ R_1 - 3R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_1 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\sin\alpha = 0$$
$$\cos\beta = 0$$
$$\tan\gamma = 0$$

From these equations,

$$\alpha = 0, \pi, 2\pi$$
$$\beta = \frac{\pi}{2}, \frac{3\pi}{2} \qquad [\because \alpha, \beta \text{ and } \gamma \text{ lie between 0 and } 2\pi]$$
$$\gamma = 0, \pi, 2\pi$$

Hence, there are $3 \cdot 2 \cdot 3 = 18$ possible solutions which satisfy the system of equations.

Example 5

For what value of λ does the following system of equations possess a non-trivial solution? Obtain the solution for real values of λ.

$$x + 2y + 3z = \lambda x$$
$$3x + y + 2z = \lambda y$$
$$2x + 3y + z = \lambda z$$

Solution

The system of equations is

$$(1-\lambda)x + 2y + 3z = 0$$
$$3x + (1-\lambda)y + 2z = 0$$
$$2x + 3y + (1-\lambda)z = 0$$

The matrix form of the system is

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system will possess a non-trivial solution if $\det(A) = 0$.

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(1-\lambda)^2 - 6] - 2(3 - 3\lambda - 4) + 3(9 - 2 + 2\lambda) = 0$$
$$(1-\lambda)(\lambda^2 - 2\lambda - 5) + 2 + 6\lambda + 21 + 6\lambda = 0$$
$$\lambda^2 - 2\lambda - 5 - \lambda^3 + 2\lambda^2 + 5\lambda + 12\lambda + 23 = 0$$
$$-\lambda^3 + 3\lambda^2 + 15\lambda + 18 = 0$$

$$\lambda = 6, \quad \lambda = -1.5 \pm 0.866\, i$$

For real value of λ, i.e. $\lambda = 6$, the augmented matrix of the system is

$$\left[\begin{array}{ccc|c} -5 & 2 & 3 & 0 \\ 3 & -5 & 2 & 0 \\ 2 & 3 & -5 & 0 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$R_2 - R_3$$

$$\sim \left[\begin{array}{ccc|c} -5 & 2 & 3 & 0 \\ 1 & -8 & 7 & 0 \\ 2 & 3 & -5 & 0 \end{array}\right]$$

$$R_{12}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -8 & 7 & 0 \\ -5 & 2 & 3 & 0 \\ 2 & 3 & -5 & 0 \end{array}\right]$$

$$R_2 + 5R_1,\ R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & -8 & 7 & 0 \\ 0 & -38 & 38 & 0 \\ 0 & 19 & -19 & 0 \end{array}\right]$$

$$\left(-\frac{1}{38}\right)R_2,\ \left(\frac{1}{19}\right)R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & -8 & 7 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -8 & 7 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 + 8R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x - z &= 0 \\ y - z &= 0 \end{aligned}$$

Solving for the leading variables,

$$\begin{aligned} x &= z \\ y &= z \end{aligned}$$

Assigning the free variable z an arbitrary value t,

$$\begin{aligned} x &= t \\ y &= t \end{aligned}$$

Hence, $x = t, y = t, z = t$ is the non-trivial solution of the system where t is a parameter.

Example 6

If the following system has a non-trivial solution, then prove that $a + b + c = 0$ *or* $a = b = c$ *and hence find the solution in each case.*

$$\begin{aligned} ax + by + cz &= 0 \\ bx + cy + az &= 0 \\ cx + ay + bz &= 0 \end{aligned}$$

Solution

The matrix form of the system is

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system has a non-trivial solution if det $(A) = 0$.

$$\begin{aligned} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} &= 0 \\ a(bc - a^2) - b(b^2 - ac) + c(ab - c^2) &= 0 \\ -a^3 + b^3 + c^3 - 3abc &= 0 \\ -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) &= 0 \end{aligned}$$

$$a+b+c=0$$

or

$$a^2+b^2+c^2-ab-bc-ca=0$$

$$\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]=0$$

$$a-b=0, b-c=0, c-a=0$$

$$a=b, b=c, c=a$$

$$a=b=c$$

Hence, the system has a non-trivial solution if $a+b+c=0$ or $a=b=c$.

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} a & b & c & 0 \\ b & c & a & 0 \\ c & a & b & 0 \end{array}\right]$$

$$R_3 + R_1 + R_2$$

$$\sim \left[\begin{array}{ccc|c} a & b & c & 0 \\ b & c & a & 0 \\ a+b+c & a+b+c & a+b+c & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} ax + by + cz &= 0 \\ bx + cy + az &= 0 \\ (a+b+c)x + (a+b+c)y + (a+b+c)z &= 0 \end{aligned}$$

(i) When $a+b+c=0$, we have only two equations.

$$ax+by+cz=0$$

$$bx+cy+az=0$$

$$\frac{x}{\begin{vmatrix} b & c \\ c & a \end{vmatrix}} = -\frac{y}{\begin{vmatrix} a & c \\ b & a \end{vmatrix}} = \frac{z}{\begin{vmatrix} a & b \\ b & c \end{vmatrix}} = t$$

$$\frac{x}{ab-c^2} = -\frac{y}{a^2-bc} = \frac{z}{ac-b^2} = t$$

Hence, $x=(ab-c^2)t, y=(bc-a^2)t, z=(ac-b^2)t$ is the solution of the system where t is a parameter.

(ii) When $a = b = c$, we have only one equation.

$$x+y+z=0$$

Let
$$y = t_1$$
$$z = t_2$$

Then
$$x = -t_1 - t_2$$

Hence, $x = -t_1 - t_2, y = t_1, z = t_2$ is the solution of the system where t_1 and t_2 are parameters.

Example 7

Discuss for all values of k, the system of equations

$$\begin{aligned} 2x + 3ky + (3k+4)z &= 0 \\ x + (k+4)y + (4k+2)z &= 0 \\ x + 2(k+1)y + (3k+4)z &= 0 \end{aligned}$$

Solution

The matrix form of the system is

$$\begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

R_{12}

$$\begin{bmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2k+2 & 3k+4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - 2R_1, R_3 - R_1$

$$\begin{bmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{vmatrix} \\ &= (k-8)(-k+2) + 5k(k-2) \\ &= (k-2)(-k+8+5k) \\ &= 4(k-2)(k+2) \end{aligned}$$

(i) When $k \neq \pm 2, \det(A) \neq 0,$ the system has a trivial solution, i.e. $x = 0$, $y = 0$, $z = 0$.

(ii) When $k = \pm 2, \det(A) = 0,$ the system has non-trivial solutions.

Case I: When $k = 2$, the augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 6 & 10 & 0 \\ 0 & -6 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left(-\frac{1}{6}\right)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 6 & 10 & 0 \\ 0 & 1 & \frac{10}{6} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 - 6R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{10}{6} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x = 0$$

$$y + \frac{10}{6}z = 0$$

Solving for the leading variables,

$$x = 0$$

$$y = -\frac{10}{6}z$$

Assigning the free variable z any arbitrary value t,

$$y = -\frac{10}{6}t = -\frac{5}{3}t$$

Hence, $x = 0, y = -\frac{5}{3}t, z = t$ is the solution of the system where t is a parameter.

Case II: When $k = -2$, the augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & -6 & 0 \\ 0 & -10 & 10 & 0 \\ 0 & -4 & 4 & 0 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$\left(-\frac{1}{10}\right)R_2, \left(-\frac{1}{4}\right)R_3$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & -6 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 - R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & -6 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 - 2R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x - 4z &= 0 \\ y - z &= 0 \end{aligned}$$

Solving for the leading variables,

$$\begin{aligned} x &= 4z \\ y &= z \end{aligned}$$

Assigning the free variable z any arbitrary value t,

$$\begin{aligned} x &= 4t \\ y &= t \end{aligned}$$

Hence, $x = 4t, y = t, z = t$ is the solution of the system where t is a parameter.

EXERCISE 10.4

1. Solve the following equations:

(i)
$$\begin{aligned} x - y + z &= 0 \\ x + 2y + z &= 0 \\ 2x + y + 3z &= 0 \end{aligned}$$

(ii)
$$\begin{aligned} x - 2y + 3z &= 0 \\ 2x + 5y + 6z &= 0 \end{aligned}$$

(iii)
$$\begin{aligned} 2x - 2y + 5z + 3w &= 0 \\ 4x - y + z + w &= 0 \\ 3x - 2y + 3z + 4w &= 0 \\ x - 3y + 7z + 6w &= 0 \end{aligned}$$

(iv)
$$\begin{aligned} 2x - y + 3z &= 0 \\ 3x + 2y + z &= 0 \\ x - 4y + 5z &= 0 \end{aligned}$$

(v)
$$\begin{aligned} 7x + y - 2z &= 0 \\ x + 5y - 4z &= 0 \\ 3x - 2y + z &= 0 \\ 2x - 7y + 5z &= 0 \end{aligned}$$

(vi)
$$\begin{aligned} 3x + 4y - z - 9w &= 0 \\ 2x + 3y + 2z - 3w &= 0 \\ 2x + y - 14z - 12w &= 0 \\ x + 3y + 13z + 3w &= 0 \end{aligned}$$

(vii)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 &= 0 \\ x_1 + x_2 - x_3 - x_4 &= 0 \\ 3x_1 - x_2 + 2x_3 + 3x_4 &= 0 \end{aligned}$$

(viii)
$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 0 \\ 3x_1 + 2x_2 + x_3 &= 0 \\ x_1 - 4x_2 + 5x_3 &= 0 \end{aligned}$$

Ans.: (i) $x = 0, y = 0, z = 0$ (ii) $x = -3t, y = 0, z = t$

(iii) $x = \frac{211}{9}t, y = 4t, z = \frac{7}{9}t, w = t$

(iv) $x = -t, y = t, z = t$ (v) $x = \frac{3}{17}t, y = \frac{13}{17}t, z = t$

(vi) $x = 11t, y = -8t, z = t, w = 0$

(vii) $x_1 = -\frac{1}{3}t, x_2 = \frac{2}{3}t, x_3 = -\frac{2}{3}t, x_4 = t$

(viii) $x_1 = -x_2 = -x_3 = t$

2. For what value of λ does the following system of equations possess a non-trivial solution? Obtain the solution for real values of λ.

(i)
$$\begin{aligned} 3x + y - \lambda z &= 0 \\ 4x - 2y - 3z &= 0 \\ 2\lambda x + 4y - \lambda z &= 0 \end{aligned}$$

(ii)
$$\begin{aligned} (1-\lambda)x_1 + 2x_2 + 3x_3 &= 0 \\ 3x_1 + (1-\lambda)x_2 + 2x_3 &= 0 \\ 2x_1 + 3x_2 + (1-\lambda)x_3 &= 0 \end{aligned}$$

Ans.:

(i) Non-trivial solution $\lambda = 1, -9$

For $\lambda = 1$, $x = -t, y = -t, z = -2t$

For $\lambda = -9$, $x = -3t, y = -9t, z = 2t$

(ii) $\lambda = 6$, $x = y = z = t$

3. Show that the system of equations $2x-2y+z=\lambda x, 2x-3y+2z=\lambda y,$ $-x+2y=\lambda z$ can posses a non-trivial solution only if $\lambda = 1$, $\lambda = -3$. Obtain the general solution in each case.

$$\left[\begin{array}{l}\textbf{Ans.: } \text{For } \lambda=1, x=2t_2-t_1, y=t_2, z=t_1 \\ \qquad\quad \text{For } \lambda=-3, x=-t, y=-2t, z=t\end{array}\right]$$

10.10 EIGENVALUES AND EIGENVECTORS

Eigenvalues and eigenvectors are important concepts in linear algebra. They are derived from the German word '*eigen*' which means proper or characteristic. Eigenvectors are nonzero vectors that get mapped into scalar multiples of themselves under a linear operator.

Any nonzero vector **x** is said to be a characteristic vector or eigenvector of a matrix A if there exists a number λ such that

$$A\mathbf{x} = \lambda \mathbf{x}$$

where $A=[a_{ij}]_{n\times n}$ is an n–rowed square matrix and $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$ is a non-zero column vector.

Also, λ is said to be characteristic root or characteristic value or eigenvalue of the matrix A.

Depending on the sign and the magnitude of the eigenvalue λ corresponding to **x**, the linear operator $A\mathbf{x} = \lambda\mathbf{x}$ compresses or stretches the eigenvector **x** by a factor λ. If λ is negative, the direction of the eigenvector reverses. (Fig. 10.1).

Now

$$A\mathbf{x} = \lambda\mathbf{x} = \lambda I\mathbf{x}$$

$$(A-\lambda I)\mathbf{x} = \mathbf{0}$$

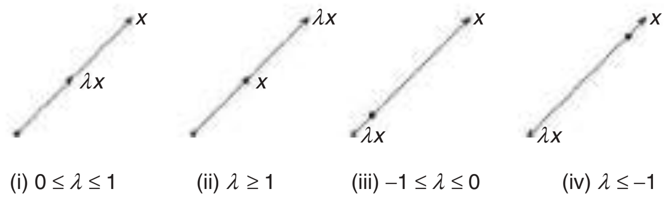

Fig. 6.1

The matrix $A - \lambda I$ is called the *characteristic matrix* of A where I is the unit matrix of order n.

$$\det(A-\lambda I)=\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

which is an ordinary polynomial in λ of degree n, is called the *characteristic polynomial* of A.

The equation $\det(A-\lambda I)=0$ is called the *characteristic equation* of A and the roots of this equation are called the eigenvalues of the matrix A. The set of all eigenvectors is called the *eigenspace* of A corresponding to λ. The set of all eigenvalues of A is called the *spectrum* of A.

Note: (1) The characteristic equation of the matrix A of order 2 can be obtained from

$$\lambda^2 - S_1\lambda + S_2 = 0$$

where S_1 = sum of principal diagonal elements and
S_2 = determinant A

(2) The characteristic equation of the matrix A of order 3 can be obtained from

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = sum of principal diagonal elements,
S_2 = sum of minors of principal diagonal elements and
S_3 = determinant A

(3) The sum of the eigenvalues of a matrix is the sum of its principal diagonal elements.
(4) The product of the eigenvalues of a matrix is the determinant of the matrix.

10.11 PROPERTIES OF EIGENVALUES

Property 1: If λ is an eigenvalue of the matrix A then λ is also an eigenvalue of A^T.

Proof: Let λ be an eigenvalue of the matrix A.

The characteristic equation of A is $\det(A-\lambda I)=0$.

The characteristic equation of A^T is $\det(A^T-\lambda I)=0$.

The determinant value does not change by the interchange of rows and columns.

$$\det(A-\lambda I)=\det(A^T-\lambda I)$$

The characteristic equations are same for both A and A^T.

Hence, λ is also an eigenvalue of A^T.

Property 2: If λ is an eigenvalue of the non-singular matrix A then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Proof: Let λ be an eigenvalue of the non-singular matrix A.

$$A\mathbf{x}=\lambda\mathbf{x} \qquad \ldots(10.1)$$

Premultiplying both sides of Eq. (10.1) by A^{-1},

$$A^{-1}A\mathbf{x} = A^{-1}\lambda\mathbf{x}$$
$$I\mathbf{x} = \lambda A^{-1}\mathbf{x}$$
$$\mathbf{x} = \lambda A^{-1}\mathbf{x}$$
$$A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$$

Hence, $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Property 3: If λ is an eigenvalue of the matrix A then λ^k is an eigenvalue of A^k.

Proof: Let λ be an eigenvalue of the matrix A.

$$A\mathbf{x} = \lambda\mathbf{x} \qquad \text{...(10.2)}$$

Premultiplying both sides of Eq. (10.2) by A,

$$AA\mathbf{x} = A\lambda\mathbf{x}$$
$$A^2\mathbf{x} = \lambda(A\mathbf{x})$$
$$= \lambda(\lambda\mathbf{x})$$
$$= \lambda^2\mathbf{x}$$

Similarly, $A^3\mathbf{x} = \lambda^3\mathbf{x}$

In general, $A^k\mathbf{x} = \lambda^k\mathbf{x}$

Hence, λ^k is an eigenvalue of A^k.

Property 4: If λ is an eigenvalue of the matrix A then $\lambda \pm k$ is an eigenvalue of $A \pm kI$.

Proof: Let λ be an eigenvalue of the matrix A.

$$A\mathbf{x} = \lambda\mathbf{x} \qquad \text{...(10.3)}$$

Adding $kI\mathbf{x}$ on both sides of Eq. (10.3),

$$A\mathbf{x} + kI\mathbf{x} = \lambda\mathbf{x} + kI\mathbf{x}$$
$$(A + kI)\mathbf{x} = \lambda\mathbf{x} + k\mathbf{x}$$
$$= (\lambda + k)\mathbf{x}$$

Similarly, $(A - kI)\mathbf{x} = (\lambda - k)\mathbf{x}$

In general, $(A \pm kI)x = (\lambda \pm k)x$

Hence, $\lambda \pm k$ is an eigenvalue of $A \pm kI$.

Property 5: If λ is an eigenvalue of the matrix A then $k\lambda$ is an eigenvalue of kA.

Proof: Let λ be an eigenvalue of the matrix A.

$$A\mathbf{x} = \lambda\mathbf{x} \qquad \text{...(10.4)}$$

Multiplying both sides of Eq. (10.4) by the scalar k,

$$kA\mathbf{x} = k\lambda\mathbf{x}$$

Hence, $k\lambda$ is an eigenvalue of kA.

Property 6: The eigenvalues of a triangular matrix are the diagonal elements of the matrix.

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ be a triangular matrix of order n.

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn}-\lambda \end{bmatrix} = 0$$

$$(a_{11}-\lambda)(a_{22}-\lambda)\ldots(a_{nn}-\lambda) = 0$$

$$\lambda = a_{11}, a_{22}, \ldots, a_{nn}$$

Hence, the eigenvalues of A are $a_{11}, a_{22}, \ldots, a_{nn}$ which are the diagonal elements of the matrix.

Property 7: The eigenvalues of a real symmetric matrix are real.

Proof: Let λ be an eigenvalue of the real symmetric matrix.

$$A\mathbf{x} = \lambda\mathbf{x} \qquad \text{... (10.5)}$$

Premultiplying both sides of Eq. (10.5) by $\bar{\mathbf{x}}^T$,

$$\bar{\mathbf{x}}^T A\mathbf{x} = \lambda\bar{\mathbf{x}}^T\mathbf{x} \qquad \text{... (10.6)}$$

Taking complex conjugate on both sides of Eq. (10.6),

$$\mathbf{x}^T \bar{A}\bar{\mathbf{x}} = \bar{\lambda}\mathbf{x}^T\bar{\mathbf{x}}$$

$$\mathbf{x}^T A\bar{\mathbf{x}} = \bar{\lambda}\mathbf{x}^T\bar{\mathbf{x}} \quad (\because \bar{A} = A \text{ for real matrix}) \qquad \text{... (10.7)}$$

Taking transpose on both sides of Eq. (10.7),

$$\bar{\mathbf{x}}^T A^T\mathbf{x} = \bar{\lambda}\bar{\mathbf{x}}^T\mathbf{x}$$

$$\bar{\mathbf{x}}^T A\mathbf{x} = \bar{\lambda}\bar{\mathbf{x}}^T\mathbf{x} \quad (\because A^T = A \text{ for symmetric matrix}) \qquad \text{... (10.8)}$$

From Eqs (10.6) and (10.8),

$$\lambda\bar{x}^T x = \bar{\lambda}\bar{x}^T x$$

$$(\lambda - \bar{\lambda})\bar{x}^T x = 0$$

$\bar{\mathbf{x}}^T\mathbf{x}$ is a 1×1 matrix, i.e., a single element which is positive,

$$\lambda - \bar{\lambda} = 0$$

i.e., λ is real.

Hence, the eigenvalues of a real symmetric matrix are real.

Example 1

Find the sum and product of the eigenvalues of the matrix $A = \begin{bmatrix} 2 & -3 \\ 4 & -2 \end{bmatrix}$.

Solution

$$\begin{aligned} \text{Sum of the eigenvalues of } A &= \text{Sum of principal diagonal elements of } A \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{Product of the eigenvalues of } A &= \text{Determinant of } A \\ &= \begin{vmatrix} 2 & -3 \\ 4 & -2 \end{vmatrix} \\ &= -4 + 12 \\ &= 8 \end{aligned}$$

Example 2

Find the sum and product of the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 3 & 6 & 7 \end{bmatrix}$.

Solution

$$\begin{aligned} \text{Sum of the eigenvalues of } A &= \text{Sum of principal diagonal elements of } A \\ &= 1 + 3 + 7 \\ &= 11 \end{aligned}$$

$$\begin{aligned} \text{Product of the eigenvalues of } A &= \text{Determinant of } A \\ &= \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 3 & 6 & 7 \end{vmatrix} \\ &= 1(21 - 24) - 2(14 - 12) + 5(12 - 9) \\ &= -3 - 4 + 15 \\ &= 8 \end{aligned}$$

Example 3

The product of two eigenvalues of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ *is* 16. *Find the third eigenvalue.*

Solution

Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A.

$$\lambda_1 \lambda_2 = 16$$

Product of the eigenvalues of A = Determinant of A

$$\lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$16\lambda_3 = 6(9-1) + 2(-6+2) + 2(2-6)$$

$$= 48 - 8 - 8$$

$$\lambda_3 = 2$$

Example 4

Two eigenvalues of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ *are equal and are* $\frac{1}{5}$ *times to the third. Find the eigenvalues.*

Solution

Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A.

$$\lambda_1 = \lambda_2$$

$$\lambda_1 = \frac{1}{5}\lambda_3$$

$$\lambda_2 = \frac{1}{5}\lambda_3$$

Sum of the eigenvalues of A = Sum of principal diagonal elements of A

$$\lambda_1 + \lambda_2 + \lambda_3 = 2 + 3 + 2$$

$$\frac{1}{5}\lambda_3 + \frac{1}{5}\lambda_3 + \lambda_3 = 7$$

$$\frac{7}{5}\lambda_3 = 7$$

$$\lambda_3 = 5$$

$$\therefore \lambda_1 = \lambda_2 = 1$$

Hence, the eigenvalues of A are 1, 1, 5.

Example 5

If 2 is an eigenvalue of the matrix $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$, *find the other two eigen values.*

Solution

Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A.

$$\lambda_1 = 2$$

Sum of the eigenvalues of A = Sum of principal diagonal elements of A

$$2 + \lambda_2 + \lambda_3 = 2 + 1 - 1$$

$$\lambda_2 + \lambda_3 = 0 \qquad \ldots(1)$$

Product of the eigenvalues of A = Determinant of A

$$2\lambda_2\lambda_3 = \begin{vmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix}$$

$$= 2(-1-3) + 2(-1-1) + 2(3-1)$$

$$= -8 - 4 + 4$$

$$= -8$$

$$\lambda_2\lambda_3 = -4 \qquad \ldots(2)$$

Solving Eqs (1) and (2),

$$\lambda_2 = 2,\ \lambda_3 = -2$$

Hence, the other two eigenvalues are 2, –2.

Example 6

For a given matrix A of order 3, $|A| = 32$ *and two of its eigenvalues are 8 and 2. Find the sum of the eigenvalues.*

Solution

Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A.

$$\lambda_1 = 8,\ \lambda_2 = 2$$

Product of the eigenvalues of A = Determinant of A

$$\lambda_1\lambda_2\lambda_3 = |A|$$

$$8 \times 2 \times \lambda_3 = 32$$

$$\lambda_3 = 2$$

Hence, the sum of the eigenvalues = 8 + 2 + 2 = 12

Example 7

For a singular matrix of order three, 2 and 3 are the eigenvalues. Find its third eigenvalue.

Solution

Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A.

$$\lambda_1 = 2, \lambda_2 = 3$$

For a singular matrix, $|A| = 0$

Product of the eigenvalues of A = Determinant of A

$$\lambda_1\lambda_2\lambda_3 = |A|$$
$$2 \times 3 \times \lambda_3 = 0$$
$$\lambda_3 = 0$$

Hence, the third eigenvalue = 0

Example 8

If 2, 3 are the eigenvalues of $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$, *find the value of a.*

Solution

Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A.

$$\lambda_1 = 2, \lambda_2 = 3$$

Sum of the eigenvalues of A = Sum of principal diagonal elements of A

$$2 + 3 + \lambda_3 = 2 + 2 + 2$$
$$\lambda_3 = 1$$

Product of the eigenvalues of A = Determinant of A

$$\lambda_1\lambda_2\lambda_3 = |A|$$
$$2 \times 3 \times 1 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{vmatrix}$$
$$6 = 2(4-0) - 0(0-0) + 1(0-2a)$$
$$= 8 - 2a$$
$$2a = 2$$
$$a = 1$$

Example 9

If 3 and 15 are the two eigenvalues of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, *find* $|A|$ *without expanding the determinant.*

Solution

Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A.

$$\lambda_1 = 3, \lambda_2 = 15$$

Sum of the eigenvalues of A = Sum of principal diagonal elements of A

$$3 + 15 + \lambda_3 = 8 + 7 + 3$$
$$\lambda_3 = 0$$

Product of the eigenvalues of A = Determinant of A

$$3 \times 15 \times 0 = |A|$$
$$|A| = 0$$

Example 10

Find a and b such that $A = \begin{bmatrix} a & 4 \\ 1 & b \end{bmatrix}$ *has 3 and –2 as eigenvalues.*

Solution

Let λ_1 and λ_2 be the eigenvalues of the matrix A.

$$\lambda_1 = 3, \lambda_2 = -2$$

Sum of the eigenvalues of A = Sum of principal diagonal elements of A

$$3 - 2 = a + b$$
$$a + b = 1 \qquad \ldots(1)$$

Product of the eigenvalues of A = Determinant of A

$$\lambda_1 \lambda_2 = |A|$$
$$(3)(-2) = \begin{vmatrix} a & 4 \\ 1 & b \end{vmatrix}$$
$$= ab - 4$$
$$ab = -2 \qquad \ldots(2)$$

From Eq. (1), $a = 1 - b$

Substituting in Eq. (2),

$$(1-b)b = -2$$
$$b - b^2 = -2$$
$$b^2 - b - 2 = 0$$
$$b = 2 \quad \text{or} \quad b = -1$$
$$\therefore a = -1 \quad \text{or} \quad a = 2$$

Example 11

Two eigenvalues of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ *are* 1 *and* 1. *Find the eigenvalues of* A^{-1}.

Solution

Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A.

$$\lambda_1 = \lambda_2 = 1$$

Sum of the eigenvalues of A = Sum of principal diagonal elements of A

$$1 + 1 + \lambda_3 = 2 + 3 + 2$$
$$\lambda_3 = 5$$

Hence, the eigenvalues of A are 1, 1, 5.

The eigenvalues of A^{-1} are $\frac{1}{1}, \frac{1}{1}, \frac{1}{5}$, i.e., $1, 1, \frac{1}{5}$.

Example 12

Two of the eigenvalues of the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ *are* 3 *and* 6. *Find the eigenvalues of* A^{-1} *and* A^3.

Solution

Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A.

$$\lambda_1 = 3, \lambda_2 = 6$$

Sum of the eigenvalues of A = Sum of the principal diagonal elements of A

$$3 + 6 + \lambda_3 = 3 + 5 + 3$$
$$\lambda_3 = 2$$

The eigenvalues of A are 3, 6, 2.

Hence, the eigenvalues of A^{-1} are $\frac{1}{3}, \frac{1}{6}, \frac{1}{2}$, and the eigenvalues of A^3 are $3^3, 6^3, 2^3$, i.e., 27, 216, 8.

Example 13

Form the matrix whose eigenvalues are $\alpha - 5$, $\beta - 5$, $\gamma - 5$ where α, β, γ are the eigenvalues of $A = \begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix}$.

Solution

If λ_1, λ_2 and λ_3 are eigenvalues of the matrix A then $\lambda_1 - k$, $\lambda_2 - k$ and $\lambda_3 - k$ are the eigenvalues of $A - kI$.

$$\text{Required matrix} = A - 5I = \begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -2 & -3 \\ 4 & 0 & -6 \\ 7 & -8 & 4 \end{bmatrix}$$

Example 14

If α and β are the eigenvalues of $\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$, *form a matrix whose eigenvalues are α^3 and β^3.*

Solution

If λ_1 and λ_2 are the eigenvalues of the matrix A then λ_1^k and λ_2^k are the eigenvalues of A^k.

Let $$A = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 10 & -8 \\ -8 & 26 \end{bmatrix}$$

$$\text{Required matrix} = A^3 = A^2A = \begin{bmatrix} 10 & -8 \\ -8 & 26 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 38 & -50 \\ -50 & 138 \end{bmatrix}$$

Example 15

If $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, *find the eigenvalues for the following matrices:*

(i) A *(ii)* A^T *(iii)* A^{-1} *(iv)* $4A^{-1}$ *(v)* A^2 *(vi)* $A^2 - 2A + I$ *(vii)* $A^3 + 2I$

Solution

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 3 + 5 + 3 = 11$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$$= (15-1) + (9-1) + (15-1)$$

$$= 14 + 8 + 14$$

$$= 36$$

$$S_3 = \det(A) = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 3(15-1) + 1(-3+1) + 1(1-5)$$

$$= 42 - 2 - 4$$

$$= 36$$

Hence, the characteristic equation is

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = 2, 3, 6$$

(i) Eigenvalues of $A = \lambda$: 2, 3, 6

(ii) Eigenvalues of $A^T = \lambda$: 2, 3, 6

(iii) Eigenvalues of $A^{-1} = \lambda^{-1}$: $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

(iv) Eigenvalues of $4A^{-1} = 4\lambda^{-1}$: $2, \frac{4}{3}, \frac{2}{3}$

(v) Eigenvalues of $A^2 = \lambda^2$: 4, 9, 36

(vi) Eigenvalues of $A^2 - 2A + I = \lambda^2 - 2\lambda + 1$: 1, 4, 25

(vii) Eigenvalues of $A^3 + 2I = \lambda^3 + 2$: 10, 29, 218

10.12 LINEAR DEPENDENCE AND INDEPENDENCE OF EIGENVECTORS

10.12.1 Linear Dependence

A set of r eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots \mathbf{x}_r$ is said to be linearly dependent if there exist r scalars (numbers) $k_1, k_2, \dots k_r$ not all zero, such that

$$k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_r\mathbf{x}_r = \mathbf{0}$$

10.12.2 Linear Independence

A set of r eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots \mathbf{x}_r$ is said to be linearly independent if there exist r scalars (numbers) $k_1, k_2, \dots k_r$ such that if

$$k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_r\mathbf{x}_r = \mathbf{0}$$

then $k_1 = k_2 = \cdots = k_r = 0$

Note: (1) If a set of eigenvectors is linearly dependent then at least one eigenvector of the set can be expressed as a linear combination of the remaining eigenvectors.

(2) If a set of eigenvectors is linearly independent then no eigenvector of the set can be expressed as a linear combination of the remaining eigenvectors.

10.13 PROPERTIES OF EIGENVECTORS

Property 1: If $\mathbf{x}$ is an eigenvector of a matrix A corresponding to the eigenvalue λ then $k\mathbf{x}$ is also an eigenvector of A corresponding to the same eigenvalue λ, where k is a nonzero scalar.

Proof: Let $\mathbf{x}$ be an eigenvector of a matrix A corresponding to the eigenvalue λ.

$$A\mathbf{x} = \lambda\mathbf{x}$$

and $$\mathbf{x} \neq 0$$

If k is any nonzero scalar then $k\mathbf{x} \neq 0$.

Also, $$A(k\mathbf{x}) = k(A\mathbf{x}) = k(\lambda\mathbf{x}) = \lambda(k\mathbf{x})$$

Hence, $k\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue λ. Thus, corresponding to an eigenvalue λ, there is more than one eigenvectors.

Property 2: If $\mathbf{x}$ is an eigenvector of a matrix A then $\mathbf{x}$ cannot correspond to more than one eigenvalue of A.

Proof: Let $\mathbf{x}$ be an eigenvector of a matrix A corresponding to two eigenvalues λ_1 and λ_2.

$$A\mathbf{x} = \lambda_1\mathbf{x}$$

and

$$A\mathbf{x} = \lambda_2\mathbf{x}$$

$$\therefore \quad \lambda_1\mathbf{x} = \lambda_2\mathbf{x}$$

$$(\lambda_1 - \lambda_2)\mathbf{x} = 0$$

$$\lambda_1 - \lambda_2 = 0 \qquad (\because \mathbf{x} \neq 0)$$

$$\lambda_1 = \lambda_2$$

Hence, $\mathbf{x}$ cannot correspond to more than one eigenvalue of A.

Property 3: The eigenvectors corresponding to distinct eigenvalues of a matrix are linearly independent.

Proof: Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ be the eigenvectors of a matrix A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$.

$$A\mathbf{x}_i = \lambda_i\mathbf{x}_i, i = 1, 2, \ldots, m$$

Let the eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ be linearly dependent. Choose r such that the eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$ are linearly independent but $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r, \mathbf{x}_{r+1}$ are linearly dependent, where $1 \le r < m$.

Then

$$k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_r\mathbf{x}_r + k_{r+1}\mathbf{x}_{r+1} = 0 \qquad \ldots (10.9)$$

where $k_1, k_2, \ldots, k_r, k_{r+1}$ are scalars not all zero.

$$A(k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_r\mathbf{x}_r + k_{r+1}\mathbf{x}_{r+1}) = 0$$

$$k_1(A\mathbf{x}_1) + k_2(A\mathbf{x}_2) + \cdots + k_r(A\mathbf{x}_r) + k_{r+1}(A\mathbf{x}_{r+1}) = 0$$

$$k_1(\lambda_1\mathbf{x}_1) + k_2(\lambda_2\mathbf{x}_2) + \cdots + k_r(\lambda_r\mathbf{x}_r) + k_{r+1}(\lambda_{r+1}\mathbf{x}_{r+1}) = 0 \qquad \ldots (10.10)$$

Multiplying Eq. (10.9) by λ_{r+1} and subtracting from Eq. (10.10),

$$k_1(\lambda_1 - \lambda_{r+1})\mathbf{x}_1 + k_2(\lambda_2 - \lambda_{r+1})\mathbf{x}_2 + \cdots + k_r(\lambda_r - \lambda_{r+1})\mathbf{x}_r = 0 \qquad \ldots (10.11)$$

Since $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$ are linearly independent,

$$k_1(\lambda_1 - \lambda_{r+1}) = 0, k_2(\lambda_2 - \lambda_{r+1}) = 0, \ldots k_r(\lambda_r - \lambda_{r+1}) = 0$$

But $\lambda_1, \lambda_2, \ldots, \lambda_r, \lambda_{r+1}$ are distinct.

$$\therefore k_1 = 0, k_2 = 0, \ldots, k_r = 0$$

Putting in Eq. (10.9),

$$k_{r+1}\mathbf{x}_{r+1} = 0$$

$$k_{r+1} = 0 \quad [\because \quad \mathbf{x}_{r+1} \neq 0]$$

Thus, for Eq. (10.9),

$$k_1 = 0, k_2 = 0, k_r = 0, k_{r+1} = 0$$

which contradicts our assumption that $k_1, k_2, \ldots, k_r, k_{r+1}$ are not all zero.

Thus, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{r+1}$ are linearly independent and hence $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ are linearly independent.

Note: If two or more eigenvalues are equal then the corresponding eigenvectors may or may not be linearly independent.

Property 4: The eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal.

Proof: Let λ_1 and λ_2 be the two distinct eigenvalues of a real symmetric matrix A and $\mathbf{x}_1$ and $\mathbf{x}_2$ be the corresponding eigenvectors respectively.

$$A\mathbf{x}_1 = \lambda_1\mathbf{x}_1 \qquad \ldots(10.12)$$

and

$$A\mathbf{x}_2 = \lambda_2\mathbf{x}_2 \qquad \ldots(10.13)$$

Premultiplying both sides of Eq. (10.12) by $\mathbf{x}_2^T$,

$$\mathbf{x}_2^T A\mathbf{x}_1 = \lambda_1\mathbf{x}_2^T\mathbf{x}_1$$

Taking the transpose on both sides,

$$\mathbf{x}_1^T A\mathbf{x}_2 = \lambda_1\mathbf{x}_1^T\mathbf{x}_2 \quad (\because A^T = A) \qquad \ldots(10.14)$$

Premultiplying both sides of Eq. (10.13) by $\mathbf{x}_1^T$,

$$\mathbf{x}_1^T A\mathbf{x}_2 = \lambda_2\mathbf{x}_1^T\mathbf{x}_2 \qquad \ldots(10.15)$$

From Eqs (10.14) and (10.15),

$$\lambda_1\mathbf{x}_1^T\mathbf{x}_2 = \lambda_2\mathbf{x}_1^T\mathbf{x}_2$$

$$(\lambda_1 - \lambda_2)\mathbf{x}_1^T\mathbf{x}_2 = 0$$

$$\mathbf{x}_1^T\mathbf{x}_2 = 0 \quad (\because \lambda_1 \neq \lambda_2)$$

Hence, the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

Working Rule for Finding the Eigenvalues and Eigenvectors

(i) Write characteristic equation det $(A - \lambda I) = 0$ for the given square matrix.

(ii) Find the eigenvalues of the matrix by solving characteristic equation.

(iii) Find eigenvectors corresponding to each eigenvalues from the equation $[A - \lambda I]\,\mathbf{x} = \mathbf{0}$.

Example 1

Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

Solution

Let

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - S_1\lambda + S_2 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 5 + 2 = 7$

$$S_2 = \det(A) = \begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} = 10 - 4 = 6$$

Hence, the characteristic equation is

$$\lambda^2 - 7\lambda^2 + 6 = 0$$
$$\lambda = 6, 1$$

(a) For $\lambda = 6$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$-x + 4y = 0$$

Let

$$y = t$$
$$x = 4t$$

$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 6$.

(b) For $\lambda = 1$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x + y = 0$$

Let

$$y = t$$
$$x = -t$$

$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix} = t\mathbf{x}_2$ where $\mathbf{x}_2$ is an eigenvector corresponding to $\lambda = 1$.

Example 2

Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$.

Solution

Let

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix}=0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 4 + 3 - 3 = 4$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 3 & 2 \\ -4 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}$$

$$= (-9+8)+(-12+6)+(12-6)$$

$$= -1-6+6$$

$$= -1$$

$$S_3 = \det(A) = \begin{vmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{vmatrix}$$

$$= 4(-9+8)-6(-3+2)+6(-4+3)$$

$$= -4+6-6$$

$$= -4$$

Hence, the characteristic equation is

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

$$\lambda = -1,\ 1,\ 4$$

(a) For $\lambda = -1$, $$[A-\lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 5 & 6 & 6 \\ 1 & 4 & 2 \\ -1 & -4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x+6y+6z=0$$

$$x+4y+2z=0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 6 & 6 \\ 4 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & 6 \\ 1 & 4 \end{vmatrix}} = t$$

$$\frac{x}{-12}=\frac{y}{-4}=\frac{z}{14}=t$$

$$\frac{x}{-6}=\frac{y}{-2}=\frac{z}{7}=t$$

$$\mathbf{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-6t\\-2t\\7t\end{bmatrix}=t\begin{bmatrix}-6\\-2\\7\end{bmatrix}=t\mathbf{x}_1 \text{ where } \mathbf{x}_1 \text{ is an eigenvector corresponding to } \lambda=-1.$$

(b) For $\lambda = 1$,

$$[A-\lambda I]\mathbf{x}=\mathbf{0}$$

$$\begin{bmatrix}3 & 6 & 6\\1 & 2 & 2\\-1 & -4 & -4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$x+2y+2z=0$$
$$-x-4y-4z=0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix}2 & 2\\-4 & -4\end{vmatrix}}=-\frac{y}{\begin{vmatrix}1 & 2\\-1 & -4\end{vmatrix}}=\frac{z}{\begin{vmatrix}1 & 2\\-1 & -4\end{vmatrix}}=t$$

$$\frac{x}{0}=\frac{y}{2}=\frac{z}{-2}=t$$

$$\frac{x}{0}=\frac{y}{1}=\frac{z}{-1}=t$$

$$\mathbf{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\t\\-t\end{bmatrix}=t\begin{bmatrix}0\\1\\-1\end{bmatrix}=t\mathbf{x}_2 \text{ where } \mathbf{x}_2 \text{ is an eigenvector corresponding to } \lambda=1.$$

(c) For $\lambda = 4$,

$$[A-\lambda I]\mathbf{x}=\mathbf{0}$$

$$\begin{bmatrix}0 & 6 & 6\\1 & -1 & 2\\-1 & -4 & -7\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$0x+6y+6z=0$$
$$x-\ y+2z=0$$
$$x-4y-7z=0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix}6 & 6\\-1 & 2\end{vmatrix}}=-\frac{y}{\begin{vmatrix}0 & 6\\1 & 2\end{vmatrix}}=\frac{z}{\begin{vmatrix}0 & 6\\1 & -1\end{vmatrix}}=t$$

$$\frac{x}{18}=\frac{y}{6}=\frac{z}{-6}=t$$

$$\frac{x}{3}=\frac{y}{1}=\frac{z}{-1}=t$$

$$\mathbf{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3t\\t\\-t\end{bmatrix}=t\begin{bmatrix}3\\1\\-1\end{bmatrix}=t\mathbf{x}_3 \text{ where } \mathbf{x}_3 \text{ is an eigenvector corresponding to } \lambda=4.$$

Example 3

Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix}1 & 0 & -1\\1 & 2 & 1\\2 & 2 & 3\end{bmatrix}$.

Solution

Let
$$A=\begin{bmatrix}1 & 0 & -1\\1 & 2 & 1\\2 & 2 & 3\end{bmatrix}$$

The characteristic equation is
$$\det(A-\lambda I)=0$$

$$\begin{vmatrix}1-\lambda & 0 & -1\\1 & 2-\lambda & 1\\2 & 2 & 3-\lambda\end{vmatrix}=0$$

$$\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 + 2 + 3 = 6$

S_2 = Sum of the minors of principal diagonal elements of A

$$=\begin{vmatrix}2 & 1\\2 & 3\end{vmatrix}+\begin{vmatrix}1 & -1\\2 & 3\end{vmatrix}+\begin{vmatrix}1 & 0\\1 & 2\end{vmatrix}$$

$$=(6-2)+(3+2)+(2-0)$$

$$=4+5+2$$

$$=11$$

$$S_3=\det(A)=\begin{vmatrix}1 & 0 & -1\\1 & 2 & 1\\2 & 2 & 3\end{vmatrix}$$

$$=1(6-2)-0-1(2-4)$$

$$=6$$

Hence, the characteristic equation is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$
$$\lambda = 1, 2, 3$$

(a) For $\lambda = 1$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x + 0y - z = 0$$
$$x + \ \ y + z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix}} = t$$

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{0} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ -t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = t\mathbf{x}_1$$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 1$.

(b) For $\lambda = 2$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 0y + z = 0$$
$$2x + 2y + z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix}} = t$$

$$\frac{x}{-2} = \frac{y}{1} = \frac{z}{2} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ 2t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = t\mathbf{x}_2$$ where $\mathbf{x}_2$ is an eigenvector corresponding to $\lambda = 2$.

(c) For $\lambda = 3$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x + 0y - z = 0$$
$$x - y + z = 0$$
$$2x + 2y + 0z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix}} = t$$

$$\frac{x}{-1} = \frac{y}{1} = \frac{z}{2} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 2t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = t\mathbf{x}_3$$ where $\mathbf{x}_3$ is an eigenvector corresponding to $\lambda = 3$.

Example 4

Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Solution

Let

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & -2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 8 + 7 + 3 = 18$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$$= (21-16)+(24-4)+(56-36)$$

$$= 5+20+20$$

$$= 45$$

$$S_3 = \det(A) = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8(21-16)+6(-18+8)+2(24-14)$$

$$= 40-60+20$$

$$= 0$$

Hence, the characteristic equation is

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda = 0,\ 3,\ 15$$

(a) For $\lambda = 0$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$8x - 6y + 2z = 0$$

$$-6x + 7y - 4z = 0$$

$$2x - 4y + 3z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}} = t$$

$$\frac{x}{10} = \frac{y}{20} = \frac{z}{20} = t$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{2} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ 2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = t\mathbf{x}_1$$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 0$.

(b) For $\lambda = 3$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x - 6y + 2z = 0$$
$$-6x + 4y - 4z = 0$$
$$2x - 4y + 0z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 5 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}} = t$$

$$\frac{x}{16} = \frac{y}{8} = \frac{z}{-16} = t$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ -2t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = t\mathbf{x}_2$$ where $\mathbf{x}_2$ is an eigenvector corresponding to $\lambda = 3$.

(c) For $\lambda = 15$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x - 6y + 2z = 0$$
$$-6x - 8y - 4z = 0$$
$$2x - 4y - 12z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \frac{y}{\begin{vmatrix} -7 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}} = t$$

$$\frac{x}{40} = -\frac{y}{40} = \frac{z}{20} = t$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = t\mathbf{x}_3$$ where $\mathbf{x}_3$ is an eigenvector corresponding to $\lambda = 15$.

Note: The eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal which can be verified with this example.

$$\mathbf{x}_1^T\mathbf{x}_2 = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = [0] = \mathbf{0}$$

$$\mathbf{x}_2^T\mathbf{x}_3 = \begin{bmatrix} 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = [0] = \mathbf{0}$$

$$\mathbf{x}_3^T\mathbf{x}_1 = \begin{bmatrix} 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = [0] = \mathbf{0}$$

Thus, $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are orthogonal to each other.

Example 5

Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$.

Solution

Let $$A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 4 + 3 + 1 = 8$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 2 \\ -5 & 3 \end{vmatrix}$$

$$= (3-8) + (4-4) + (12+10)$$

$$= -5 + 0 + 22$$

$$= 17$$

$$S_3 = \det(A) = \begin{vmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{vmatrix}$$

$$= 4(3-8) - 2(-5+4) - 2(-20+6)$$

$$= -20 + 2 + 28$$

$$= 10$$

Hence, the characteristic equation is

$$\lambda^3 - 8\lambda^2 + 17\lambda - 10 = 0$$

$$\lambda = 1, 2, 5$$

(a) For $\lambda = 1$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x + 2y - 2z = 0$$

$$-5x + 2y + 2z = 0$$

$$-2x + 2y + 0z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & -2 \\ 2 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 3 & -2 \\ -5 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 2 \\ -5 & 2 \end{vmatrix}} = t$$

$$\frac{x}{8} = \frac{y}{4} = \frac{z}{16} = t$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{4} = t$$

$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ 4t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = t\mathbf{x}_1$ where vector $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 1$.

(b) For $\lambda = 2$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x + 2y - 2z = 0$$

$$-5x + y + 2z = 0$$

$$-2x + 2y - z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 2 & -2 \\ -5 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & 2 \\ -5 & 1 \end{vmatrix}} = t$$

$$\frac{x}{6} = \frac{y}{6} = \frac{z}{12} = t$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{2} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} t \\ t \\ 2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = t\mathbf{x}_2 \text{ where vector } \mathbf{x}_2 \text{ is an eigenvector corresponding to } \lambda = 2.$$

(c) For $\lambda = 5$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x + 2y - 2z = 0$$

$$-5x - 2y + 2z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -1 & -2 \\ -5 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 2 \\ -5 & -2 \end{vmatrix}} = t$$

$$\frac{x}{0} = \frac{y}{12} = \frac{z}{12} = t$$

$$\frac{x}{0} = \frac{y}{1} = \frac{z}{1} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = t\mathbf{x}_3 \text{ where } \mathbf{x}_3 \text{ is an eigenvector corresponding to } \lambda = 5.$$

Example 6

Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

Solution

Let

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix}=0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = -2 + 1 + 0 = -1$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= (0-12)+(0-3)+(-2-4)$$

$$= -12-3-6$$

$$= -21$$

$$S_3 = \det(A) = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$

$$= (-2)(0-12)-2(0-6)-3(-4+1)$$

$$= 24+12+9$$

$$= 45$$

Hence, the characteristic equation is

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\lambda = 5, -3, -3$$

(a) For $\lambda = 5$,

$$[A-\lambda I]\mathbf{x} = 0$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x+2y-3z=0$$

$$2x-4y-6z=0$$

$$-x-2y-5z=0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}} = t$$

$$\frac{x}{-24} = \frac{y}{-48} = \frac{z}{24} = t$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = t\mathbf{x}_1 \text{ where } \mathbf{x}_1 \text{ is an eigenvector corresponding to } \lambda = 5.$$

(b) For $\lambda = -3$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 2y - 3z = 0$$

Let $y = t_1$ and $z = t_2$

$$x = -2t_1 + 3t_2$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t_1 + 3t_2 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} -2t_1 \\ t_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3t_2 \\ 0 \\ t_2 \end{bmatrix} = t_1\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = t_1\mathbf{x}_2 + t_2\mathbf{x}_3$$

where $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent eigenvectors corresponding to $\lambda = -3$.

Example 7

Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Solution

Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 0 + 0 + 0 = 0$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= (0-1)+(0-1)+(0-1)$$

$$= -3$$

$$S_3 = \det(A) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 0 - 1(0-1) + 1(1-0)$$

$$= 2$$

Hence, the characteristic equation is

$$\lambda^3 - 3\lambda - 2 = 0$$

$$\lambda = 2, -1, -1$$

(a) For $\lambda = 2$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x + y + z = 0$$

$$x - 2y + z = 0$$

$$x + y - 2z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}} = t$$

$$\frac{x}{3} = \frac{y}{3} = \frac{z}{3} = t$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = t\mathbf{x}_1$$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 2$.

(b) For $\lambda = -1$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + y + z = 0$$

Let $y = t_1$ and $z = t_2$

$$\mathbf{x} = -t_1 - t_2$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t_1 - t_2 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} -t_1 \\ t_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -t_2 \\ 0 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = t_1\mathbf{x}_2 + t_2\mathbf{x}_3$$

where $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent eigenvectors corresponding to $\lambda = -1$.

Example 8

Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$.

Solution

Let $$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = -9 + 3 + 7 = 1$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 3 & 4 \\ 8 & 7 \end{vmatrix} + \begin{vmatrix} -9 & 4 \\ -16 & 7 \end{vmatrix} + \begin{vmatrix} -9 & 4 \\ -8 & 3 \end{vmatrix}$$

$$= (21-32) + (-63+64) + (-27+32)$$

$$= -11 + 1 + 5$$

$$= -5$$

$$S_3 = \det(A) = \begin{vmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{vmatrix}$$

$$= -9(21-32) - 4(-56+64) + 4(-64+48)$$

$$= 99 - 32 - 64$$

$$= 3$$

Hence, the characteristic equation is

$$\lambda^3 - \lambda^2 - 5\lambda - 3 = 0$$

$$\lambda = -1, -1, 3$$

(a) For $\lambda = -1$,
$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-8x + 4y + 4z = 0$$

Let
$$y = t_1 \text{ and } z = t_2$$

$$x = \frac{1}{2}t_1 + \frac{1}{2}t_2$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t_1 + \frac{1}{2}t_2 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t_1 \\ t_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}t_2 \\ 0 \\ t_2 \end{bmatrix} = t_1\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent eigenvectors corresponding to $\lambda = -1$.

(b) For $\lambda = 3$,
$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-12x + 4y + 4z = 0$$
$$-8x + 0y + 4z = 0$$
$$-16x + 8y + 4z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 4 & 4 \\ 0 & 4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -12 & 4 \\ -8 & 4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -12 & 4 \\ -8 & 0 \end{vmatrix}} = t$$

$$\frac{x}{16} = \frac{y}{16} = \frac{z}{32} = t$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{2} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ 2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = t\mathbf{x}_3$$ where $\mathbf{x}_3$ is an eigenvector corresponding to $\lambda = 3$.

Example 9

Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$.

Solution

Let $$A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & -6 & -4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 + 4 - 3 = 2$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 4 & 2 \\ -6 & -3 \end{vmatrix} + \begin{vmatrix} 1 & -4 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 1 & -6 \\ 0 & 4 \end{vmatrix}$$

$$= (-12 + 12) + (-3 + 0) + (4 + 0)$$

$$= 1$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{vmatrix}$$

$$= 1(-12 + 12) - 6(0 - 0) - 4(0 - 0)$$

$$= 0$$

Hence, the characteristic equation is

$$\lambda^3 - 2\lambda^2 + \lambda = 0$$

$$\lambda = 0, 1, 1$$

(a) For $\lambda = 0$, $$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - 6y - 4z = 0$$

$$0x + 4y + 2z = 0$$

$$0x - 6y - 3z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix}-6 & -4\\ 4 & 2\end{vmatrix}} = -\frac{y}{\begin{vmatrix}1 & -4\\ 0 & 2\end{vmatrix}} = \frac{z}{\begin{vmatrix}1 & -6\\ 0 & 4\end{vmatrix}} = t$$

$$\frac{x}{4} = \frac{y}{-2} = \frac{z}{4} = t$$

$$\frac{x}{2} = \frac{y}{-1} = \frac{z}{2} = t$$

$$\mathbf{x} = \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}2t\\ -t\\ 2t\end{bmatrix} = t\begin{bmatrix}2\\ -1\\ 2\end{bmatrix} = t\mathbf{x}_1$$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 0$.

(b) For $\lambda = 1$,

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix}0 & -6 & -4\\ 0 & 3 & 2\\ 0 & -6 & -4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$0x + 3y + 2z = 0$$

Let

$$x = t_1 \text{ and } z = t_2$$

$$y = -\frac{2}{3}t_2$$

$$\mathbf{x} = \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}t_1\\ -\frac{2}{3}t_2\\ t_2\end{bmatrix} = \begin{bmatrix}t_1\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}0\\ -\frac{2}{3}t_2\\ t_2\end{bmatrix} = t_1\begin{bmatrix}1\\ 0\\ 0\end{bmatrix} + t_2\begin{bmatrix}0\\ -\frac{2}{3}\\ 1\end{bmatrix} = t_1\mathbf{x}_2 + t_2\mathbf{x}_3$$

where $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent eigenvectors corresponding to $\lambda = 1$.

Example 10

Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix}1 & 2 & 2\\ 0 & 2 & 1\\ -1 & 2 & 2\end{bmatrix}$.

Solution

Let

$$A = \begin{bmatrix}1 & 2 & 2\\ 0 & 2 & 1\\ -1 & 2 & 2\end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix}1-\lambda & 2 & 2\\ 0 & 2-\lambda & 1\\ -1 & 2 & 2-\lambda\end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 + 2 + 2 = 5$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}$$

$$= (4-2)+(2+2)+(2-0)$$

$$= 2+4+2$$

$$= 8$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{vmatrix}$$

$$= 1(4-2) - 2(0+1) + 2(0+2)$$

$$= 2-2+4$$

$$= 4$$

Hence, the characteristic equation is

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\lambda = 1, 2, 2$$

(a) For $\lambda = 1$, $[A - \lambda I]\,\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x + \ \ y + z = 0$$

$$-x + 2y + z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix}} = t$$

$$\frac{x}{-1} = \frac{y}{-1} = \frac{z}{1} = t$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{-1} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = t\mathbf{x}_1$$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 1$.

(b) For $\lambda = 2$, $[A - \lambda I]\,\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x + 2y + 2z = 0$$
$$0x + 0y + \quad z = 0$$
$$-x + 2y + 0z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 2 \\ 0 & 0 \end{vmatrix}} = t$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{0} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = t\mathbf{x}_2$$ where $\mathbf{x}_2$ is an eigenvector corresponding to $\lambda = 2$.

Hence, there is only one eigenvector corresponding to repeated root $\lambda = 2$.

Example 11

Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$.

Solution

Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 0 + 0 + 3 = 3$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 0 & 1 \\ -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= (0 + 3) + (0) + (0)$$

$$= 3$$

$$S_3 = \det(A) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{vmatrix}$$

$$= 0 - 1(0-1) + 0$$

$$= 1$$

Hence, the characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

$$\lambda = 1, 1, 1$$

For $\lambda = 1$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x + y + 0z = 0$$

$$0x - y + z = 0$$

$$x - 3y + 2z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}} = t$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = t\mathbf{x}_1$$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 1$.

Hence, there is only one eigenvector corresponding to repeated root $\lambda = 1$.

Example 12

Find the values of μ *which satisfy the equation* $A^{100}\mathbf{x} = \mu\mathbf{x}$

where

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & -2-\lambda & -2 \\ 1 & 1 & -\lambda \end{vmatrix}=0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 2 - 2 + 0 = 0$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} -2 & -2 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix}$$

$$= (0+2)+(0+1)+(-4-0)$$

$$= -1$$

$$S_3 = \det(A) = \begin{vmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 2(0+2)-1(0+2)-1(0+2)$$

$$= 4-2-2$$

$$= 0$$

Hence, the characteristic equation is

$$\lambda^3 - \lambda = 0$$

$$\lambda = 0, 1, -1$$

If λ is an eigen value of A, it satisfies the equation $A\mathbf{x} = \lambda\mathbf{x}$.
For equation $A^{100}\mathbf{x} = \mu\mathbf{x}$, μ represents eigen values of A^{100}. Eigenvalues of $A^{100} = \lambda^{100}$, i.e., 0, 1, 1.
Hence, values of μ are 0, 1, 1.

Example 13

Find the characteristic roots and characteristic vectors of $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ *and verify that characteristic roots are of unit modulus.*

Solution

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} \cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda \end{vmatrix}=0$$

$$(\cos\theta-\lambda)^2+\sin^2\theta=0$$

$$(\cos\theta-\lambda)^2=-\sin^2\theta$$

$$\cos\theta-\lambda=\pm i\sin\theta$$

$$\lambda=\cos\theta\pm i\sin\theta$$

$$|\lambda|=\sqrt{\cos^2\theta+\sin^2\theta}=1$$

Hence, the characteristic roots are of unit modules.

(a) For $\lambda=\cos\theta+i\sin\theta$, $[A-\lambda I]\mathbf{x}=\mathbf{0}$

$$\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-i\sin\theta x-\sin\theta y=0$$

Let

$$y=t$$
$$x=it$$

$\mathbf{x}=\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} it \\ t \end{bmatrix}=t\begin{bmatrix} i \\ 1 \end{bmatrix}=t\mathbf{x}_1$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda=\cos\theta+i\sin\theta$.

(b) For $\lambda=\cos\theta-i\sin\theta$, $[A-\lambda I]\mathbf{x}=\mathbf{0}$

$$\begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$i\sin\theta x-\sin\theta y=0$$

Let

$$y=t$$
$$x=-it$$

$\mathbf{x}=\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -it \\ t \end{bmatrix}=t\begin{bmatrix} -i \\ 1 \end{bmatrix}=t\mathbf{x}_2$ where $\mathbf{x}_2$ is an eigenvector corresponding to $\lambda=\cos\theta-i\sin\theta$.

Example 14

Find orthogonal eigenvectors for the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

Solution

Let
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$
$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{vmatrix} = 0$$
$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 + 4 + 9 = 14$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 4 & 6 \\ 6 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$$
$$= (36-36) + (9-9) + (4-4)$$
$$= 0$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{vmatrix}$$
$$= 1(36-36) - 2(18-18) + 3(12-12)$$
$$= 0$$

Hence, the characteristic equation is

$$\lambda^3 - 14\lambda^2 = 0$$
$$\lambda = 0, 0, 14$$

(a) For $\lambda = 14$,
$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} -13 & 2 & 3 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$-13x + 2y + 3z = 0$$
$$2x - 10y + 6z = 0$$
$$3x + 6y - 5z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & 3 \\ -10 & 6 \end{vmatrix}} = \frac{y}{\begin{vmatrix} -13 & 3 \\ 2 & 6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -13 & 2 \\ 2 & -10 \end{vmatrix}} = t$$

$$\frac{x}{42} = \frac{y}{84} = \frac{z}{126} = t$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ 3t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = t\mathbf{x}_1$$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 14$.

(b) For $\lambda = 0$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 2y + 3z = 0$$

Let $y = t_1$ and $z = t_2$

.
$x = -2t_1 - 3t_2$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t_1 - 3t_2 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} -2t_1 \\ t_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3t_2 \\ 0 \\ t_2 \end{bmatrix} = t_1\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = t_1\mathbf{x}_2 + t_2\mathbf{x}_3$$

where $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent eigenvectors corresponding to $\lambda = 0$.

Since $\mathbf{x}_2$ and $\mathbf{x}_3$ are not orthogonal, we must choose $\mathbf{x}_3$ such that $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are orthogonal.

Let
$$\mathbf{x}_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

For $\mathbf{x}_1$ and $\mathbf{x}_3$ to be orthogonal, $\mathbf{x}_1^T\mathbf{x}_3 = 0$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$$l + 2m + 3n = 0 \qquad \ldots(1)$$

For $\mathbf{x}_2$ and $\mathbf{x}_3$ to be orthogonal, $\mathbf{x}_2^T\mathbf{x}_3 = 0$

$$\begin{bmatrix} -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$$-2l + m = 0 \qquad \ldots(2)$$

Solving Eqs (1) and (2) by Cramer's rule,

$$\frac{l}{\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix}} = -\frac{m}{\begin{vmatrix} 1 & 3 \\ -2 & 0 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}} = t$$

$$\frac{l}{-3} = \frac{m}{-6} = \frac{n}{5} = t$$

$$\frac{l}{3} = \frac{m}{6} = \frac{n}{5} = t$$

$$\mathbf{x} = \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 3t \\ 6t \\ 5t \end{bmatrix} = t\begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix} = t\mathbf{x}_3 \text{ where } \mathbf{x}_3 \text{ is an eigenvector corresponding to } \lambda = 14.$$

EXERCISE 10.5

1. Find the sum and product of the eigenvalues of the following matrices:

(i) $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

[Ans.: (i) –3, 4 (ii) –1, 45**]**

2. The matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ 1 & 2 & 3 \end{bmatrix}$ is singular. One of its eigenvalues is 2. Find the other two eigenvalues.

[Ans.: $1+\sqrt{5}, 1-\sqrt{5}$**]**

3. If two of the eigenvalues of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ are 2 and 8, find the third eigenvalue.

[Ans.: 2**]**

4. Find the eigenvalues of the matrix $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$. Hence, form the matrix whose eigenvalues are $\frac{1}{6}$ and -1.

$$\left[\textbf{Ans.:}\ 6, -1, -\frac{1}{6}\begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}\right]$$

5. If 2 and 3 are the eigenvalues of $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$, find the eigenvalues of A^{-1} and A^3.

$$\left[\textbf{Ans.:}\ \text{(i)}\ \frac{1}{2}, \frac{1}{2}, \frac{1}{3} \quad \text{(ii)}\ 2^3, 2^3, 3^3\right]$$

6. Two eigenvalues of $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$ are equal and are double the third. Find the eigenvalues of A^2.

[**Ans.:** 1, 4, 4]

7. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$, find the eigenvalues of $3A^3 + 5A^2 - 6A + 2I$

[**Ans.:** 4, 110, 10]

8. If $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, find the eigenvalues of the following matrices:

(i) $A^3 + I$ (ii) A^{-1} (iii) $A^2 - 2A + I$ (iv) $A^3 - 3A^2 + A$

$$\left[\textbf{Ans.:}\ \text{(i)}\ 2, 2, 126 \quad \text{(ii)}\ 1, 1, \frac{1}{5} \quad \text{(iii)}\ 0, 0, 16 \quad \text{(iv)}\ -1, -1, 55\right]$$

9. Find the eigenvalues and eigenvectors for the following matrices:

(i) $\begin{bmatrix} 9 & -1 & 9 \\ 3 & -1 & 3 \\ -7 & 1 & -7 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$

(iv) $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ (vi) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

(vii) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ (viii) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$ (ix) $\begin{bmatrix} 2 & 4 & -6 \\ 4 & 2 & -6 \\ -6 & -6 & -15 \end{bmatrix}$

(x) $\begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{bmatrix}$ (xi) $\begin{bmatrix} 7 & -2 & -2 \\ -2 & 1 & 4 \\ -2 & 4 & 1 \end{bmatrix}$ (xii) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

(xiii) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (xiv) $\begin{bmatrix} 7 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 7 \end{bmatrix}$ (xv) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(xvi) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ (xvii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ (xviii) $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

(xix) $\begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$ (xx) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Ans.:

(i) $-1, 0, 2; \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$ (ii) $-1, 2, 1; \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

(iii) $-1, -2, -3; \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ (iv) $1, 2, 3; \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

(v) $1, 1, 7; \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (vi) $5, 1, 1; \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(vii) $5, -3, -3; \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ (viii) $-1, 1, 1; \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(ix) $-2, 9, -18; \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ (x) $3, 6, 9; \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

(xi) $-3, 3, 9; \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ (xii) $2, 3, 6; \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

(xiii) $8, 2, 2; \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ (xiv) $12, 6, 6; \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(xv) $4, 1, 1; \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ (xvi) $1, 3, 3; \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

(xvii) $1, 2, 2; \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ (xviii) $3, 2, 2; \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}$

(xix) $1, 1, 1; \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ (xx) $2, 2, 2; \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

10. Verify that $\mathbf{x} = [2, 3, -2, -3]^T$ is an eigenvector corresponding to the eigenvalue $\lambda = 2$ of the matrix

$$A = \begin{bmatrix} 1 & -4 & -1 & -4 \\ 2 & 0 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{bmatrix}$$

11. If $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ then check whether eigenvectors of A are orthogonal.

12. If $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ then verify whether eigenvectors are linearly independent or not.

10.14 CAYLEY–HAMILTON THEOREM

Statement: Every square matrix satisfies its own characteristic equation.
Proof: Let A be an n-rowed square matrix. Its characteristic equation is

$$|A-\lambda I| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + p + a_n)$$

$$(A-\lambda I)\, adj\, (A-\lambda I) = |A-\lambda I|\, I \qquad \ldots(10.16)$$

$$[\{A\, adj(A) = |A|\, I]$$

Since adj $(A - \lambda I)$ has element as cofactors of elements of $|A - \lambda I|$, the elements of adj $(A - \lambda I)$ are polynomials in λ of degree n – 1 or less. Hence, adj $(A - \lambda I)$ can be written as a matrix polynomial in λ.

adj $(A - \lambda \mathrm{I}) = B_0\lambda^{n-1} + B_1\lambda^{\mathrm{n}-2} + \cdots + B_{\mathrm{n}-2}\lambda + B_{\mathrm{n}-1}$

where $B_0, B_1, \ldots, B_{\mathrm{n}-1}$ are matrices of order n.

$$(A-\lambda I)\, adj\, (A-\lambda I) = (A-\lambda I)[B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots + B_{n-2}\lambda + B_{n-1}]$$

$$|A-\lambda I|\, I = (A-\lambda I)[B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots + B_{n-2}\lambda + B_{n-1}]$$

$$(-1)^n[I\lambda^n + a_1 I\lambda^{n-1} + a_2 I\lambda^{n-2} + \cdots + a_{n-1}I\lambda + a_n I]$$

$$= (-IB_0)\lambda^n + (AB_0 - IB_1)\lambda^{n-1} + (AB_1 - IB_2)\lambda^{n-2} + \cdots + (AB_{n-2} - IB_{n-1})\lambda + AB_{n-1}$$

Equating corresponding coefficients,

$$-IB_0 = (-1)^n I$$

$$AB_0 - IB_1 = (-1)^n a_1 I$$

$$AB_1 - IB_2 = (-1)^n a_2 I$$

$$\vdots \qquad \vdots$$

$$AB_{n-2} - IB_{n-1} = (-1)^n a_{n-1} I$$

$$AB_{n-1} = (-1)^n a_n I$$

Premultiplying the above equations successively by A^n, A^{n-1}, A^{n-2},… I and adding,

$$(-1)^n [A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I] = \mathbf{0}$$

Hence,

$$A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I = \mathbf{0} \qquad \ldots(10.17)$$

Corollary: If A is a non-singular matrix, i.e. det $(A) \neq 0$ then premultiplying equation (1) by A^{-1}, we get

$$A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \ldots + a_n A^{-1} = \mathbf{0}$$

$$A^{-1} = -\frac{1}{a^n}[A^{n-1} + a_1 A^{n-2} + \ldots + a_{n-1} I]$$

Example 1

Apply Cayley–Hamilton theorem to $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ *and deduce that* $A^8 = 625I$.

Solution

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - S_1\lambda + S_2 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 - 1 = 0$

$$S_2 = \det(A) = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

$$= -1 - 4$$

$$= -5$$

Hence, the characteristic equation is

$$\lambda^2 - 5 = 0$$

By Cayley–Hamilton theorem, the matrix A satisfies its own characteristic equation.

$$A^2 - 5I = \mathbf{0}$$

$$A^2 = 5I$$

$$A^4 = 25I$$

$$A^8 = 625I$$

Example 2

Verify Cayley–Hamilton theorem for the following matrix and hence, find A^{-1} *and* A^4.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}=0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 2+2+2=6$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (4-1)+(4-1)+(4-1)$$

$$= 9$$

$$S_3 = \det(A) = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= 2(4-1)+1(-2+1)+1(1-2)$$

$$= 6-1-1$$

$$= 4$$

Hence, the characteristic equation is

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0} \qquad \text{...(1)}$$

The matrix A satisfies its own characteristic equation. Hence, Cayley–Hamilton theorem is verified.

Premultiplying Eq. (1) by A^{-1},

$$A^{-1}(A^3 - 6A^2 + 9A - 4I) = \mathbf{0}$$

$$A^2 - 6A + 9I - 4A^{-1} = \mathbf{0}$$

$$4A^{-1} = (A^2 - 6A + 9I)$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Multiplying Eq. (1) by A,

$$A(A^3 - 6A^2 + 9A - 4I) = \mathbf{0}$$

$$A^4 - 6A^3 + 9A^2 - 4A = \mathbf{0}$$

$$A^4 = 6A^3 - 9A^2 + 4A$$

$$= \begin{bmatrix} 132 & -126 & 126 \\ -126 & 132 & -126 \\ 126 & -126 & 132 \end{bmatrix} - \begin{bmatrix} 54 & -45 & 45 \\ -45 & 54 & -45 \\ 45 & -45 & 54 \end{bmatrix} + \begin{bmatrix} 8 & -4 & 4 \\ -4 & 8 & -4 \\ 4 & -4 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{bmatrix}$$

Example 3

Show that matrix $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ *satisfies Cayley–Hamilton theorem and hence find* A^{-1}*, if it exists.*

Solution

$$A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} -\lambda & c & -b \\ -c & -\lambda & a \\ b & -a & -\lambda \end{vmatrix}=0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 0$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} + \begin{vmatrix} 0 & -b \\ b & 0 \end{vmatrix} + \begin{vmatrix} 0 & c \\ -c & 0 \end{vmatrix}$$

$$= (0+a^2)+(0+b^2)+(0+c^2)$$

$$= a^2+b^2+c^2$$

$$S_3 = \det(A) = \begin{vmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix}$$

$$= 0 - c(0-ab) - b(ac-0)$$

$$= abc - abc$$

$$= 0$$

Hence, the characteristic equation is

$$\lambda^3 + (a^2+b^2+c^2)\lambda = 0$$

$$A^2 = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} = \begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -c^3-cb^2-ca^2 & b^3+bc^2+ba^2 \\ c^3+ca^2+cb^2 & 0 & -ab^2-ac^2-a^3 \\ -bc^2-b^3-a^2b & ac^2+ab^2+a^3 & 0 \end{bmatrix}$$

$$= -(a^2+b^2+c^2)\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= -(a^2+b^2+c^2)A$$

$$A^3 + (a^2+b^2+c^2)A = \mathbf{0}$$

The matrix A satisfies its own characteristic equation. Hence, Cayley–Hamilton theorem is verified.

$$\det(A)=\begin{vmatrix}0 & c & -b\\ -c & 0 & a\\ b & -a & 0\end{vmatrix}=-c(0-ab)-b(ac-0)$$

$$=abc-abc=0$$

Hence, A^{-1} does not exist.

Example 4

Find the characteristic roots of the matrix $A=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}$ *and verify Cayley–Hamilton theorem for this matrix. Find* A^{-1} *and also express* $A^5-4A^4-7A^3+11A^2-A-10I$ *as a linear polynomial in* A.

Solution

$$A=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix}1-\lambda & 4\\ 2 & 3-\lambda\end{vmatrix}=0$$

$$\lambda^2-S_1\lambda+S_2=0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 + 3 = 4$

$$S_2=\det(A)=\begin{vmatrix}1 & 4\\ 2 & 3\end{vmatrix}$$

$$=3-8$$

$$=-5$$

Hence, the characteristic equation is

$$\lambda^2-4\lambda-5=0$$

$$\lambda=-1,5$$

$$A^2=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}=\begin{bmatrix}9 & 16\\ 8 & 17\end{bmatrix}$$

$$A^2-4A-5=\begin{bmatrix}9 & 16\\ 8 & 17\end{bmatrix}-\begin{bmatrix}4 & 16\\ 8 & 12\end{bmatrix}-\begin{bmatrix}5 & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\mathbf{0} \qquad \text{... (1)}$$

The matrix A satisfies its own characteristic equation. Hence, Cayley–Hamilton theorem is verified.

Premultiplying Eq. (1) by A^{-1},

$$A^{-1}(A^2-4A-5)=0$$
$$A-4I-5A^{-1}=0$$
$$4A^{-1}=\frac{1}{5}(A-4I)$$
$$=\frac{1}{5}\begin{bmatrix}-3 & 4\\ 2 & -1\end{bmatrix}$$

Now, $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A^3(A^2 - 4A - 5I) - 2A(A^2 - 4A - 5I) + 3(A^2 - 4A - 5I) + A + 5I$
$$= (A^2-4A-5I)(A^3-2A+3I)+(A+5I)$$
$$= 0+(A+5I) \qquad \text{[Using Eq. (1)]}$$
$$= A+5I$$

which is a linear polynomial in A.

Example 5

Find the characteristic equation of the matrix $A=\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}$ *and hence find the matrix represented by* $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

Solution

$$A=\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$
$$\begin{vmatrix}2-\lambda & 1 & 1\\ 0 & 1-\lambda & 0\\ 1 & 1 & 2-\lambda\end{vmatrix}=0$$
$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 2 + 1 + 2 = 5$

S_2 = Sum of the minors of principal diagonal elements of A

$$=\begin{vmatrix}1 & 0\\ 1 & 2\end{vmatrix}+\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}+\begin{vmatrix}2 & 1\\ 0 & 1\end{vmatrix}$$
$$=(2-0)+(4-1)+(2-0)$$
$$=7$$

$$S_3 = \det(A) = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 2(2-0) - 1(0-0) + 1(0-1)$$

$$= 4 - 0 - 1$$

$$= 3$$

Hence, the characteristic equation is

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley–Hamilton theorem,

$$A^3 - 5A^2 + 7A - 3I = 0 \qquad \ldots(1)$$

Now, $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + (A^2 + A + I)$$

$$= (A^3 - 5A^2 + 7A - 3I)(A^5 + A) + (A^2 + A + I)$$

$$= 0 + (A^2 + A + I) \qquad \text{[Using Eq.(1)]}$$

$$= A^2 + A + I$$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^2 + A + I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Example 6

If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, *prove by induction that for every integer* $n \geq 3$, $A^n = A^{n-2} + A^2 - I$. *Hence, find* A^{50}.

Solution

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix}=0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1 =$ Sum of the principal diagonal elements of $A = 1 + 0 + 0 = 1$

$S_2 =$ Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$$

$$= (0-1)+(0-0)+(0-0)$$

$$= -1$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 1(0-1)+0+0$$

$$= -1$$

Hence, the characteristic equation is

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

By Cayley–Hamilton theorem,

$$A^3 - A^2 - A + I = 0$$

$$A^3 = A^2 + A - I$$

$$= A^{3-2} + A^2 - I \qquad \ldots(1)$$

Hence, $A^n = A^{n-2} + A^2 - I$ is true for $n = 3$.

Assuming that Eq. (1) is true for $n = k$,

$$A^k = A^{k-2} + A^2 - I$$

Multiplying both the sides by A

$$A^{k+1} = A^{k+1} + A^3 - A$$

Substituting the value of A^3

$$A^{k+1} = A^{k+1} + (A^2 + A - I) - A$$

$$A^{(k+1)-2} + A^2 - I$$

Hence, $A^n = A^{n-2} + A^2 - I$ is true for $n = k + 1$

Thus, by mathematical induction, it is true for $n \geq 3$.

We have, $\qquad A^n = A^{n-2} + A^2 - I$

$$= (A^{n-4} + A^2 - I) + A^2 - I$$
$$= A^{n-4} + 2(A^2 - I)$$
$$= (A^{n-6} + A^2 - I) + 2(A^2 - I)$$
$$= A^{n-6} + 3(A^2 - I)$$
....
$$A^n = A^{n-2r} + r(A^2 - I)$$

Putting $n = 50$ and $r = 24$,

$$A^{50} = A^{50-2(24)} + 24(A^2 - I)$$
$$= A^2 + 24A^2 - 24I$$
$$= 25A^2 - 24I$$
$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$
$$A^{50} = \begin{bmatrix} 25 & 0 & 0 \\ 25 & 25 & 0 \\ 25 & 0 & 25 \end{bmatrix} - \begin{bmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

EXERCISE 10.6

1. Verify Cayley–Hamilton theorem for the matrix A and hence, find A^{-1} and A^4.

(i) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$$\left[\begin{array}{l}\textbf{Ans.:}\\ \text{(i)}\begin{bmatrix}3&2&6\\1&1&2\\2&2&5\end{bmatrix},\begin{bmatrix}-55&104&24\\-20&-15&32\\32&-42&13\end{bmatrix}\quad \text{(ii)}\ \frac{1}{40}\begin{bmatrix}-3&5&11\\14&-10&2\\5&5&-5\end{bmatrix},\begin{bmatrix}248&101&218\\272&109&50\\104&98&204\end{bmatrix}\\ \text{(iii)}\ \frac{1}{6}\begin{bmatrix}4&0&2\\0&3&0\\2&0&4\end{bmatrix},\begin{bmatrix}41&0&-40\\0&16&0\\-40&0&41\end{bmatrix}\quad \text{(iv)}\begin{bmatrix}3&2&6\\1&1&2\\2&2&5\end{bmatrix},\begin{bmatrix}-55&104&24\\-20&-15&32\\32&-40&-23\end{bmatrix}\end{array}\right]$$

2. Verify that the matrix $A=\begin{bmatrix}1&2&0\\2&-1&0\\0&0&-1\end{bmatrix}$ satisfies its characteristic equation and hence, find A^{-2}.

$$\left[\textbf{Ans.:}\ A^3+A^2-5A-5I=\mathbf{0},\ \frac{1}{5}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right]$$

3. Use Cayley–Hamilton theorem to find $2A^5-3A^4+A^2-4I$, where $A=\begin{bmatrix}3&1\\-1&2\end{bmatrix}$.

$$\left[\textbf{Ans.:}\ 138A-403I=\begin{bmatrix}11&138\\-138&127\end{bmatrix}\right]$$

4. If $A=\begin{bmatrix}1&4\\1&1\end{bmatrix}$, find A^7-9A^2+I.

[**Ans.:** $609A+640I$]

5. Verify Cayley–Hamilton theorem for (i) $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$ (ii) $A=\begin{bmatrix}2&4\\1&2\end{bmatrix}$ and hence, find A^{-1} and A^3-5A^2.

$$\left[\textbf{Ans.:}\ \text{(i)}\begin{bmatrix}-2&1\\1.5&-0.5\end{bmatrix},\ 2A\ \ \text{(ii)}\ A^{-1}\ \text{does not exist},\ A^2\right]$$

6. Compute $A^9-6A^8+10A^7-3A^6+A+I$, where $A=\begin{bmatrix}1&2&3\\-1&3&1\\1&0&2\end{bmatrix}$.

$$\left[\textbf{Ans.:}\begin{bmatrix}2&2&3\\-1&4&1\\1&0&3\end{bmatrix}\right]$$

7. Verify Cayley-Hamilton theorem for $A=\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}$ and evaluate $2A^4 - 5A^3 - 7A - 6I$.

$$\left[\text{Ans.}: \begin{bmatrix}36 & 32\\ 32 & 52\end{bmatrix}\right]$$

10.15 SIMILARITY TRANSFORMATION

If A and B are two square matrices of order n then B is said to be similar to A, if there exists a nonsingular matrix P such that

$$B = P^{-1}AP$$

Note: (1) Similarity of matrices is an equivalence relation.

(2) Similar matrices have the same determinant.

(3) Similar matrices have the same characteristic polynomial and hence the same eigenvalues. If $\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue λ, then $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding to the eigenvalue λ where $B = P^{-1}AP$.

10.16 DIAGONALIZATION OF A MATRIX

A matrix A is said to be *diagonalizable* if it is similar to a diagonal matrix.

A matrix A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix, also known as *spectral matrix*. The matrix P is then said to diagonalize A or transform A to a diagonal form. P is known as the *modal matrix*.

Note (1) An $n \times n$ matrix is diagonalizable if and only if it possesses n linearly independent eigenvectors.

(2) If the eigenvalues of an $n \times n$ matrix are all distinct then it is always similar to a diagonal matrix.

(3) If A is similar to a diagonal matrix D, the diagonal elements of D are the eigenvalues of A.

10.16.1 Orthogonally Similar Matrices

If A and B are two square matrices of order n then B is said to be orthogonally similar to A if there exists an orthogonal matrix P such that

$$B = P^{-1}AP$$

Since P is orthogonal, $P^{-1} = P^T$

$$B = P^{-1}AP = P^TAP$$

Note: (1) Every real symmetric matrix is orthogonally similar to a diagonal matrix with real elements.

(2) A real symmetric matrix of order n has n mutually orthogonal real eigenvectors.

(3) Any two eigenvectors corresponding to two distinct eigenvalues of a real symmetric matrix are orthogonal.

Note: To find the orthogonal matrix P, each element of the eigenvector is divided by its norm (length).

Working Rule for Diagonalization of Square Matrix A

(i) Find the eigenvalues of the square matrix A.
(ii) Find the eigenvectors corresponding to each eigenvalue.
(iii) Find the modal matrix P having the normalized eigenvectors as its column vectors.
(iv) Find the diagonal matrix $D = P^T AP$. The diagonal matrix D has eigenvalues as its diagonal elements.

Example 1

Show that the matrix $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ *is not diagonalizable.*

Solution

Let
$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

The characteristic equation is
$$\det(A - \lambda I) = 0$$
$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$
$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 2 + 2 + 2 = 6$

S_2 = Sum of the minors of principal diagonal elements of A
$$= \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}$$
$$= (4-0) + (4-0) + (4-0)$$
$$= 12$$
$$S_3 = \det(A) = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{vmatrix}$$
$$= 2(4-0) - 1(0-0) + 0$$
$$= 8 - 0 + 0$$
$$= 8$$

Hence, the characteristic equation is

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

$$\lambda = 2, 2, 2$$

For $\lambda = 2$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x + \quad y + 0z = 0$$

$$0x + 0y + \quad z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}} = t$$

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{0} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = t\mathbf{x}_1$$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 1$.

Since the matrix A has only one linearly independent eigenvector which is less than its order 3, matrix A is not diagonalizable.

Example 2

Show that the matrix $\begin{bmatrix} 1 & -2 & 0 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ *is not diagonalizable.*

Solution

Let

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & -2 & 0 \\ 1 & 2-\lambda & 2 \\ 1 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 + 2 + 3 = 6$

$$S_2 = \text{Sum of the minors of principal diagonal elements of } A$$
$$= \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix}$$
$$= (6-4)+(3-0)+(2+2)$$
$$= 9$$
$$S_3 = \det(A) = \begin{vmatrix} 1 & -2 & 0 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$
$$= 1(6-4)+2(3-2)+0$$
$$= 4$$

Hence, the characteristic equation is

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$
$$\lambda = 1, 1, 4$$

(a) For $\lambda = 1$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} 0 & -2 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$0x - 2y + 0z = 0$$
$$x + y + 2z = 0$$
$$x + 2y + 2z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -2 & 0 \\ 1 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}} = t$$
$$\frac{x}{-4} = \frac{y}{0} = \frac{z}{2} = t$$
$$\frac{x}{2} = \frac{y}{0} = \frac{z}{-1} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix} = t\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = t\mathbf{x}_1$$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 1$.

(b) For $\lambda = 4$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} -3 & -2 & 0 \\ 1 & -2 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$-3x - 2y + 0z = 0$$
$$x - 2y + 2z = 0$$
$$x + 2y - z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix}-2 & 0\\ -2 & 2\end{vmatrix}} = -\frac{y}{\begin{vmatrix}-3 & 0\\ 1 & 2\end{vmatrix}} = \frac{z}{\begin{vmatrix}-3 & -2\\ 1 & -2\end{vmatrix}} = t$$

$$\frac{x}{-4} = \frac{y}{6} = \frac{z}{8} = t$$

$$\frac{x}{-2} = \frac{y}{3} = \frac{z}{4} = t$$

$\mathbf{x} = \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}-2t\\ 3t\\ 4t\end{bmatrix} = t\begin{bmatrix}-2\\ 3\\ 4\end{bmatrix} = t\mathbf{x}_2$ where $\mathbf{x}_2$ is an eigenvector corresponding to $\lambda = 1$.

Since the matrix A has two linearly independent eigenvectors which is less than its order 3, the matrix A is not diagonalizable.

Example 3

Determine a diagonal matrix orthogonally similar to the real symmetric matrix $\begin{bmatrix}3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}$. *Also find the modal matrix.*

Solution

Let
$$A = \begin{bmatrix}3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix}3-\lambda & -1 & 1\\ -1 & 5-\lambda & -1\\ 1 & -1 & 3-\lambda\end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 3 + 5 + 3 = 11$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix}5 & -1\\ -1 & 3\end{vmatrix} + \begin{vmatrix}3 & 1\\ 1 & 3\end{vmatrix} + \begin{vmatrix}3 & -1\\ -1 & 5\end{vmatrix}$$

$$= (15-1) + (9-1) + (15-1)$$

$$= 36$$

$$S_3 = \det(A) = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 3(15-1) + 1(-3+1) + 1(1-5)$$

$$= 42 - 2 - 4$$

$$= 36$$

Hence, the characteristic equation is

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = 2, 3, 6$$

(a) For $\lambda = 2$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - y + z = 0$$

$$-x + 3y - z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}} = t$$

$$\frac{x}{-2} = \frac{y}{0} = \frac{z}{2} = t$$

$$\frac{x}{-1} = \frac{y}{0} = \frac{z}{1} = t$$

$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 2$.

(b) For $\lambda = 3$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x - y + z = 0$$

$$-x + 2y - z = 0$$

$$x - y + 0z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix}-1 & 1\\ 2 & -1\end{vmatrix}} = -\frac{y}{\begin{vmatrix}0 & 1\\ -1 & -1\end{vmatrix}} = \frac{z}{\begin{vmatrix}0 & -1\\ -1 & 2\end{vmatrix}} = t$$

$$\frac{x}{-1} = \frac{y}{-1} = \frac{z}{-1} = t$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$$

$$\mathbf{x} = \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}t\\ t\\ t\end{bmatrix} = t\begin{bmatrix}1\\ 1\\ 1\end{bmatrix} = t\mathbf{x}_2$$ where $\mathbf{x}_2$ is an eigenvector corresponding to $\lambda = 3$.

(c) For $\lambda = 6$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix}-3 & -1 & 1\\ -1 & -1 & -1\\ 1 & -1 & -3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$-3x - y + z = 0$$

$$x + y - z = 0$$

$$x - y - 3z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix}-1 & 1\\ -1 & -1\end{vmatrix}} = -\frac{y}{\begin{vmatrix}-3 & 1\\ -1 & -1\end{vmatrix}} = \frac{z}{\begin{vmatrix}-3 & -1\\ -1 & -1\end{vmatrix}} = t$$

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{2} = t$$

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{1} = t$$

$$\mathbf{x} = \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}t\\ -2t\\ t\end{bmatrix} = t\begin{bmatrix}1\\ -2\\ 1\end{bmatrix} = t\mathbf{x}_3$$ where $\mathbf{x}_3$ is an eigenvector corresponding to $\lambda = 6$.

Since matrix A has three linearly independent eigenvectors which is same as its order, the matrix A is diagonalizable.

Length of eigenvector $\mathbf{x}_1 = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$

Length of eigenvector $\mathbf{x}_2 = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

Length of eigenvector $\mathbf{x}_3 = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$

The normalized eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

The modal matrix P has normalized eigenvectors as its column vectors.

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$P^{-1} = P^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$D = P^T AP = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Hence, the diagonal matrix D has eigenvalues as its diagonal elements.

Example 4

Determine a diagonal matrix orthogonally similar to the real symmetric matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$. *Also find the modal matrix.*

Solution

Let $$A=\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix}=0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 8 + 7 + 3 = 18$

S_2 = Sum of the minors of principal diagonal elements of A

$$=\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix}+\begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix}+\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$$=(21-16)+(24-4)+(56-36)$$

$$=5+20+20$$

$$=45$$

$$S_3=\det(A)=\begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$=8(21-16)+6(-18+8)+2(24-14)$$

$$=40-60+20$$

$$=0$$

Hence, the characteristic equation is

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda = 0, 3, 15$$

(a) For $\lambda = 0$, $$[A-\lambda I]\mathbf{x}=\mathbf{0}$$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$8x-6y+2z=0$$

$$-6x+7y-4z=0$$

$$2x-4y+3z=0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}} = t$$

$$\frac{x}{10} = \frac{y}{20} = \frac{z}{20} = t$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{2} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ 2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = t\mathbf{x}_1$$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 0$.

(b) For $\lambda = 3$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x - 6y + 2z = 0$$

$$-6x + 4y - 4z = 0$$

$$2x - 4y + 0z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 5 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}} = t$$

$$\frac{x}{16} = \frac{y}{8} = \frac{z}{-16} = t$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ -2t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = t\mathbf{x}_2$$ where $\mathbf{x}_2$ is an eigenvector corresponding to $\lambda = 3$.

(c) For $\lambda = 15$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -7x - 6y + 2z &= 0 \\ -6x - 8y - 4z &= 0 \\ 2x - 4y - 12z &= 0 \end{aligned}$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -7 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}} = t$$

$$\frac{x}{40} = \frac{y}{-40} = \frac{z}{20} = t$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = t\mathbf{x}_3$$ where $\mathbf{x}_3$ is an eigenvector corresponding to $\lambda = 15$.

Since the matrix A has three linearly independent eigenvectors which is same as its order, the matrix A is diagonalizable.

Length of eigenvector $\mathbf{x}_1 = \sqrt{1^2 + 2^2 + 2^2} = 3$

Length of eigenvector $\mathbf{x}_2 = \sqrt{2^2 + 1^2 + (-2)^2} = 3$

Length of eigenvector $\mathbf{x}_3 = \sqrt{2^2 + (-2)^2 + 1^2} = 3$

The normalized eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

The modal matrix P has normalized eigenvectors as its column vectors.

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$P^{-1} = P^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$D = P^T AP = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

Hence, the diagonal matrix D has eigenvalues as its diagonal elements.

Example 5

Determine a diagonal matrix orthogonally similar to the real symmetric matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$. *Also find the modal matrix.*

Solution

Let
$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 6 + 3 + 3 = 12$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$$= (9-1) + (18-4) + (18-4)$$

$$= 8 + 14 + 14$$

$$= 36$$

$$S_3 = \det(A) = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9-1) + 2(-6+2) + 2(2-6)$$

$$= 48 - 8 - 8$$

$$= 32$$

Hence, the characteristic equation is

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$\lambda = 2, 2, 8$$

(a) For $\lambda = 8$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x - 2y + 2z = 0$$

$$-2x - 5y - z = 0$$

$$2x - y - 5z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -2 & 2 \\ -5 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -2 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & -2 \\ -2 & -5 \end{vmatrix}} = t$$

$$\frac{x}{12} = \frac{y}{-6} = \frac{z}{6} = t$$

$$\frac{x}{2} = \frac{y}{-1} = \frac{z}{1} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = t\mathbf{x}_1$$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 8$.

Since matrix A has three linearly independent eigenvectors which is same as its order, matrix A is diagonalizable.

(b) For $\lambda = 2$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x - 2y + 2z = 0$$

Let $$y = t_1 \text{ and } z = t_2$$

$$x = \frac{1}{2}t_1 - \frac{1}{2}t_2$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t_1 - \frac{1}{2}t_2 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t_1 \\ t_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}t_2 \\ 0 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} = t_1\mathbf{x}_2 + t_2\mathbf{x}_3$$

where $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent eigenvectors corresponding to $\lambda = 2$.

The orthogonal matrix P has mutually orthogonal eigenvectors. Since $\mathbf{x}_2$ and $\mathbf{x}_3$ are not orthogonal, we must choose $\mathbf{x}_3$ such that $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are orthogonal.

Let $$\mathbf{x}_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

For orthogonality of eigenvectors,

$$\mathbf{x}_1^T\,\mathbf{x}_3 = \mathbf{0} \quad \text{and} \quad \mathbf{x}_2^T\mathbf{x}_3 = \mathbf{0}$$

$$\begin{bmatrix} 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} l \\ m \\ n \end{bmatrix} = \mathbf{0} \quad \text{and} \quad \begin{bmatrix} \frac{1}{2} & 1 & 0 \end{bmatrix}\begin{bmatrix} l \\ m \\ n \end{bmatrix} = \mathbf{0}$$

$$2l - m + n = 0 \quad \text{and} \quad \frac{1}{2}l + m + 0n = 0$$

By Cramer's rule,

$$\frac{l}{\begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix}} = -\frac{m}{\begin{vmatrix} 2 & 1 \\ \frac{1}{2} & 0 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 2 & -1 \\ \frac{1}{2} & 1 \end{vmatrix}} = t$$

$$\frac{l}{-1} = \frac{m}{\frac{1}{2}} = \frac{n}{\frac{5}{2}} = t$$

$$\frac{l}{-2} = \frac{m}{1} = \frac{n}{5} = t$$

$$\mathbf{x} = \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ 5t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} = t\mathbf{x}_3$$ where $\mathbf{x}_3$ is an eigenvector corresponding to $\lambda = 2$.

Length of eigenvector $\mathbf{x}_1 = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{6}$

Length of eigenvector $\mathbf{x}_2 = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2 + 0^2} = \sqrt{\frac{5}{2}}$

Length of eigenvector $\mathbf{x}_3 = \sqrt{(-2)^2 + (1)^2 + (5)^2} = \sqrt{30}$

The normalized eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -\frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{30}} \\ \frac{5}{\sqrt{30}} \end{bmatrix}$$

The modal matrix P has normalized eigenvectors as its column vectors.

$$P = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{bmatrix}$$

$$P^{-1} = P^T = \begin{bmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ -\frac{2}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{bmatrix}$$

$$D = P^T AP = \begin{bmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ -\frac{2}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Hence, the diagonal matrix D has eigenvalues as its diagonal elements.

Example 6

For a symmetric matrix A, the eigenvectors are $[1, 1, 1]^T$, $[1, -2, 1]^T$ *corresponding to* $\lambda_1 = 6$ *and* $\lambda_2 = 12$. *Find the eigenvector corresponding to* $\lambda_3 = 6$ *and find the matrix A.*

Solution

Let $\mathbf{x}_3 = [x_3, y_3, z_3]^T$ be the eigenvector corresponding to $\lambda_3 = 6$.

$$\mathbf{x}_1 = [1, 1, 1]^T, \quad \mathbf{x}_2 = [1, -2, 1]^T$$

Since A is real symmetric matrix, $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are orthogonal.

i.e.,
$$\mathbf{x}_1^T\mathbf{x}_3 = 0 \quad \text{and} \quad \mathbf{x}_2^T\mathbf{x}_3 = 0$$

$$[1, 1, 1]\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = 0 \quad \text{and} \quad [1, -2, 1]\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = 0$$

$$x_3 + y_3 + z_3 = 0$$
$$x_3 + 2y_3 + z_3 = 0$$

By Cramer's rule,

$$\frac{x_3}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = -\frac{y_3}{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{z_3}{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}} = t$$

$$\frac{x_3}{3} = -\frac{y_3}{0} = \frac{z_3}{-3} = t$$

$$\frac{x_3}{1} = \frac{y_3}{0} = \frac{z_3}{-1} = t$$

$$\mathbf{x} = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = t\mathbf{x}_3 \text{ where } \mathbf{x}_3 \text{ is an eigenvector corresponding to } \lambda = 8.$$

$$\mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Length of eigenvector $\mathbf{x}_1 = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

Length of eigenvector $\mathbf{x}_2 = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$

Length of eigenvector $\mathbf{x}_3 = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$

The normalized eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 6 \end{bmatrix} = P^T AP$$

$$\begin{aligned} A &= IAI \\ &= PP^T A\, PP^T \\ &= P(P^T AP)\, P^T \\ &= PDP^T \end{aligned}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 7 \end{bmatrix}$$

EXERCISE 1.9

1. Show that the following matrices are not similar to diagonal matrices.

(i) $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

2. Determine diagonal matrices orthogonally similar to the following real symmetric matrices. Also, find the modal matrix in each case.

(i) $\begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & 8 \end{bmatrix}$ (ii) $\begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{bmatrix}$

Ans.:

(i) $D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{bmatrix}$, $P = \begin{bmatrix} \frac{4}{\sqrt{8}} & 0 & \frac{1}{3} \\ \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ -\frac{1}{\sqrt{18}} & \frac{1}{\sqrt{2}} & \frac{2}{3} \end{bmatrix}$

(ii) $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix}$, $P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$

3. Find A^{11}, where $A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$

Ans.: $\begin{bmatrix} -1 & 10237 & -2047 \\ 0 & 1 & 0 \\ 0 & 10245 & -2048 \end{bmatrix}$

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. If A is an orthogonal matrix, then A^{-1} equals

(a) A (b) A^T (c) A^2 (d) none of these

2. If A is a symmetric matrix and $n \in N$, then A^n is

(a) symmetric (b) skew symmetric

(c) diagonal matrix (d) none of these

3. If $A = \begin{bmatrix} 1 & -5 & 7 \\ 0 & 7 & 9 \\ 11 & 8 & 9 \end{bmatrix}$, then trace of the matrix A is

(a) 17 (b) 25 (c) 3 (d) 12

4. All the four entries of the 2×2 matrix $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ are non-zero, and one of its eigenvalues is zero. Which of the following statements are true?

(a) $P_{11}P_{22} - P_{12}P_{21} = 1$ (b) $P_{11}P_{22} - P_{12}P_{21} = -1$

(c) $P_{11}P_{22} - P_{12}.P_{21} = 0$ (d) $P_{11}P_{22} + P_{12}P_{21} = 0$

5. The system of linear equations $4x + 2y = 7$, $2x + y = 6$ has

(a) a unique solution

(b) no solution

(c) an infinite number of solution

(d) exactly two distinct solutions

6. The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is

(a) 0 (b) 1 (c) 2 (d) 3

7. The eigenvalues and the corresponding eigenvectors of a 2×2 matrix are given by

eigenvalue	eigenvector
$\lambda_1 = 8$	$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\lambda_2 = 4$	$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The matrix is

(a) $\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix}$

8. Let $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} \frac{1}{2} & a \\ 0 & b \end{bmatrix}$. Then $(a + b)$ is

(a) $\frac{7}{20}$ (b) $\frac{3}{20}$ (c) $\frac{19}{60}$ (d) $\frac{11}{20}$

9. Given an orthogonal matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $[AA^{-1}]$ is

(a) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$

10. The characteristic equation of a 3×3 matrix P is defined as $\alpha(\lambda) = |\lambda I - P| = \lambda^3 + \lambda^2 + 2\lambda + 1 = 0$. If I denotes identity matrix, then the inverse of matrix P will be

(a) $P^2 + P + 2I$ (b) $P^2 + P + I$

(c) $-(P^2 + P + I)$ (d) $-(P^2 + P + 2I)$

11. For the matrix $P = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, one of the eigenvalue is equal to -2. Which of the following is an eigenvector?

(a) $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

12. Consider the following system of equations in three real variables x_1, x_2 and x_3.

$2x_1 - x_2 + 3x_3 = 1$

$3x_1 + 2x_2 + 5x_3 = 2$

$-x_1 + 4x_2 + x_3 = 3$

The system equation has

(a) no solution

(b) a unique solution

(c) more than one but a finite number of solutions

(d) an infinite number of solutions

13. Eigenvalues of a matrix $S = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ are 5 and 1. What are the eigenvalues of the matrix S^2?

(a) 1 and 25 (b) 6 and 4 (c) 5 and 1 (d) 2 and 10

14. Match the items in columns I and II

I	II
(P) Singular matrix	(1) Determinant is not defined
(Q) Non-square matrix	(2) Determinant is always ± 1
(R) Real symmetric matrix	(3) Determinant is zero
(S) Orthogonal matrix	(4) Eigenvalues are always real
	(5) Eigenvalues are not defined

(a) P-3, Q-1, R-4, S-2 (b) P-2, Q-3, R-4, S-1

(c) P-3, Q-2, R-5, S-4 (d) P-3, Q-4, R-2, S-1

15. A is a 3×4 real matrix and $Ax = b$ is an inconsistent system of equations. The highest possible rank of A is

(a) 1 (b) 2 (c) 3 (d) 4

16. The sum of eigenvalues of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ is

(a) 5 (b) 7 (c) 9 (d) 18

17. Let A and B be real symmetric matrices of size $n \times n$. Then which one of the following is true?

(a) $AA' = I$ (b) $A = A^{-1}$ (c) $AB = BA$ (d) $(AB)' = BA$

18. Among the following, the pair of vectors orthogonal to each other is

(a) [3, 4, 7], [3, 4, 7] (b) [1, 0, 0], [1, 1, 0]

(c) [1, 0, 2], [0, 5, 0] (d) [1, 1, 1], [−1, −1, −1]

19. The eigenvalues of the matrix $\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ are

(a) 2, –2, 1, –1 (b) 2, 3, –2, 4
(c) 2, 3, 1, 4 (d) none of these

20. The eigenvalues of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ are

(a) 0, 0, 0 (b) 0, 0, 1 (c) 0, 0, 3 (d) 1, 1, 1

21. The minimum and maximum eigenvalues of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are –2 and 6 respectively. What is the other eigenvalue?

(a) 5 (b) 3 (c) 1 (d) –1

22. The inverse of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$ (d) none of these

23. Are the following vectors linearly dependent?
$X_1 = [3, 2, 7]$, $X_2 = [2, 4, 1]$ and $X_3 = [1, -2, 6]$

(a) Dependent (b) Independent
(c) Cannot say (d) None of these

24. If $A = \begin{bmatrix} 1 & 0 \\ 1 & 7 \end{bmatrix}$, then the value of k for which $A^2 = 8A + kI$ is

(a) 5 (b) −5 (c) 7 (d) −7

25. $A = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is unitary matrix if and only if

(a) $a^2+b^2+c^2=0$ (b) $b^2+c^2+d^2=0$
(c) $a^2+b^2+c^2+d^2=1$ (d) $a^2+b^2+c^2+d^2=0$

26. Which of the following matrices is not diagonalizable?

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

27. The sum of the eigenvalues of the matrix $\begin{bmatrix} 3 & 4 \\ x & 1 \end{bmatrix}$ for real and negative value of x is

(a) greater than zero (b) less than zero
(c) zero (d) dependent values of x.

28. A is a singular matrix of order 3 with eigenvalues 2 and 3. The third eigenvalue is

(a) 1 (b) 0 (c) 4 (d) −1

29. The eigenvalues and the corresponding eigenvectors of a 2×2 matrix are given by

$$\lambda_1 = 8 \qquad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad\qquad \lambda_2 = 4 \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The matrix is

(a) $\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix}$

Answers

1. (b)	2. (a)	3. (a)	4. (c)	5. (b)	6. (c)	7. (a)	8. (a)
9. (c)	10. (d)	11. (d)	12. (b)	13. (a)	14. (a)	15. (b)	16. (b)
17. (d)	18. (c)	19. (b)	20. (b)	21. (b)	22. (a)	23. (a)	24. (d)
25. (c)	26. (d)	27. (a)	28. (b)	29. (a)			

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